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作品名稱　Can Quantum Mechanical Two-State Theory model Coulomb’s Force?

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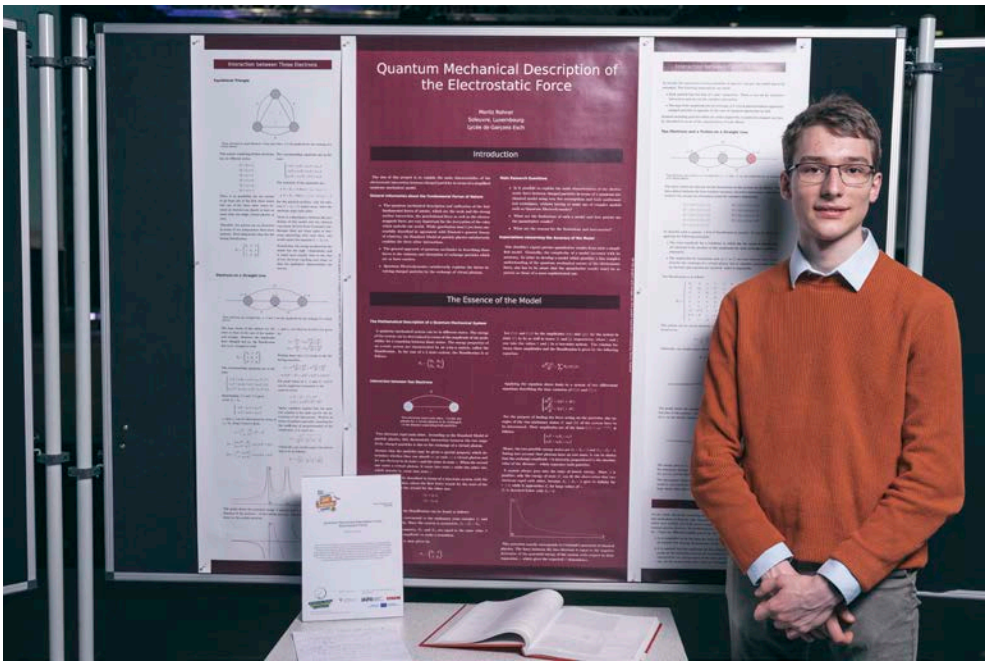
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Can Quantum Mechanical Two-State Theory model Coulomb's Force?

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1 Introduction

The quantum mechanical description of the four fundamental forces of nature is very important for the decryption of the rules which underlie our world. While Quantum Electrodynamics (QED) describes the electromagnetic force in great detail, it also uses complex mathematical techniques and advanced physical concepts.

In the following, I will analyze to what extent a quantum mechanical two-state model can be used to describe the Coulomb interaction between two charged particles. To do so, I will exclusively focus on the electrostatic interaction, leaving dynamics aside. Furthermore, the analysis is nonrelativistic and does not consider the spin of the particles. Finally, using discrete state theory allows to explore the strength of the basic concepts of early quantum mechanics.

In this sense, I will try to develop a simplified model for the quantum mechanical description of the electrostatic force. However, the analysis is not simplistic, since the traditional formalism of quantum mechanics will be used, including Dirac's Bra-ket notation, probability amplitudes, the Hamiltonian matrix as well as the Schrödinger equation.

To understand the framework of my project, it may be helpful to take a look at the source of inspiration for my analysis: In Chapter 10 of the third volume of the well-known textbook series *The Feynman Lectures on Physics*[4], the force holding the hydrogen molecular ion together is explained in terms of a two-state system. The electron of the molecular ion can be either at the first proton or at the second one. The exchange of the electron between both protons leads to an attractive force between them.

It is known from QED that the electrostatic interaction between two charged particles is due to the exchange of a virtual photon which acts as force carrier. The idea of my work is to explore whether the electrostatic force can be described by a very similar model, replacing the electron acting as force carrier in the molecular ion by a virtual photon for the description of the electrostatic force between two charged particles.

To describe a system consisting of charged particles, I will make the assumption that a charged particle can appear in two states. Either it is in state e where it can emit a photon or it is in state a which enables it to absorb a photon. Upon emission or absorption of a photon the charged particle transitions to the respective other state. This makes the approach analyzed in my work an element of discrete state theory, since two different states of the particle are used to store information about it.

Of course such a model cannot be compared to the sophisticated theory of Quantum Electrodynamics. The point is, however, that it is interesting to explore the power of the most fundamental concepts of quantum mechanics and to show that such an analysis can lead to inspiring results.

2 Remarks on Notation and Introduction to the Formalism of Quantum Mechanics

In quantum mechanics, all the information about the current configuration of a system is contained in its state. The state of a system has to be represented in terms of a linear combination of base states. The base states represent a certain basis which the state of the system is projected onto in order to be able to work with it.

In the following, we will use the possible combinations of a and e states as basis for describing a system made up of multiple charged particles. In this basis, a system consisting of two equally charged particles, for instance, has then 4 different possible base states. One in which both particles are in state e, one in which both are in state a and two in which they are in opposite states. One has to be careful not to confuse the two states of a single charged particle with the states of a system of multiple charged particles.

In particular, the information contained in the state of a system is the probability that a measurement in a certain basis of the state of the system yields a particular base state as result. Since the assumed a and e property of a charged particle is rather a theoretical tool than a measurable physical quantity, they represent a kind of pseudo-states, in the sense that they are treated in this analysis as real characteristics of charged particles while they don't seem to be measurable quantities. Treating them as real states allows, however, to use the formalism of quantum mechanics to calculate the properties of the system.

The probability for the system in state $|\psi\rangle$ to be found in a state $|1\rangle$ upon measurement is the square of the modulus of the probability amplitude $\langle 1|\psi\rangle$. The probability amplitude is a complex quantity found by taking the inner product of the state of interest $|1\rangle$ with the current state $|\psi\rangle$ of the system.

The time evolution of the state of a system is determined by Schrödinger's equation which involves its Hamiltonian. This mathematical operator contains all is needed to know about the system by describing the energy properties of the system. Solving Schrödinger's equation gives the possible states of the system and provides thus the answer to a particular question about the behaviour of the system.

3 Model of the Electrostatic Force

3.1 The Mathematical Description of Two-State Systems

The main point of the model should be introduced by using a quantum mechanical two-state system, which can be described in the following way:

The two base states $|1\rangle$ and $|2\rangle$ initially represent the system. The Hamiltonian, which is in this case a 2-by-2 matrix, describes the energy properties of the system and is given by

$$H_{ij} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{1}$$

Knowing the matrix elements of the Hamiltonian allows to solve the system and to get a description of its properties by applying Schrödinger's equation:

$$i\hbar \frac{dC_i(t)}{dt} = \sum_j H_{ij}(t) C_j(t) \tag{2}$$

$C_i(t)$ and $C_j(t)$ are the amplitudes $\langle i|\psi\rangle$ and $\langle j|\psi\rangle$ for the system in state $|\psi\rangle$ to be in states $|i\rangle$ and $|j\rangle$ respectively, where i and j can take the values 1 and 2 in a two-state system.[3]

3.2 Electron - Electron Interaction

According to the Standard Model of particle physics, the electrostatic interaction between charged particles is due to the exchange of a virtual photon.

The particles may be given a special property which determines whether they can absorb (a) or emit (e) a virtual photon. Suppose there are two electrons, one in state a and the other in state e. When the second one emits a virtual photon, it turns into state a while the other one, which absorbs it, turns into state e. This system can be described in terms of a two-state system with the following base states, where the first letter stands for the state of the first electron and the second for the other one:

$$\begin{aligned} |1\rangle &= |a; e\rangle \\ |2\rangle &= |e; a\rangle \end{aligned} \tag{3}$$

The elements of the Hamiltonian can be found as follows: If no exchange of a virtual photon were possible, both H_{21} and H_{12} would be 0 and H_{11} and H_{22} would correspond to the stationary state energies E_1 and E_2 respectively. Since the system is symmetric, $E_1 = E_2 = E_0$.

However, the possibility of a particle exchange is given and so, H_{21} and H_{12} differ from 0. Again by symmetry, H_{21} and H_{12} are both equal to the same value A which is the amplitude to make a transition. The Hamiltonian is then given by:

$$H_{ij} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix} \tag{4}$$

Applying Eq.(2) leads to a system of two differential equations describing the time variation of $C_1(t)$ and $C_2(t)$:

$$\begin{cases} i\hbar\dfrac{dC_1}{dt} = E_0C_1 + AC_2 \\ i\hbar\dfrac{dC_2}{dt} = E_0C_2 + AC_1 \end{cases} \tag{5}$$

By adding and subtracting both equations, one finds:

$$\begin{cases} C_1 + C_2 = ae^{-(i/\hbar)(E_0+A)t} \\ C_1 - C_2 = be^{-(i/\hbar)(E_0-A)t} \end{cases} \tag{6}$$

Where a and b are constants. $C_1(t)$ and $C_2(t)$ then become:

$$\begin{cases} C_1(t) = \dfrac{a}{2}e^{-(i/\hbar)(E_0+A)t} + \dfrac{b}{2}e^{-(i/\hbar)(E_0-A)t} \\ C_2(t) = \dfrac{a}{2}e^{-(i/\hbar)(E_0+A)t} - \dfrac{b}{2}e^{-(i/\hbar)(E_0-A)t} \end{cases} \tag{7}$$

The expressions for $C_1(t)$ and $C_2(t)$ show that a system which is initially in state $|1\rangle$ ($a = b = 1$), for instance, starts oscillating between states $|1\rangle$ and $|2\rangle$, which means that virtual photons are exchanged in this model.[3]

The study of the repulsive force between both negatively charged particles requires, however, to determine the allowed energy states for the system. These correspond to the energies of its stationary states $|I\rangle$ and $|II\rangle$. From Eq.(6), it follows that taking the sum and the difference of C_1 and C_2 gives stationary states because the absolute squares of these linear combinations, which represent the probability for being in these states, are independent of time since the absolute square of the pure imaginary exponential function is equal to 1.

Let $|I\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ and $|II\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ where the $\frac{1}{\sqrt{2}}$ factors are necessary for reasons of normalization. The energies of $|I\rangle$ and $|II\rangle$ are then equal to $E_I = E_0 + A$ and $E_{II} = E_0 - A$. By taking into account that photons have no rest mass, it can be shown that the exchange amplitude A is inversely proportional to the distance r which separates both particles, without referring to classical physics (see Appendix A):

$$A \propto \frac{1}{|r|} \tag{8}$$

The force between the two electrons is equal to the negative derivative of the potential energy of the system with respect to their separation r. Since E_0 doesn't change as a function of r, there is only the A term left, whose derivative is proportional to $1/r^2$. This is as it should be because the Coulomb force has the same r-dependence as well.[4]

However, the model makes two predictions, one with positive and one with negative sign of A. Obviously, since A is positive, only the energy of state $|I\rangle$ can fit the observation that two electrons repel each other, because $E_I = E_0 + A$ goes to infinity for $r \to 0$, while it approaches E_0 for a large r.

In fact, by comparing E_I to the expression derived from Coulomb's law, it turns out that the coefficient of proportionality has to be equal to $\frac{|q_1 q_2|}{4\pi\epsilon_0} = k_e|q_1 q_2| \approx 9 \cdot 10^9 |q_1 q_2| \frac{Nm^2}{C^2}$ (q_1 and q_2 are the charges of the particles)

4 Interaction between Three Electrons

4.1 Equilateral Triangle

As a first application of the model, a system consisting of three electrons put at equal distances of each other so that they form an equilateral triangle may be studied.

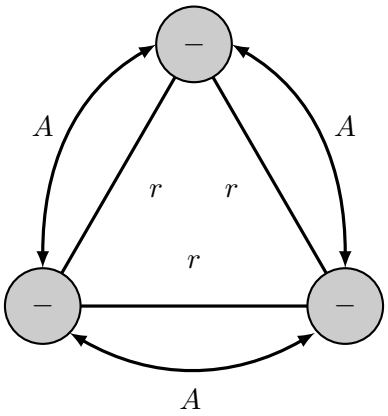


Figure 1: *Three electrons at equal distances r from each other; A is the amplitude for the exchange of a virtual photon*

Excluding the stationary states in which the three electrons are all either in state a or e leaves six different system states of the system consisting of the three electrons:

$$\begin{aligned} |1\rangle &= |a; e; e\rangle \\ |2\rangle &= |e; a; e\rangle \\ |3\rangle &= |e; e; a\rangle \\ |4\rangle &= |e; a; a\rangle \\ |5\rangle &= |a; e; a\rangle \\ |6\rangle &= |a; a; e\rangle \end{aligned} \tag{9}$$

Generally a 6-by-6 matrix is necessary to describe such a system. However, it should be noticed that, according to the model, there is no possibility for the system to go from one of the first three states into one of the three other states because an electron can absorb or emit no more than one single virtual photon at once.

Therefore, the system can be described in terms of two independent three-state systems. Both subsystems obey the following Hamiltonian, which can be derived using the fact that by symmetry, the different transitions between the states of each subsystem have the same amplitude A:

$$H_{ij} = \begin{pmatrix} E_0 & A & A \\ A & E_0 & A \\ A & A & E_0 \end{pmatrix} \tag{10}$$

The stationary states describing the different energy levels of the system are of the form $C_i(t) = a_i e^{-(i/\hbar)Et}$.[5] Putting these into Eq.(2) together with the Hamiltonian and cancelling out the common exponential factors leads to the following set of equations:

$$\begin{cases} a_I E = a_I E_0 + a_{II} A + a_{III} A \\ a_{II} E = a_{II} E_0 + a_I A + a_{III} A \\ a_{III} E = a_{III} E_0 + a_I A + a_{II} A \end{cases} \tag{11}$$

The allowed energy states are $E = E_0 + 2A$ for $a_I = a_{II} = a_{III} = 1$ and $E = E_0 - A$ for $a_I = 1$, $a_{II} = 0$ and $a_{III} = -1$, for instance. As it has already been explained in the previous section, only the solution $E = E_0 + 2A$ makes sense, since the electrons repel each other.

There is, however, a discrepancy between the prediction of this model and the classical expression derived from Coulomb's law. As there are three pairs of electrons interacting with each other, one would expect the energy to be equal to $E_c = E_0 + 3A$.

Nonetheless, the energy predicted by the model has the right r-dependence (r has the right exponent) and varies more rapidly than in the case of two electrons repelling each other, as one would expect.

4.2 Electrons on a Straight Line

In order to further check to which extent the model can be applied, a situation should be studied in which symmetry is broken. Three electrons on a straight line as shown in Fig.(2) represent such a system.

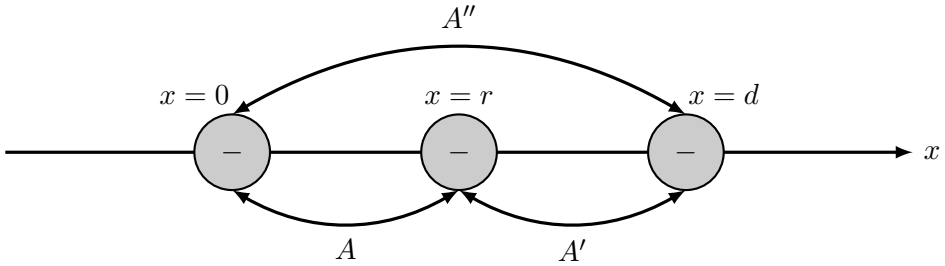


Figure 2: *Three electrons on a straight line; A, A′ and A″ are the amplitudes for the exchange of a virtual photon*

Let A be the amplitude for a virtual photon to be exchanged between the first and the second electron, A' the amplitude for an exchange between the second and the third one and finally A'' the amplitude

for the exchange between the first and the third electron. These amplitudes are proportional to the distance which separates the particles:

$$A \propto \frac{1}{|r|},\ A' \propto \frac{1}{|d-r|},\ A'' \propto \frac{1}{|d|} \tag{12}$$

The base states of the system are the same as those in the case of the equilateral triangle (excluding again the stationary states in which the three electrons are all either in state a or e). The letters represent the electrons in the same order as they are drawn in Fig.(2):

$$\begin{aligned} |1\rangle &= |a;e;e\rangle \\ |2\rangle &= |e;a;e\rangle \\ |3\rangle &= |e;e;a\rangle \\ |4\rangle &= |e;a;a\rangle \\ |5\rangle &= |a;e;a\rangle \\ |6\rangle &= |a;a;e\rangle \end{aligned} \tag{13}$$

However, the amplitudes have changed and so, the Hamiltonian has to be changed as well. Following the same arguments as those used in the previous section, it becomes clear that the system can again be described by a three-state subsystem with a Hamiltonian as follows:

$$H_{ij} = \begin{pmatrix} E_0 & A & A'' \\ A & E_0 & A' \\ A'' & A' & E_0 \end{pmatrix} \tag{14}$$

The set of equations which describes the system is given by applying Eq.(2) and letting $C_i(t) = a_i e^{-(i/\hbar)Et}$ be the amplitudes for the stationary states:[5]

$$\begin{cases} a_I E = a_I E_0 + a_{II} A + a_{III} A'' \ (L1) \\ a_{II} E = a_{II} E_0 + a_I A + a_{III} A' \ (L2) \\ a_{III} E = a_{III} E_0 + a_I A'' + a_{II} A' \ (L3) \end{cases} \tag{15}$$

Solving these equation requires some mathematics. One approach is to find a_I and a_{II} in terms of a_{III} using $(L1)$ and $(L2)$ and putting the results into $(L3)$. Since the E_0 term only represents a shift in the definition of the 0-energy level, it can be taken to be equal to 0. Rearranging the terms allows to write $(L1)$ and $(L2)$ as follows:

$$\begin{cases} a_I E - a_{II} A = a_{III} A'' \\ -a_I A + a_{II} E = a_{III} A' \end{cases} \tag{16}$$

Using Cramer's Rule to solve this system requires the following determinants:

$$\Delta = \begin{vmatrix} E & -A \\ -A & E \end{vmatrix} = E^2 - A^2$$

$$\Delta a_I = \begin{vmatrix} a_{III}A'' & -A \\ a_{III}A' & E \end{vmatrix} = a_{III}(EA'' + AA') \tag{17}$$

$$\Delta a_{II} = \begin{vmatrix} E & a_{III}A'' \\ -A & a_{III}A' \end{vmatrix} = a_{III}(EA' + AA'')$$

a_I and a_{II} can then be found to be given by:

$$\begin{aligned} a_I &= \frac{\Delta a_I}{\Delta} = a_{III}\frac{EA'' + AA'}{E^2 - A^2} \\ a_{II} &= \frac{\Delta a_{II}}{\Delta} = a_{III}\frac{EA' + AA''}{E^2 - A^2} \end{aligned} \tag{18}$$

Putting these into $(L3)$ leads to the following equation (with $E_0 = 0$):

$$\begin{aligned} E &= A''\frac{EA'' + AA'}{E^2 - A^2} + A'\frac{EA' + AA''}{E^2 - A^2} \\ \Leftrightarrow E(E^2 - A^2) &= EA'^2 + EA''^2 + 2AA'A'' \end{aligned} \tag{19}$$

It can be shown that this equation is coherent with the results in section 3.1 (see Appendix B).

For small values of A, A' and A'', $2AA'A''$ can be neglected compared to the squared terms. Then Eq.(19) reads:

$$\begin{aligned} E^2 &= A^2 + A'^2 + A''^2 \\ \Leftrightarrow E &= \pm\sqrt{A^2 + A'^2 + A''^2} \end{aligned} \tag{20}$$

Again, repulsion implies that the positive solution is the right one for the description of the interaction. Written in terms of position and with c standing for the coefficient of proportionality of the amplitudes, Eq.(20) gives the following result:

$$E = c\sqrt{\frac{1}{r^2} + \frac{1}{(d-r)^2} + \frac{1}{d^2}} \tag{21}$$

Classically, one would expect the potential to be as follows:

$$E_c = c\left(\frac{1}{|r|} + \frac{1}{|d-r|} + \frac{1}{|d|}\right) \tag{22}$$

Both E and E_c go to infinity when r tends to 0 or to d and they approach $\frac{c}{|d|}$ for large values of r. However, in between there is some discrepancy with a relative error of $f = E/E_c - 1$. Fig.(3) shows the potenial energy of the system as well as the discrepancy between both predictions. c is taken to be equal to 1. The c cancels out in the expression for the relative error, whose maxima seem to be at -0.4 regardless of the spacing between both fixed electrons. At infinity, the relative error approaches zero.

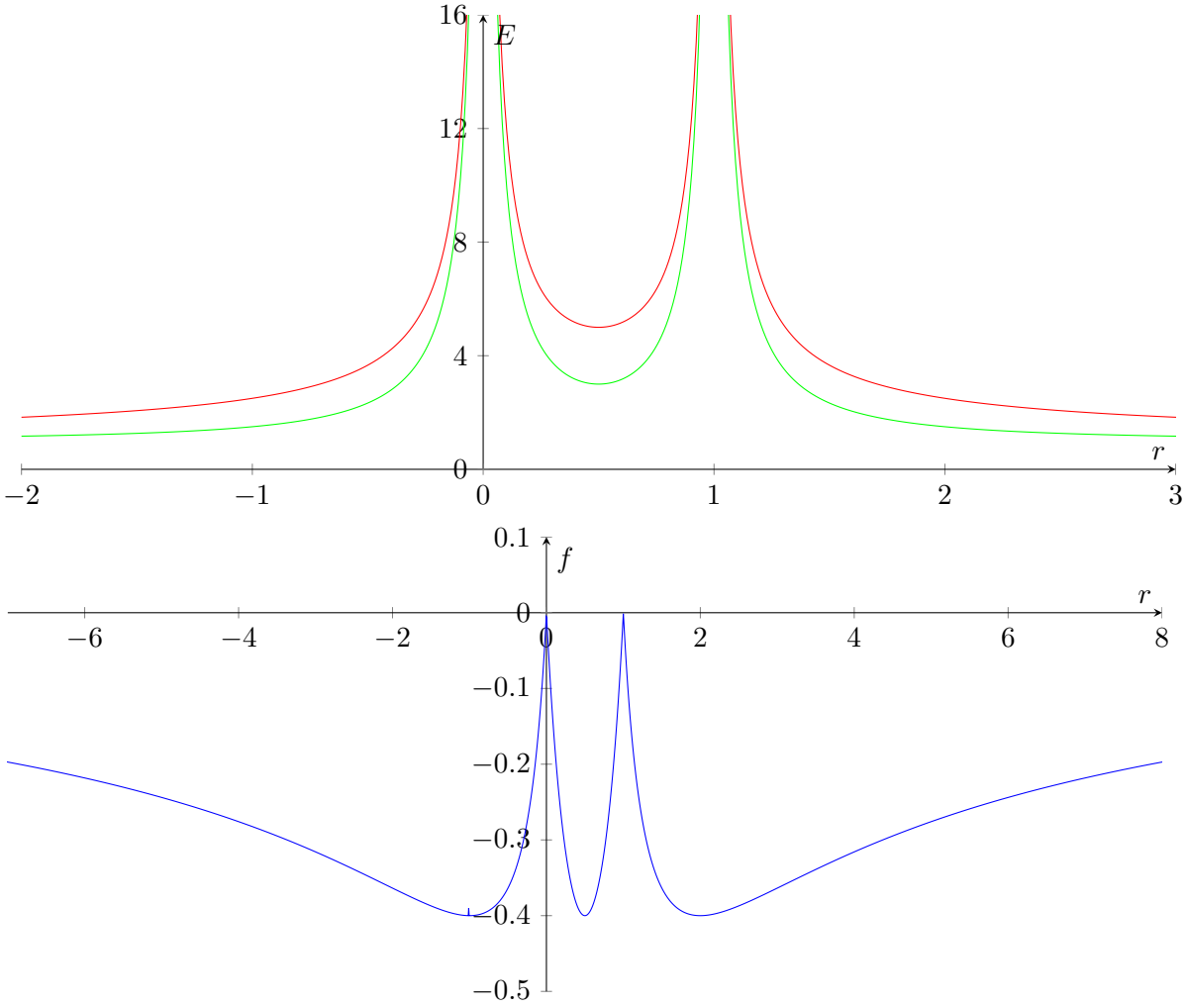


Figure 3: *The first graph shows the potential energy of the system E (green) computed quantum mechanically in terms of the a-e-model and E_c (red) computed from the classical Coulomb potential as a function of the position r of the middle electron of Fig.(2) while the other electrons are fixed at $x = 0$ and $x = 1$ respectively. The second graph shows the error of E with respect to E_c. The constant c is taken to be equal to* 1.

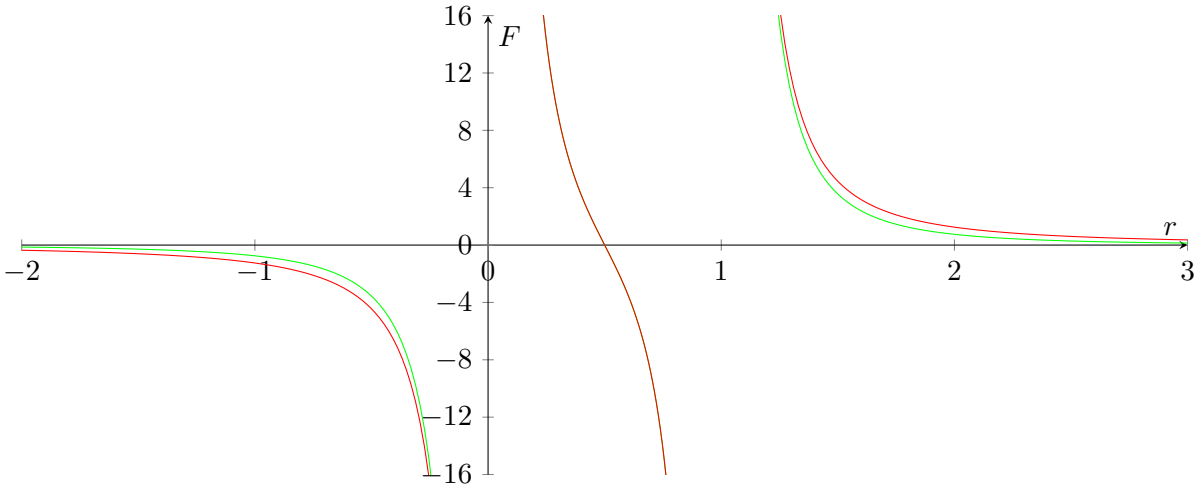
The force acting on the particles is the negative derivative of the potential energy with respect to position. The force calculated from the quantum mechanical model can be determined as follows:

$$\begin{aligned} F &= -c\,\frac{d}{dr}\sqrt{\frac{1}{r^2}+\frac{1}{(d-r)^2}+\frac{1}{d^2}} \\ &= -c\,\frac{1}{2}\left(\frac{1}{r^2}+\frac{1}{(d-r)^2}+\frac{1}{d^2}\right)^{-\frac{1}{2}}\left(-\frac{2}{r^3}-\frac{2}{(d-r)^3}(-1)\right) \\ &= c\,\frac{\frac{1}{r^3}-\frac{1}{(d-r)^3}}{\sqrt{\frac{1}{r^2}+\frac{1}{(d-r)^2}+\frac{1}{d^2}}} \end{aligned} \tag{23}$$

Using $(x^2)^{\frac{1}{2}} = |x|$, the classically derived force turns out to be:

$$\begin{aligned} F_c &= -c\,\frac{d}{dr}\left(\frac{1}{|r|}+\frac{1}{|d-r|}+\frac{1}{|d|}\right) \\ &= -c\,\left[\frac{d}{dr}\left(r^2\right)^{-\frac{1}{2}}+\frac{d}{dr}\left((d-r)^2\right)^{-\frac{1}{2}}\right] \\ &= -c\,\left[-\frac{1}{2}\left(r^2\right)^{-\frac{3}{2}}2r-\frac{1}{2}\left((d-r)^2\right)^{-\frac{3}{2}}2(d-r)(-1)\right] \\ &= c\,\left(\frac{r}{|r|^3}-\frac{d-r}{|d-r|^3}\right) \end{aligned} \tag{24}$$

Fig.(4) shows F and F_c as well as the relative error of F with respect to F_c defined by $f = F/F_c - 1$. Interestingly enough, f is equal to 0 between both fixed electrons ($0 < r < d$) while it tends to 1 for large absolute values of r. For every d, f seems to be equal to -0.4, for instance, at $r = 2d$ or at $r = -d$. As before, f is independent of c.



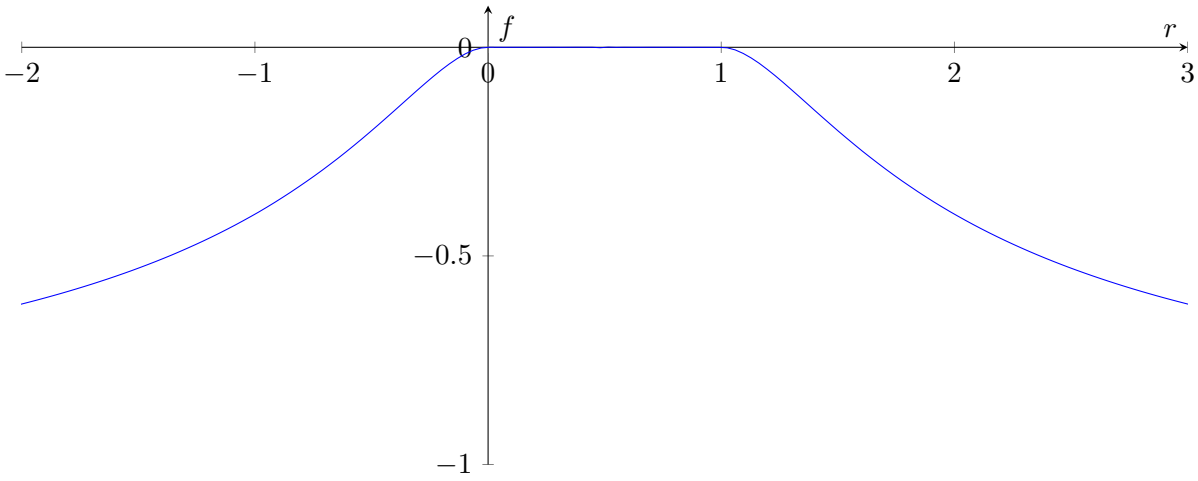


Figure 4: *The first graph shows the force on the middle electron of Fig.(2) F (green) computed quantum mechanically in terms of the a-e-model and F_c (red) computed from the classical Coulomb potential as a function of its position r while the other electrons are fixed at $x = 0$ and $x = 1$ respectively. The second graph shows the error of F with respect to F_c. The constant c is taken to be equal to* 1.

5 Interaction between Particles of Opposite Charges

To include the interaction between particles of opposite charges, the model has to be extended.

One may assume that each particle has two sets of a and e properties. There is one set for attractive interaction and one set for repulsive interaction. The sign of the amplitudes for an exchange of a virtual photon between oppositely charged particles is opposite to the case of repulsive interaction as well. Systems including particles which are either negatively or positively charged can then be described in terms of the superposition of both effects.

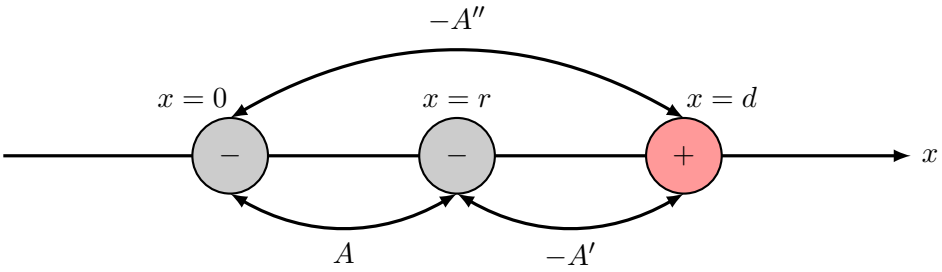


Figure 5: *Two electrons and a proton on a straight line; A, $-A'$ and $-A''$ are the amplitudes for an exchange of a virtual photon.*

The system in Fig.(5) consists of two electrons and a proton on a straight line. The amplitudes for an exchange of a virtual photon between the electrons and the proton are negative while the amplitude for an exchange between both electrons is positive. A, A' and A'' are still given by the following relations, where c is again a constant of proportionality:

$$A = \frac{c}{|r|}, \quad A' = \frac{c}{|d-r|}, \quad A'' = \frac{c}{|d|} \tag{25}$$

There are different subsystems which give the same (or other, for this purpose irrelevant) solutions, as discussed earlier. One of these subsystems can be described in terms of the following nine base states:

$$
\begin{aligned}
|1\rangle &= |(a,e,e);(a,e,e)\rangle \\
|2\rangle &= |(a,e,e);(e,a,e)\rangle \\
|3\rangle &= |(a,e,e);(e,e,a)\rangle \\
|4\rangle &= |(e,a,e);(a,e,e)\rangle \\
|5\rangle &= |(e,a,e);(e,a,e)\rangle \\
|6\rangle &= |(e,a,e);(e,e,a)\rangle \\
|7\rangle &= |(e,e,a);(a,e,e)\rangle \\
|8\rangle &= |(e,e,a);(e,a,e)\rangle \\
|9\rangle &= |(e,e,a);(e,e,a)\rangle
\end{aligned}
\tag{26}
$$

The three letters between the first brackets represent attractive interaction while those between the second set of brackets stand for repulsive interaction.

To describe such a system, a 9-by-9 Hamiltonian is required. However, quite a number of its elements are equal to 0 because some transitions like $|1\rangle$ to $|3\rangle$ are impossible. Such a transition would describe the exchange of a virtual photon due to repulsive interaction while an electron and a proton are involved. The total amplitude for a transition in which the two types of interactions are involved is the product of the amplitudes for both exchanges considered separately. Applying these principles gives the following Hamiltonian:

$$
H_{ij} = \begin{pmatrix}
E_0 & A & 0 & 0 & 0 & 0 & -A'' & -A''A & 0 \\
A & E_0 & 0 & 0 & 0 & 0 & -A''A & -A'' & 0 \\
0 & 0 & E_0 & 0 & 0 & 0 & 0 & 0 & -A'' \\
0 & 0 & 0 & E_0 & A & 0 & -A' & -A'A & 0 \\
0 & 0 & 0 & A & E_0 & 0 & -A'A & -A' & 0 \\
0 & 0 & 0 & 0 & 0 & E_0 & 0 & 0 & -A' \\
-A'' & -A''A & 0 & -A' & -A'A & 0 & E_0 & A & 0 \\
-A''A & -A'' & 0 & -A'A & -A' & 0 & A & E_0 & 0 \\
0 & 0 & -A'' & 0 & 0 & -A' & 0 & 0 & E_0
\end{pmatrix}
\tag{27}
$$

The set of equations which describe the system is given by applying Eq.(2) and letting $C_i(t) = a_i e^{-(i/\hbar)Et}$ be the amplitudes for the stationary states (E_0 is taken to be equal to 0):[5]

$$
\begin{cases}
a_1 E = a_2 A - a_7 A'' - a_8 A''A \ (L1) \\
a_2 E = a_1 A - a_7 A''A - a_8 A'' \ (L2) \\
a_3 E = -a_9 A'' \ (L3) \\
a_4 E = a_5 A - a_7 A' - a_8 A'A \ (L4) \\
a_5 E = a_4 A - a_7 A'A - a_8 A' \ (L5) \\
a_6 E = -a_9 A' \ (L6) \\
a_7 E = -a_1 A'' - a_2 A''A - a_4 A' - a_5 A'A + a_8 A \ (L7) \\
a_8 E = -a_1 A''A - a_2 A'' - a_4 A'A - a_5 A' + a_7 A \ (L8) \\
a_9 E = -a_3 A'' - a_6 A' \ (L9)
\end{cases}
\tag{28}
$$

There are multiple solutions which fit those equations, but which don't give the right potential. However, at least one solution does. It can be found by taking the values of a_i as follows:

$$\begin{aligned} a_1 &= a_2 = a' \\ a_4 &= a_5 = a'' \\ a_7 &= a_8 = a''' \\ a_3 &= a_6 = a_9 = 0 \end{aligned} \tag{29}$$

Putting these coefficients into Eq.(28) gives a set of three equations:

$$\begin{cases} a'E = a'A - a'''A''(1+A) \ (L') \\ a''E = a''A - a'''A'(1+A) \ (L'') \\ a'''E = -a'A''(1+A) - a''A'(1+A) + a'''A \ (L''') \end{cases} \tag{30}$$

(L') is equivalent to $a' = -a'''\frac{A''(1+A)}{E-A}$ while (L'') is equivalent to $a'' = -a'''\frac{A'(1+A)}{E-A}$. (L''') then gives the following equation:

$$\begin{aligned} E &= \frac{A''(1+A)}{E-A}A''(1+A) + \frac{A'(1+A)}{E-A}A'(1+A) + A \\ \Leftrightarrow E^2 - EA &= (A''(1+A))^2 + (A'(1+A))^2 + EA - A^2 \\ \Leftrightarrow 0 &= E^2 - 2EA - \left[(1+A)^2(A'^2 + A''^2) - A^2\right] \end{aligned} \tag{31}$$

This is an equation of the second degree which has two solutions. For some reason, the solution with the negative sign describes the system in this case:

$$\begin{aligned} E &= \frac{2A - \sqrt{4A^2 + 4\left[(1+A)^2(A'^2 + A''^2) - A^2\right]}}{2} \\ \Leftrightarrow E &= A - \sqrt{(1+A)^2(A'^2 + A''^2)} \\ \Leftrightarrow E &= A - |1+A|\sqrt{A'^2 + A''^2} \end{aligned} \tag{32}$$

Replacing the amplitudes in Eq.(32) by Eq.(25) gives the solution for the potential energy of the system:

$$E = \frac{c}{|r|} - \left|1 + \frac{c}{|r|}\right| \sqrt{\left(\frac{c}{d-r}\right)^2 + \left(\frac{c}{d}\right)^2} \tag{33}$$

Classically, one would expect the following expression:

$$E_c = c \left(\frac{1}{|r|} - \frac{1}{|d-r|} - \frac{1}{|d|} \right) \tag{34}$$

For values of c higher than 1, the second term of Eq.(33) becomes predominant, which doesn't fit the classical solution. However, as already described, c is equal to $9 \cdot 10^9 |q_1 q_2| \frac{Nm^2}{C^2}$ with $q_1 = q_2 = 1.602 \cdot 10^{-19} C$ in this case so that it becomes very small. Fig.(6) shows E and E_c for $d = 1$ and $c = 0.1$. The energy goes to $-\infty$ at $x = 1$ where the proton is located, which is sensible since the proton attracts the electron.

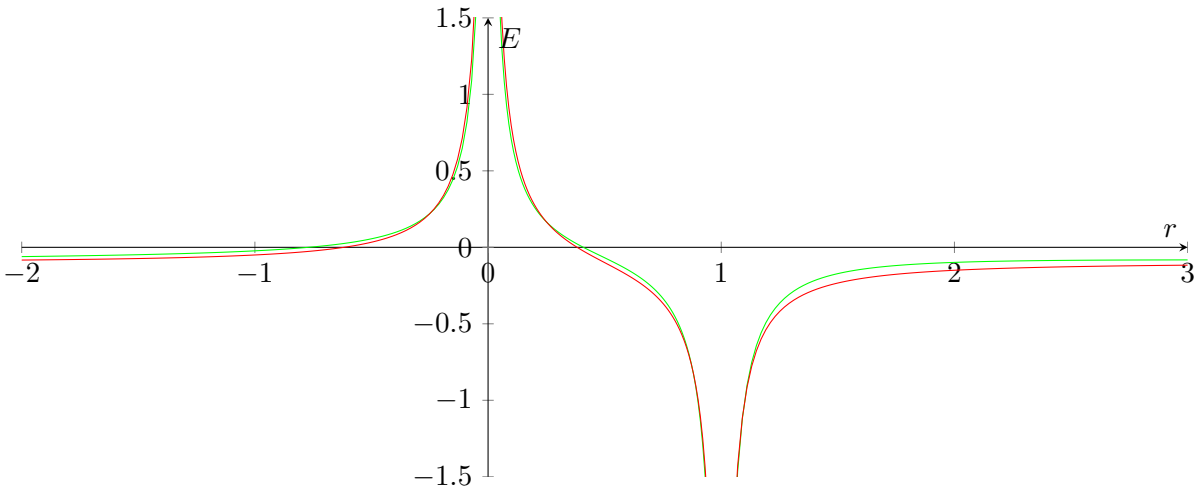


Figure 6: *The graph shows the potential energy of the system E (green) computed quantum mechanically in terms of the a-e-model and E_c (red) computed from the classical Coulomb potential as a function of the position r of the middle electron of Fig.(5) while the other electron and the proton are fixed at $x = 0$ and $x = 1$ respectively. The constant c is taken to be equal to* 0.1.

In analogy to Eq.(24), the classical expectation for the force reads as follows:

$$F_c = c \left(\frac{r}{|r|^3} + \frac{d-r}{|d-r|^3} \right) \tag{35}$$

Since c is very small $\left|1+\frac{c}{|r|}\right|$ may be taken equal to 1 in Eq.(33). Then the force predicted by the quantum mechanical model can be calculated in the following way:

$$\begin{aligned} F &= -\frac{d}{dr}\left(\frac{c}{|r|} - \sqrt{\left(\frac{c}{d-r}\right)^2 + \left(\frac{c}{d}\right)^2}\right) \\ &= c\left(-\frac{d}{dr}\frac{1}{|r|} + \frac{d}{dr}\sqrt{\left(\frac{1}{d-r}\right)^2 + \left(\frac{1}{d}\right)^2}\right) \\ &= c\left(\frac{r}{|r|^3} + \frac{1}{2}\left(\frac{1}{(d-r)^2} + \frac{1}{d^2}\right)^{-\frac{1}{2}}\left(-\frac{2}{(d-r)^3}(-1)\right)\right) \\ &= c\left(\frac{r}{|r|^3} + \frac{1}{(d-r)^3\sqrt{\frac{1}{(d-r)^2} + \frac{1}{d^2}}}\right) \end{aligned} \tag{36}$$

Fig.(7) shows F and F_c again for $d = 1$ and $c = 0.1$. The relative error between both predictions is very small between $r = 0$ and $r = d$ and takes on large values when the force goes to 0. It is independent of the constant c.

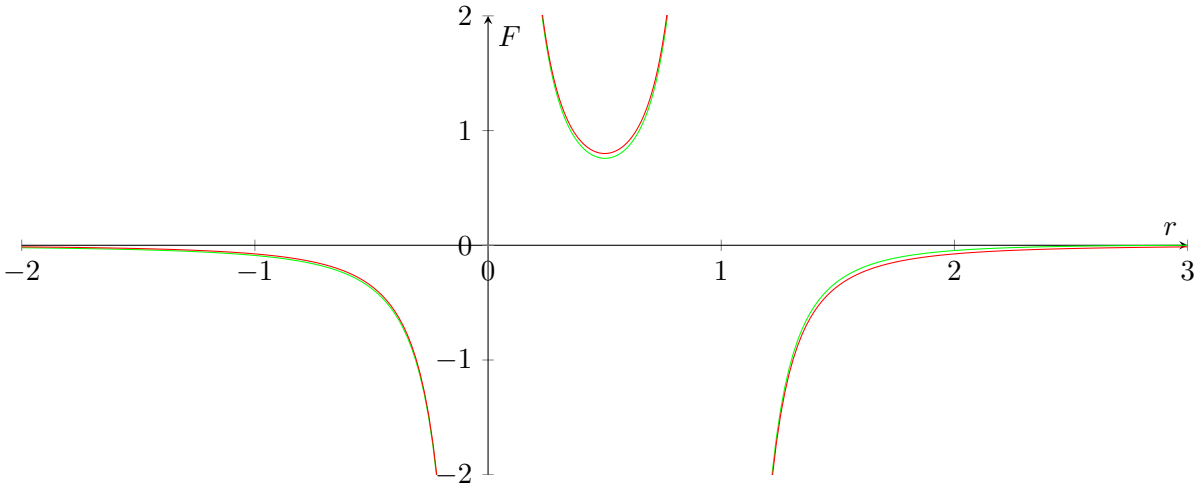


Figure 7: *The graph shows the force on the middle electron of Fig.(5) F (green) computed quantum mechanically in terms of the a-e-model and F_c (red) computed from the classical Coulomb potential as a function of its position r while the other electron and the proton are fixed at $x = 0$ and $x = 1$ respectively. The constant c is taken to be equal to* 0.1.

6 Conclusion

On the whole, the model describing the electrostatic force developed in the course of this work allows to find the right dependence of the potential energies of the systems which were studied and of the forces acting on one of their particles. Compared to classical physics, however, the predictions are accurate only for some ranges of values of r.

One reason for these inaccuracies might be the fact that E_0 is considered to be the same for every state of the system, which might not be entirely true because of asymmetry of the location of the particles. Futhermore this simplified model assumes that the amplitude for the exchange of a virtual photon only depends on the separation between the involved pair of particles, while it might be influenced by the current states of the other particles which are part of the system. So the model works with a kind of "Independent Particle Approximation".[6]

To conclude, it is interesting to see that basic discrete-state theory can be used to get at least a qualitative description of the electrostatic force between charged particles. The two state model based on the assumption of the a and e characteristics of charged particles allows therefore to describe the electrostatic interaction of charged particles in an inspiringly fundamental manner.

Even if this model can by no means be compared to the sophisticated theory of Quantum Electrodynamics, it is interesting to see and to explore how much is already contained in the basics of early quantum mechanics.

7 Bibliography

References

[1] https://home.cern/

[2] The Feynman Lectures on Physics, Volume III (2010)
Feynman - Leighton - Sands, Caltech, Millennium Edition
Chapter 3: Probability Amplitudes

[3] The Feynman Lectures on Physics, Volume III (2010)
Feynman - Leighton - Sands, Caltech, Millennium Edition
Chapter 8: The Hamiltonian Matrix

[4] The Feynman Lectures on Physics, Volume III (2010)
Feynman - Leighton - Sands, Caltech, Millennium Edition
Chapter 10: Other Two-State Systems

[5] The Feynman Lectures on Physics, Volume III (2010)
Feynman - Leighton - Sands, Caltech, Millennium Edition
Chapter 12: The Hyperfine Splitting in Hydrogen

[6] The Feynman Lectures on Physics, Volume III (2010)
Feynman - Leighton - Sands, Caltech, Millennium Edition
Chapter 15: The Independent Particle Approximation

Note: References [3] and [4] refer to identical calculations presented in *The Feynman Lectures on Physics*, which are applied to a different system studied in terms of another model.
The Feynman Lectures on Physics can be found on the following website:
https://www.feynmanlectures.caltech.edu/

A further helpful source for the work on this project was the textbook *Introduction to Quantum Mechanics, Third Edition* (2018) by David J. Griffiths and Darrell F. Schroeter.

8 Appendix A: Inverse Proportionality of A to the Separation between the Particles

In section 2.2, it is stated that it can be shown that the amplitude A for the exchange of a virtual photon is inversely proportional to $\frac{1}{r}$. Following arguments very similar to those presented in section 10.2 of Volume III of *The Feynman Lectures on Physics* leads to the result stated above[4]:

The exchange of a virtual photon between two electrons can be described by the following equations:

$$\begin{aligned} e^- &\to e^- + \gamma \\ e^- + \gamma &\to e^- \end{aligned} \tag{37}$$

The energy of the electrons stays constant during this process. This means that the virtual photon is given no energy ($E = 0$). Since photons have no rest mass ($m_\gamma = 0$), the equation for the energy of a particle therefore implies that the momentum of the photon is zero:

$$\begin{aligned} E^2 &= p^2c^2 + m_\gamma^2 c^4 \\ \Leftrightarrow p &= 0 \end{aligned} \tag{38}$$

The amplitude to reach position $\mathbf{r_2}$ from position $\mathbf{r_1}$ for a free particle of definite energy follows the relation below:[2]

$$A = \langle \mathbf{r_2} | \mathbf{r_1} \rangle \propto \frac{e^{i\mathbf{p}\cdot\mathbf{r_{12}}/\hbar}}{r_{12}} \tag{39}$$

Putting $\mathbf{p} = \mathbf{0}$ into this expression for A finally leads to:

$$A \propto \frac{1}{r_{12}} \quad (r_{12} \text{ positive}) \tag{40}$$

9 Appendix B: Coherence of the Model

Even if a system of three electrons on a straight line was analyzed in section 3.2, this arrangement does not directly affect the equations. In fact, only the amplitudes, which depend on the separation between the particles, change them. Taking the equations of section 3.2 and assuming that the separation between each pair of particles is the same (the amplitudes for the exchange of a virtual photon are then equal) leads therefore again to the system of electrons located on the corners of an equilateral triangle.
Putting $A = A' = A''$ into Eq.(19) leads to the following expression:

$$\begin{aligned} E(E^2 - A^2) &= 2EA^2 + 2A^3 \\ \Leftrightarrow E^3 - 3EA^2 - 2A^3 &= 0 \end{aligned} \tag{41}$$

The mathematical solutions for the system of the equilateral triangle found in section 3.1 are $E = E_0 + 2A$ and $E = E_0 - A$. With the adjustment of the zero-energy-level $E_0 = 0$ in section 3.2, the solutions become $E = 2A$ and $E = -A$.

Putting these into Eq.(41) shows that the solutions of section 3.1 verify the equations of section 3.2 for the case of equal separation:

$$\begin{aligned} (-A)^3 - 3(-A)A^2 - 2A^3 &\stackrel{!}{=} 0 \\ (2A)^3 - 3(2A)A^2 - 2A^3 &\stackrel{!}{=} 0 \end{aligned} \tag{42}$$

This shows that the calculations of both sections are coherent.

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This theoretical project aims to examine whether the classical electric forces between two or three stationary charges could be derived. By assuming a virtual photon is exchanged between charges oscillating between two degenerate single particle states, the author has derived a good approximated result. It provides a simple way to understand the effect of virtual photon. Maybe the author could think about the relationship of his virtual photon coupling with the gauge field.