2023 年臺灣國際科學展覽會
優勝作品專輯

作品編號　**010050**

參展科別　**數學**

作品名稱　**Explorative Development of Ford Circle and Sierpinski Triangle in Hyperbolic Geometry**

得獎獎項

國　　家　**Indonesia**

就讀學校　**SMP Cita Hati - West Campus**

指導教師　**Kuncoro Kohar**

Yon Hermana Anggowarsito

作者姓名　**Tristanio Zedric Yosa Anggowarsito**

關鍵詞　**Farey Sequence、Ford Circle、Apollonian Gasket**

作者照片

Explorative Development of Ford Circle and Sierpinski Triangle in Hyperbolic Geometry

Tristanio Zedric Yosa Anggowarsito
Cita Hati Christian Senior School West Campus, Surabaya, East Java

ABSTRACT

Explorations in mathematics are limited due to the negative image and perspective about Mathematics itself in high school. However, some topics in mathematics are found interesting in high schools such as geometry and sequences. Therefore, this research will look at the explorative development of Ford Circle that creates some interesting results while combined with other theories and geometry. The main focus of this research is to address and explore the ford circle with its connection to the Sierpinski Triangle in hyperbolic geometry. The investigation and exploration will focus on the properties of geometry in the hyperbolic plane, the fractal geometry of Ford Circle and Euclidean fractals through a hyperbolic perspective that brings a fascinating correlation between all the topics discussed in this research.

Keywords:

Farey Sequence
Ford Circle
Apollonian Gasket
Segitiga Sierpinski

Corresponding Author:

Tristanio Zedric Yosa Anggowarsito
Cita Hati Christian Senior School West Campus
Surabaya, Jawa Timur, Indonesia.
Email: ssw_tristanio.anggowarsito@bchati.sch.id

1. INTRODUCTION

1.1. Background Research

Mathematics is often categorized as a difficult subject in school. This causes mathematics not to be explored much and preferred for further research investigations. Many basic Maths topics attract attention and are taught in secondary schools. One of which is geometry and mathematical series. These two topics caught my attention to investigate further.

Geometry is the topic in Mathematics that relates to shape, size, relative position of figures, and those related to it. A series is a group of number sequences that have a pattern in changing numbers. Both are closely related to the Ford Circle which will be the focus of the discussion of this scientific paper.

The discussion of mathematics in this scientific work focuses on exploration. Exploration in the field of mathematics is necessary and interesting to study. Forms of exploration with different methods will produce different and interesting results, so that mathematics, which has minimal exploration and is rich in interrelationships and relationships with other fields, will have a major impact on today's modern science. The exploration in this scientific paper will focus on the development and elaboration of the Ford Circle and its relationship with several other mathematical theories.

1.2 Background Theory

In Mathematics, there are some properties or statements that define a valid element in Geometry, which are called Postulates. Geometry has been divided into 2 groups, which is euclidean geometry, and non-euclidean geometry. The Euclidean Geometry follow the Euclid's Postulates:

I. To draw a straight line from any point to any point.
II. To produce a finite straight line continuously in a straight line.
III. To describe a circle with any center and distance (i.e., radius).
IV. That all right angles are equal to each other.
V. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Mathematicians over the years have been trying to prove the Fifth postulates, since it is the most complicated than the rest since it is hard to prove where 3 angels equals less than 2 right angles (180 degrees). After years of discoveries, mathematicians have found a new concept of non-euclidean geometry which the fifth postulates apply.

1.2.1 Hyperbolic Geometry

Non-euclidean geometry has been divided into 2, Hyperbolic geometry and elliptic geometry. In this research, hyperbolic geometry will be our main discussion. Hyperbolic geometry is a negative-gaussian curvature plane, which one of the 3 principles of curvature is positive. Which is imagined as a hyperbolic plane with a saddle shaped design.

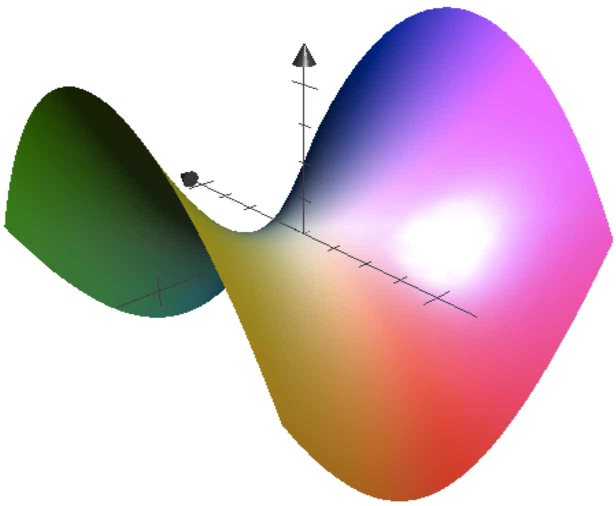


Figure 1. Saddle-shaped hyperbolic plane, z = $x^2 - y^2$ (plotted on "Grapher")

Hyperbolic Geometry has infinite lines, and a triangle in hyperbolic geometry does not add to 180 degrees.

1.2.2 Farey Sequence

$$F_1 = \left(\frac{0}{1}, \frac{1}{1}\right), \qquad |F_1| = 2,$$

$$F_2 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right), \qquad |F_2| = 3,$$

$$F_3 = \left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right), \qquad |F_3| = 5,$$

$$F_4 = \left(\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right), \qquad |F_4| = 7,$$

$$F_5 = \left(\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right), \qquad |F_5| = 11.$$

Figure 2. Farey Sequence

Farey Sequence, is a set of sequence where every F_{n+1} , the denominator increased by n+1. Meanwhile the numerator increases, as well as the amount of fraction as shown from the diagram above.

1.2.3 Ford Circle and Apollonian Gasket

Ford Circle and Appolonian gasket are both fractal patterns that involve circles in geometry. Ford Circles itself is a visual pattern which involves 2 tangent circles that creates another smaller circle inside of both tangent circles that happen tangent along the x-axis. Meanwhile Appolonian gaskets are a fractal pattern that is similar to a ford circle, but happen tangentially to a circle. In this case, each circles in Ford Circle follows a diameter of $\frac{1}{n^2}$ for every nth term in the circle and follows a placement from the farey sequence values.

Meanwhile, the apollonian gasket follows values from the Descartes theorem and radii as their base measurements as shown below,

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

Each variables is the curvature of each circles in the apollonian gasket

$$r = \frac{1}{d}$$

As it shows here, “r” symbolizes the variable of the curvature and “d” symbolizes the diameter.

Both figures below displays the fractal pattern of both Ford Circles and Apollonian Gasket

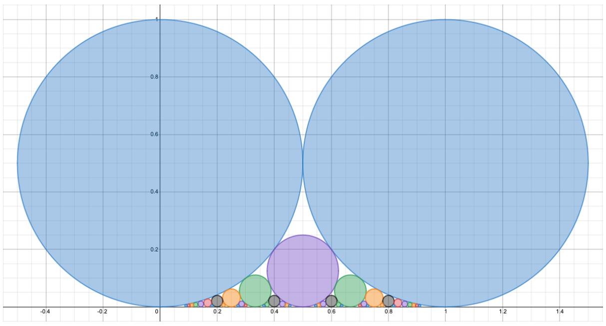


Figure 3. Ford Circles, Plotted on ("Desmos")

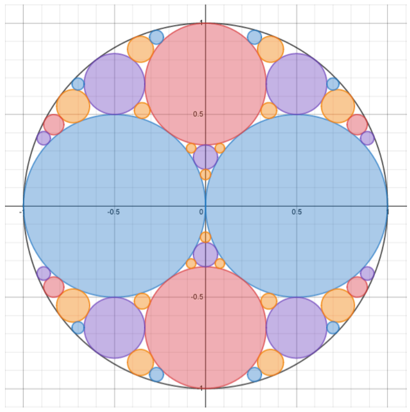


Figure 4. Apollonian Gasket, Plotted on ("Desmos")

1.2.4 Sierpinski Triangle

Another Fractal Geometry Pattern is Sierpinski Triangle, here, it is based off a Triangle, where every term, the number of triangle increases as follow using nth term: 3^{n-1}

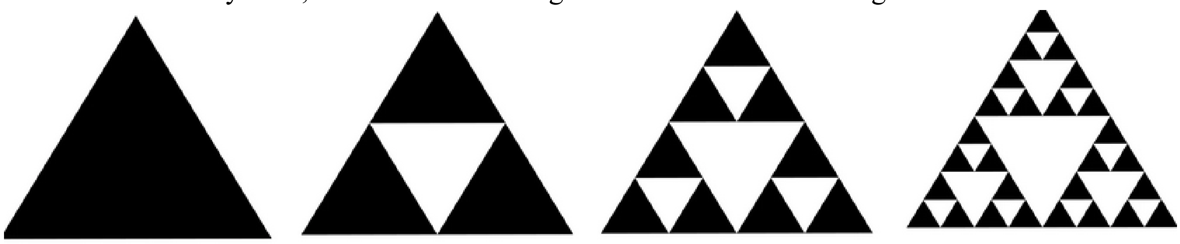


Figure 5. Sierpinski triangle from 1st to 4th Term

2. RESEARCH GOAL

The formulation of the problem to be examined in this scientific work is:

1. Developing and Analyzing the Ford Circle and Sierpinski Triangle in Hyperbolic Geometry.
2. Using Mobius transform Properties to develop the Ford Circle.

3. METHODOLOGY

Since this research is based on non-euclidean geometry, it is important to start off with an interest and gather as much information as possible, especially in hyperbolic geometry. Since hyperbolic geometry is rarely introduced to the public.

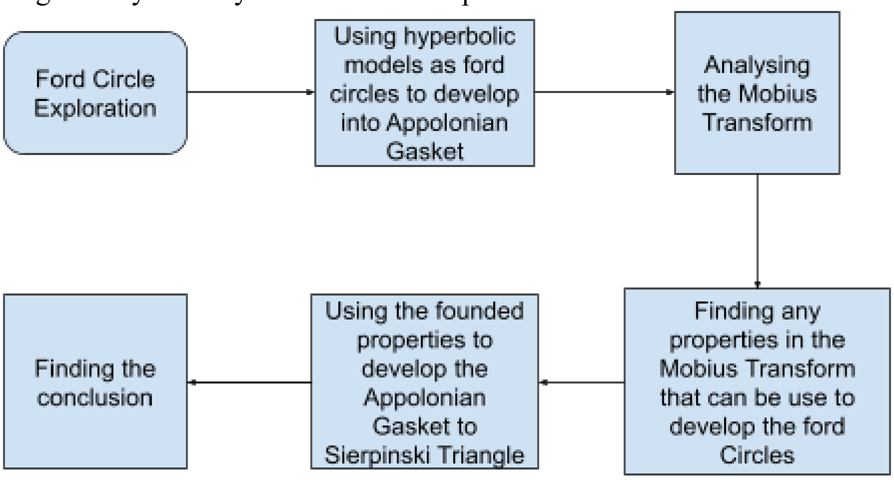


Figure 6. Methodology Diagram

This research is based on exploration to expand boundaries of ford circles using mathematical theories to develop into Sierpinski Triangle and find any connection possible using conformal mapping through mapping Ford Circles into Apollonian Gasket.

When visualizing these functions, a mathematical graphing software (Geogebra) is used to provide visualization to acquire more information on this research

4. RESULT AND ANALYSIS

4.1 Models of Hyperbolic Planes

Since it is already known that the properties of Hyperbolic Geometry do not follow the properties of Euclidean geometry that mostly people familiar with. That reason made mathematicians make models of hyperbolic geometry, one of them is Henri poincare. These models are representations of two-dimensional models on a hyperbolic plane.

This two dimensional hyperbolic plane has different properties from normal euclidean shapes, which has a curve on every distance drawn inside of each model. It is the reason why euclidean geometry is different from hyperbolic geometry.Which in this case, these circular hyperbolic models has a decrease in the value of π

4.1.1 Poincare Half Plane

$$H = \{(x, y) \mid y > 0; x, y \in \mathbb{R}\}$$

Metric tensor of Poincare half plane model:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Where ds is the distance inside the poincare half plane model and dx is the distance differences between the two points in the x axis and dy is the distance differences between the two points in the y axis.

In this case, Interpret a Poincare Half-plane model to draw a semi circle design across a point where 2 circles are tangent in the Ford Circle.

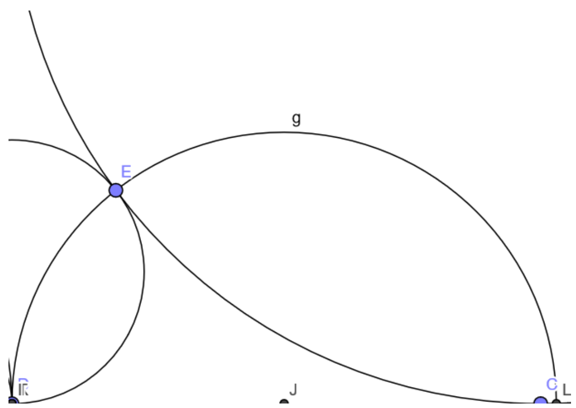


Figure 7. Poincare Half Plane Model Application on Ford Circle. Plotted on (“Geogebra”)

4.1.2 Poincare Disk

$$P = \{(x, y) \mid x^2 + y^2 = 1 < 1; x, y \in \mathbb{R}\}$$

Metric tensor of Poincare Disk Model using the same variable as the metric tensor of poincare half plane

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)}$$

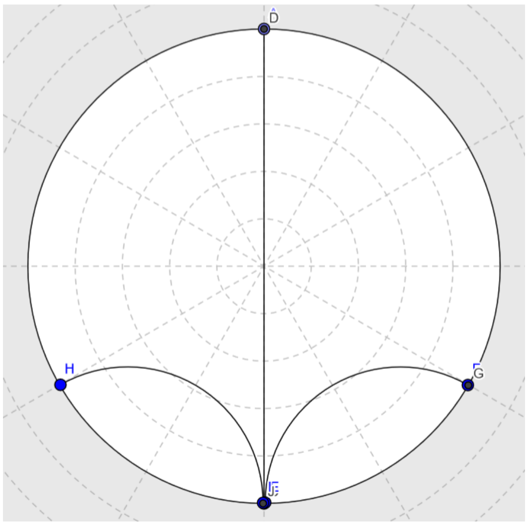


Figure 8. Poincare Disk Model Plotted on ("Geogebra")

Here, the Poincare model shares a 2-dimensional circle as in hyperbolic geometry. It is plotted the new semicircle from the poincare disk model. It is chapped like a circle, which shows that the more near a line drawn, the more curve it is produced, but it only applies to non circular objects. Since, circle is a shape with curvatures, it has no changes inside the poincare disk plane. It shows that the more edge of a line placement on the hyperbolic plane, the more curvy the line resulted.

4.2 Mapping into Apollonian Gasket

A conformal mapping is an angle perceiving transformation, in this case, mobius transformation will be used followed by circle inversion to transform Ford Circle into a new fractal.

Before starting the transformation, it is needed to understand the basic inversion, here, symbolize "z" as the variable for the mobius equation later on this paper

$$z = re^{i\theta}$$

While the inverse of "z" is

$$\frac{1}{z} = \frac{1}{r}ei^{(-\theta)}$$

The "z" variable itself symbolizes the vector where the inversion is forwarding to.

4.2.1 Mobius Transform

The mobius Transform follows as the equation shown below

$$T(z) = \frac{az+b}{cz+d}$$

If the formula expanded correctly ($c \neq 0$):

$$= \frac{c(az+b)}{c(cz+d)}$$

$$= \frac{acz+ad+bc-ad}{c(cz+d)}$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{(z+d/c)}$$

In which $\frac{a}{c}$ only measures the translation of the transform, $\frac{bc-ad}{c^2}$ measures the enlargement and shrinking rotation, and $\frac{1}{(z+d/c)}$ measures the inversion and reflection. Where as in mapping reciprocal mobius transform on the complex plane, the equation shown below is shown,

$$T(z) = z \rightarrow T(z) = \frac{1}{z}$$

By the reciprocal function above, when transforming into a riemann sphere, the function below must be used.

$$T(z) = \frac{iz+tan(\frac{\theta}{2})}{(tan(\frac{\theta}{2}))z + i}$$

Using the function above s inputing θ as to find the correct composition when rotating the sphere correctly.

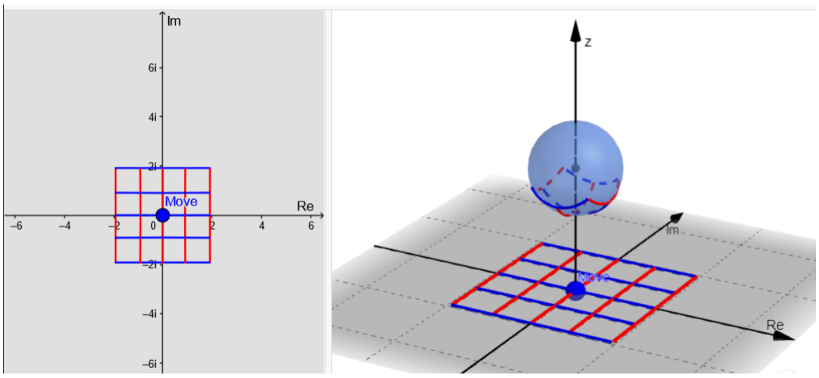


Figure 9. Riemann Sphere and Mobius Transform. Plotted on (“Geogebra")

As seen on Figure 9, it is seen that when the sphere is not rotated (Reimann sphere), it shows an euclidean design on the complex plane, but if the sphere is rotated, it shows an reciprocal design on the complex plane.

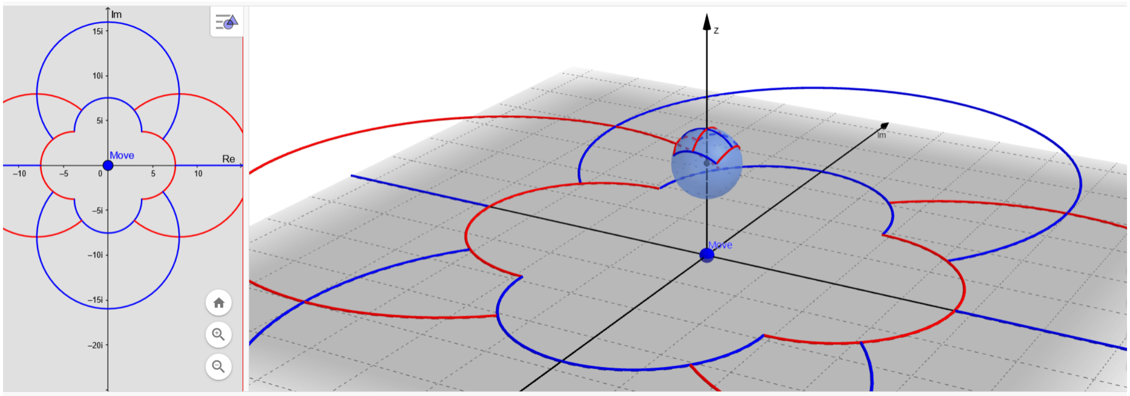


Figure 10 .Riemann Sphere and Mobius Transform. Plotted on (“Geogebra”)

All the lines shown on the complex plane is the result of a concept called inversion from the sphere, from here, it is possible use the concept of inversion to develop from ford circles to Appolonian gasket

4.2.2 Circle Inversion 1

Circle Inversion is an inversion that happens on the euclidean plane that maps circles and perceives the angles. Although this main research is focused on the Non-euclidean plane, it is possible to bring this concept to hyperbolic geometry while keeping the properties alive.

In the concept of Circle inversion, it has 3 main properties that is need to be apply on circle inversion:

(i) Angle Perceiving, Reverse Orientation
(ii) Preserve Symmetric Points
(iii) Orthogonal Circles Fixed

Since Circle inversions originally design to happen in euclidean geometry, cross out the 3rd properties since in this case, circles is the main focus for Non-euclidean geometry since the goal is to find a tangent circle and arrange it into an Appolonian gasket and since the object that is used is a circle which has neutral angle and only the value of π decreased.

In this concept, to create a new point or distance, the formula below is used:

$$r^2 = OA \cdot OA'$$

The variable "r" is the radius of the biggest circle, OA and OA’ both are distances of a straight line between both points. in this case A and A’ is equal because it is the tangent point of both circle. The variable A can be replaced by any variable along the circle.

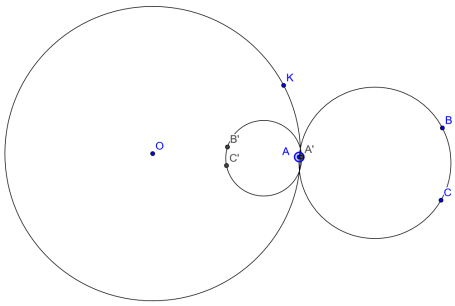


Figure 11. Circle Inversion Example. Plotted on (“Geogebra")

For example, when finding OB. Here let say the radius of the circle is 4, and B’ lies on 2:

$$r^2 = OB \cdot OB'$$
$$4^2 = OB \cdot 2$$
$$16 = OB \cdot 2$$
$$OB = \frac{16}{2}$$
$$OB = 8$$

Means that OB length is 8 after inverted from OB’ parallel to each point.

Our main focus here in the equation is to find each point of the inverted circles on the left side. Now, if the original circle on the left side, small enough and automatically, the inverted circle on the right will enlarge.

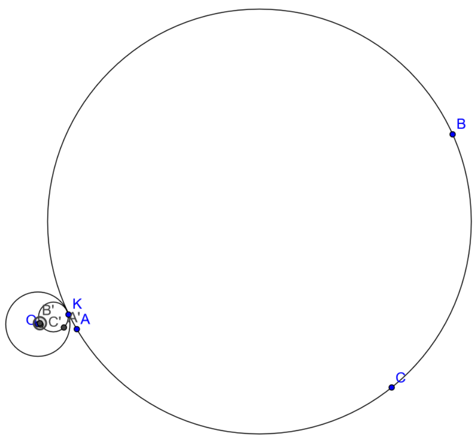


Figure 12. Circle Inversion Example. Plotted on (“Geogebra”)

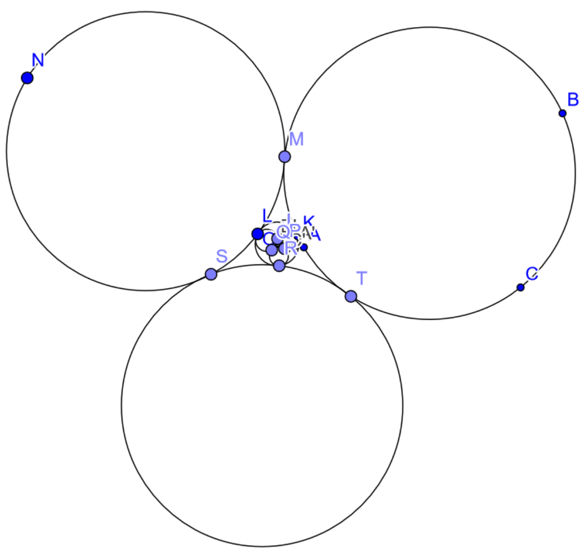


Figure 13. Circle Inversion example. Plotted on (“Geogebra”)

From here, it can replicate this 2 more times. From here, it is seen an apollonian gasket (1st term) formed by using Circle inversion, only by inputting a circle to the equation above.

4.2.3 Circle Inversion 2

Aside from creating an apollonian gasket model, There is another application of circle inversion from inverting a group of ford circles. It is known already, that ford circles usually stand on from the x-axis of 0 to 1.

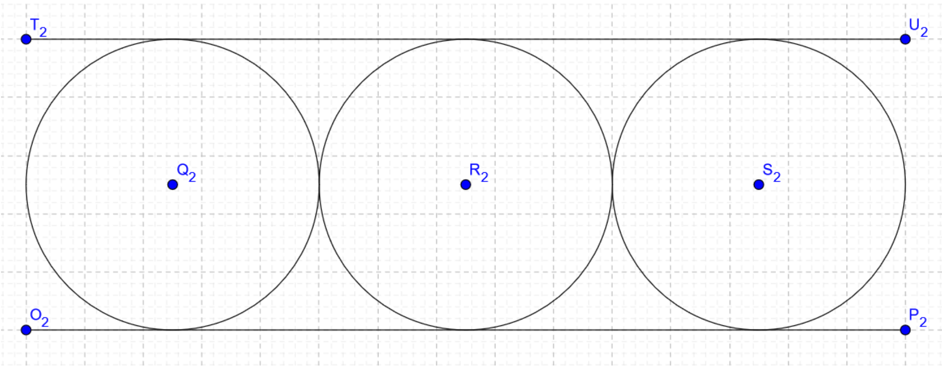
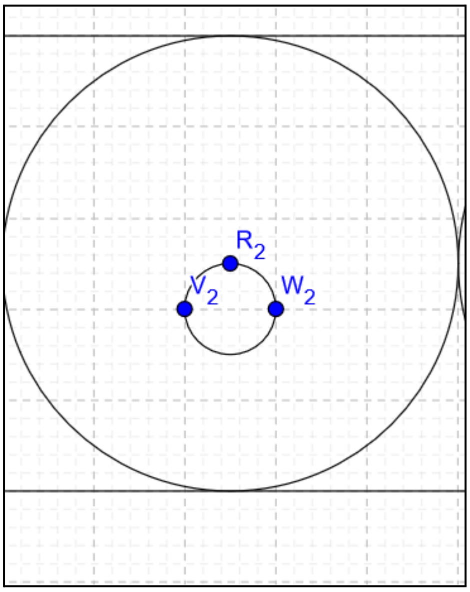


Figure 14. 3 series of ford circle (1st term). Plotted on ("Geogebra")

Here is 3 sets of ford circle, ranging from the farey sequence only F_1. From here, the line on the x axis is inverted to a circle. It is known that the total distance on the line is 3. If it is converted into a circle, it needs to see the line as the circumference.

In circle inversion, when the circle touches the center of the bigger circle, it will have a line as the inverted product, but if the smaller circle is bring to the front, the inverted product changed from a line to a circle, in which the other 2 circles position also changed.



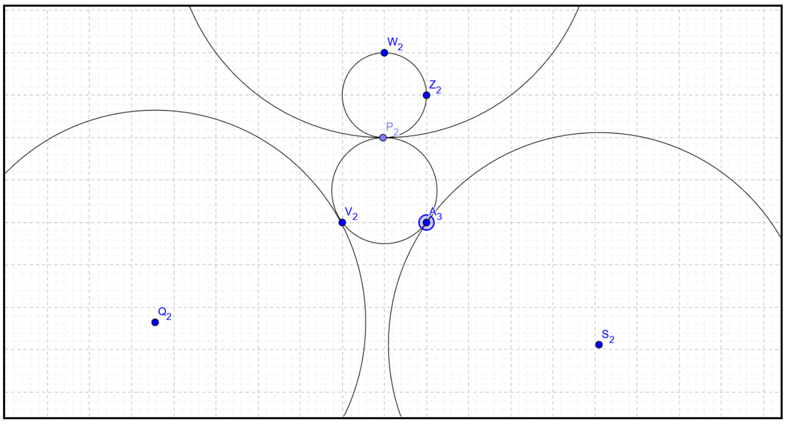


Figure 15. Circle Inversion. Plotted on ("Geogebra") Figure 16 Circle Inversion example. Plotted on ("Geogebra")

As it is seen on figure 16, the position of the circle changed due to the inverted circle. From here a similar problem with the problem that Appolonius created more than 2 thousand years ago is shown. Which is how to make all 4 circles tangent to each other. From here, decrease the size of the circle by an "x" amount. Since decreasing the smaller circle decreases, the larger ones increase. Hence, apollonian gasket is constructed.

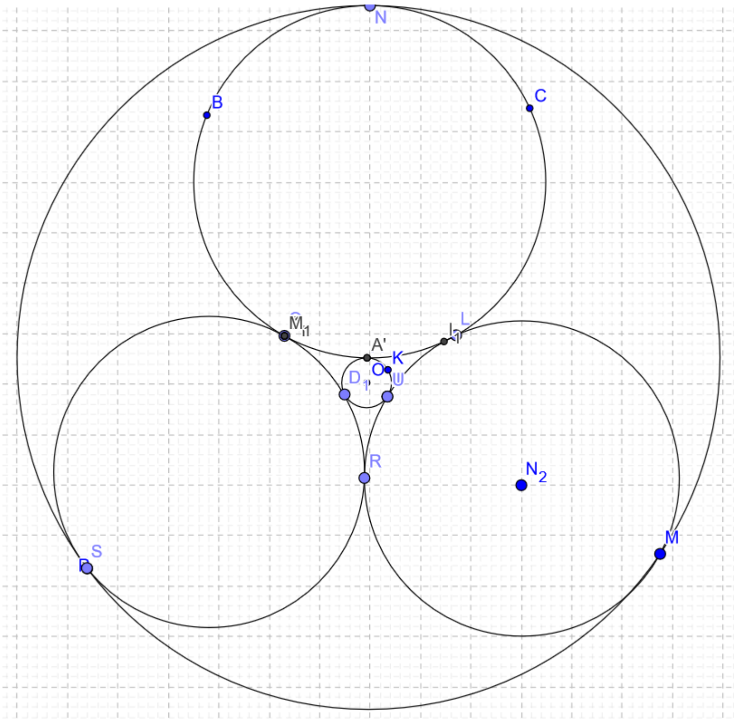


Figure 17. Circle Inversion example. Plotted on ("Geogebra")

The center circle has decreased, while the larger ons increased.

4.3 Mapping into Sierpinski Triangle

If using the ford circle transformation, draw smaller circles to support the inversion transformation

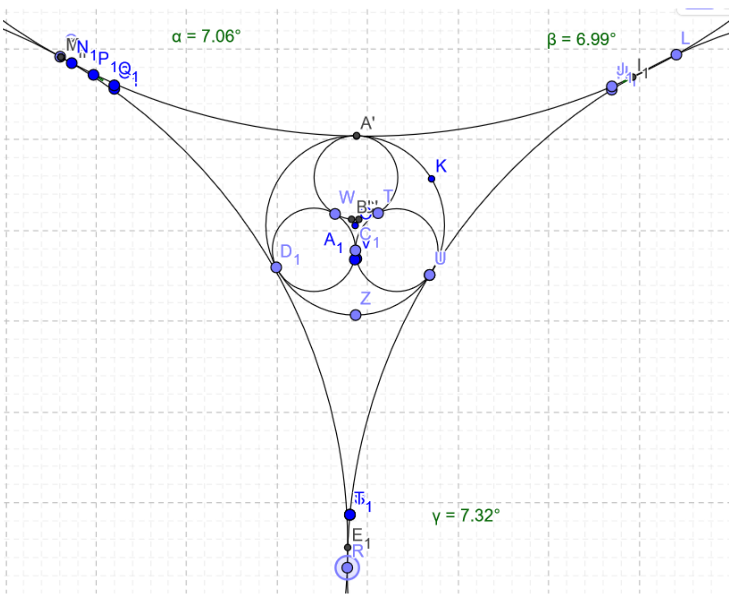


Figure 18. Sierpinski Triangle Shaped on hyperbolic Geometry. Plotted on ("Geogebra")

All of the three different angles are measured. It shows that both 3 angles have an average 7.12°. It has a similar amount of angle which may symbolize an equilateral triangle, but in a hyperbolic plane. If each of the curves were created in a poincare half plane model, it will not change the shape, it will still remain as a curve. Therefore, invert it back, to the euclidean plane, a sierpinski triangle pattern is discovered on figure 19.

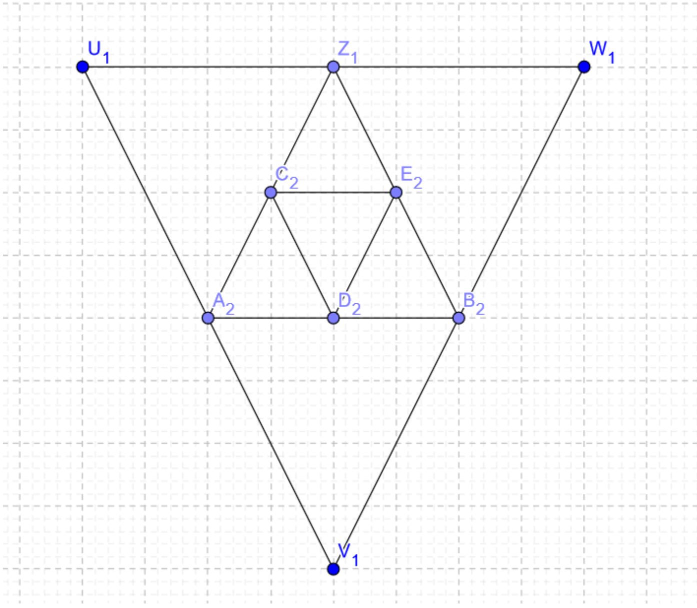


Figure 19. Sierpinski Triangle . Plotted on ("Geogebra")

5. CONCLUSION

From all of the explorations above, it is discovered that from looking at the hyperbolic geometry perspective and using a few hyperbolic models, transforming a series of Ford Circles into an Appolonian gasket using the concept from Mobius Transformation into Appolonian gasket and using it again to map it into a Sierpinski triangle by simply creating another circles and adding terms in the apollonian gasket fractal nth term and by converting the center point where the 3 big circles tangent and creates the small area and taking that to a hyperbolic design of a triangle using the properties of hyperbolic geometry of a negative gaussian curvature.it is seen mathematical concepts can be used by each other to change a fractal pattern in one of the beautiful ways.

Referring back to the title of this research paper, it is seen and proven that exploration and development on fractal geometry and hyperbolic geometry could be connected together. It is all connected by the concept from mobius transformation that connects both concepts.

6. REFERENCES

- Anthony Shaw. (n.d.)Inversion in a Circle of a Circle. Available from:<https://www.geogebra.org/m/FdZVb7pV >
- *Apollonian Gasket* (no date) *Apollonian gasket*. Available at: https://www.wikiwand.com/en/Apollonian_gasket
- Desmos. (2011). Desmos Graphing Calculator [Online]. Retrieved October 19, 2020, from https://www.desmos.com/calculator
- Elika Kurniadi. (November 14, 2021) The Difference Between Euclidean and Non Euclidean Geometry. Palembang Available from:<https://elikakurniadi.wordpress.com/2011/11/14/the-difference-between-euclidean-and-non-euclidean-geometry/>
- Ford Circles. (n.d.). Retrieved from https://mathworld.wolfram.com/FordCircle.html

- Juan Carlos Ponce Campuzano. (n.d.)Möbius transformation. Available from:<https://www.geogebra.org/m/GhaSJw3t>
- Mathemaniac. (2021, December 17). What if we define 1/0 = ∞? | Möbius transformations visualized [Video]. YouTube. https://www.youtube.com/watch?v=hhI8fVxvmaw&t=938s
- Mathemaniac. (2020, September 21). Problem of Apollonius - what does it teach us about problem solving?[Video]. YouTube. https://www.youtube.com/watch?v=Z6GG8zsMWH8&t=542s
- Mathemaniac. (2021, September 12).The 5 ways to visualize complex functions | Essence of complex analysis #3 [Video]. YouTube. https://www.youtube.com/watch?v=NtoIXhUgqSk&t=16s
- Mathemaniac. (2021, October 25).What do complex functions look like? | Essence of complex analysis #4 [Video]. YouTube.hhttps://www.youtube.com/watch?v=TaWBXRaNEcc&t=326s
- *Poincaré's half-plane model* (2013) *ThatsMaths*. Available at: https://thatsmaths.com/2013/10/11/poincares-half-plane-model/ (Accessed: December 31, 2022).
- Poincaré Hyperbolic Disk. (n.d.). Retrieved from https://mathworld.wolfram.com/PoincareHyperbolicDisk.html
- Sukhoy, V. and Stoytchev, A. (2021) *Formulas and algorithms for the length of a Farey sequence*. Nature Publishing Group. Available at: https://www.nature.com/articles/s41598-021-99545-w#article-info (Accessed: December 31, 2022).
- *The Sierpinski Triangle* (n. d.). Available at: https://math.bu.edu/DYSYS/chaos-game/node2.html (Accessed: December 31, 2022).

【評語】010050

The author observed a very nice application of hyperbolic geometry by considering Mobius transformation of circles and lines. Based on the observation，the author built up a connection between Mobius transformation of circles and Feray sequences. Moreover，Ford circle and Sierpinski Triangle and can derived from the above-mentioned observation by clever translation as well. These observations built a nice structure of the work. It would be even nicer if the author can formulate more mathematical statements and possibly prove them.