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參展科別 數學

作品名稱 **Maximum Isosceles Sets**

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關鍵詞　**Isosceles Sets、Two-Distance Sets、Maximum Cardinality**

作者簡介

I am Charles Jan, a senior from Taipei American School. I have loved math since I could remember. Math research is very different from math taught in schools; this project has widened my definition of what math really can achieve. It has also let me explore my interests and taught me resilience. I also learned how valuable short rest periods can be; when I am in a bottleneck, I resort to my favorite leisurely activities like badminton, cooking, or even working out. Switching things up often gives me the jolt of inspiration I need to power on. With my persistent efforts, I hope I can make meaningful contributions to the field of math and promote math as an easily accessible and understandable subject.

摘要

ABSTRACT

An isosceles set is a collection of points in which any subset of three points forms an isosceles triangle. We want to find the upper bound for the size of isosceles sets in any *n*-dimensional Euclidean space. Kido has already completed the study of isosceles sets in 3 and 4-dimensional space. We study the upper bound of spherical two-distance sets, a special type of isosceles sets, to help us find the upper bound of isosceles sets. More specifically, Musin's Linear Programming technique on spherical two-distance sets could be used to study isosceles sets if a consistent relationship between isosceles sets and two-distance sets can be characterized. We offer a conjecture of this relationship. We also offer non-trivial lower bounds of isosceles sets in dimension 5 with 17 points and dimension 7 with 30 points as examples.

壹、前言

Introduction

一、研究動機/目的

Research Motive / Purpose

An isosceles set is a collection of points in which any subset of three points forms an isosceles triangle.
We want to know the upper bound of the size of isosceles sets.

二、研究背景

Research Background

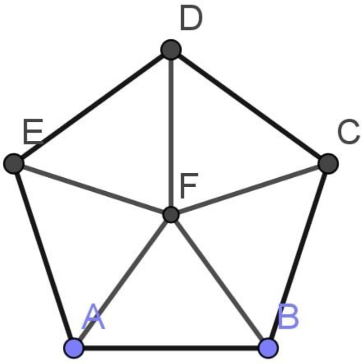

Figure 1

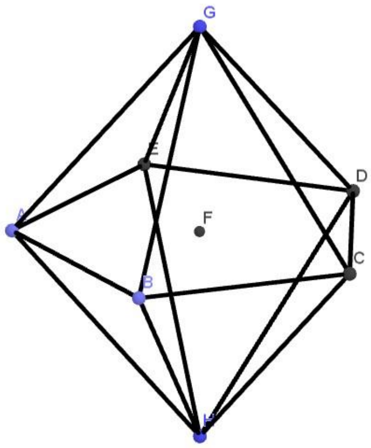

Figure 2

Over 7 decades ago, in 1947, Erdos and Kelly [6] proved 6 points (Figure 1) as the upper bound of isosceles sets in 2-dimensional Euclidean space. Then, in 1961, Croft [4] suggested 8 points as the upper bound in 3-dimensional Euclidean space. The configuration is made by adding a point directly above and below center point the plane of Figure 1 (Figure 2). Kido [9] proved the uniqueness of this 8-point configuration as the upper bound configuration of isosceles sets in 3-dimensional Euclidean space. Kido [10] then determined configurations of isosceles sets up to 4-dimensional Euclidean space. He suggested 11 points as the upper bound. We know very little when the dimension is above 4. We want to study the upper bound of the size of isosceles sets in *n*-dimensional Euclidean space when *n>4*. We noticed the study of spherical two-distance sets has been extended up to *n* equals *infinity*. Thus, we restrict our discussion to spherical two-distance sets first.

A set S of vectors in n-dimensional Euclidean space is called two-distance set, if there are two numbers *c* and *d* so that the distance of distinct vectors of S are either *c* or *d*. A two-distance set is obviously an isosceles set since the triangles formed from any three points are either of side lengths *c,c,d* or *c,d,d*. Larman, Rogers, and Seidel [11] have discovered a rule governing the ratios of distances of two-distance sets. The rule is as such: in n-dimensional space, if the cardinality of a two-distance set is greater than $2n+3$ with distances *c* and *d*, $c<d$, then the ratio $\frac{c^2}{d^2}=\frac{k-1}{k}$ for an integer *k* with $2 \le k \le \frac{1+\sqrt{2n}}{2}$.

Here we introduce a specific type of two-distance sets known as spherical two-distance sets. It is a two-distance set on a unit sphere, with unit vectors pointing from the sphere center outwards. The points of a spherical two-distance set are where the unit vectors located in the surface of the unit sphere. It logically follows that the vectors that compose of a spherical two-distance set may only have inner products of two distinct magnitudes. This is because the inner product between two vectors is defined by the following equation, $ab\cos\theta$, with *a* and *b* being the vector lengths and θ being the angle measure. The angles between the vectors of a spherical two distance-set can only be of two distinct magnitudes by the definition of a two-distance set, and vector lengths in a spherical two-distance set are equal (all vectors are unit vectors).

As proven by Delsarte, Goethals, and Seidel [5], the largest cardinality, g(n), of a spherical two-distance set in *n* dimensions is given by $g(n) \le \frac{n(n+3)}{2}$.

Now that we have an upper bound of *g(n)*, we reference a well-known method to find the lower bound of *g(n)*. Let $e_1,...,e_{n+1}$ represent the standard unit vectors that form an orthogonal basis of a *n+1*-dimensional space. Also let *X* be the regular simplex with vertices $2e_1,...,2e_{n+1}$. The set of mid-points of the edges of X would thus represent a spherical two-distance set in n-dimensional space. Thus, $|X|=\frac{n(n+1)}{2}$. This explicit construction offers the lower bound of g(n); namely, $g(n) \ge \frac{n(n+1)}{2}$.

When n<7 the largest cardinality of two-distance sets is g(n), where g(n) is bounded in the aforementioned range: Table 1 shows the specific results.

Table 1

N	g(n)
2	**5**
3	**6**
4	**10**
5	**16**
6	**27**

For 6<n<40, n $\neq$ 22,23, Musin [12] narrowed down the range to prove that g(n) = $\frac{n(n+1)}{2}$, a single value.

Barg and Yu [1] then used the semidefinite programming method to find the exact values for g(n) for all *n* (except 46 and 78) with $n \leq 93$. Yu [13] then further extended that range to $n \leq 417$. Finally, Glazyrin and Yu [7] extended that range to infinity.

The maximum cardinality of spherical two-distance sets got much attention recently: Zhao et al. [8] proved that when the two inner product values *a* and *b* are given, and as the dimension *n* approaches infinity, the cardinality of such spherical two-distance sets is at most a constant times dimension *n*. Compared to the results of Glazyrin and Yu ($|S| \leq \frac{n(n+1)}{2}$), which is in the order of big O of n^2, the results of Zhao et al. are in the order of big O of n, although their results are based on the conditions that *a* and *b* are given and *n* is large enough.

貳、研究過程與方法

Materials and Methods

MATLAB, GeoGebra, and MathType were used.

Here we explain clearly Musin's methods that led him to this conclusion: Musin improved the upper bound of g(n) given by Delsarte, Goethals, and Seidel, under the condition that *a* +

$b \geq 0$. By some counting argument of dimension in conjunction with the $a+b>=0$ condition, we can reduce the upper bound to n(n+1)/2. Denote by $\rho(n)$ the largest possible cardinality of spherical two-distance sets in □^{n} with a + b ≥ 0.

Then $\rho(n) \leq \frac{n(n+1)}{2}$. The set of mid-points of the edges of a regular simplex has $\frac{n(n+1)}{2}$ points, and we will show that $a+b \geq 0$ for $n \geq 7$. In our calculations, an *n*-dimensional simplex will be represented in *n+1* dimensional space as $\{2e_i\}$ for *i=1 to n+1* and e_i as the standard orthonormal basis, where e_i is a vector with *i*th component as one and all others as zero. The midpoints between the vertices of the simplex would thus have coordinates with two components being 1 and all others being 0. The center point of the simplex would be (2/(n+1), 2/(n+1), …) since the center point is a vector with its coordinates as an average of all others. To obtain the values of *a* and *b*, we need to calculate the inner products between the vectors representing the midpoints between the vertices of the simplex. The vectors can be obtained by subtracting the coordinates of the midpoints from the center point. The vectors would thus have two components being $1-\frac{2}{n+1}=\frac{n-1}{n+1}$ and all other components being $-\frac{2}{n+1}$. We know that there exists only two distinct inner product values, *a* and *b*, and it does match with the calculations performed with this example: the only two possible distinct magnitudes are created by taking the inner products of vectors with each of their $\frac{n-1}{n+1}$ components completely staggered or with one of their $\frac{n-1}{n+1}$ components overlapping, respectively. Therefore, we obtain $a=\frac{n-3}{2(n-1)}$, $b=\frac{-2}{n-1}$ and $a+b=\frac{n-7}{2(n-1)} \geq 0$.

When $a+b<0$, however, the methods mentioned above are no longer applicable. Delsarte's method is needed to find the maximum cardinality of spherical distance sets when $a+b<0$. We use the polynomial method to estimate the maximum cardinality of spherical two-distance sets.

We define Gegenbauer (or ultraspherical) polynomials $G_k^{(n)}(t)$. $G_k^{(n)}$ as the following:

$$G_0^{(n)}=1, G_1^{(n)}=t, \ldots, G_k^{(n)}=\frac{(2k+n-4)tG_{k-1}^{(n)}-(k-1)G_{k-2}^{(n)}}{k+n-3}.$$

For instance,

$$G_2^{(n)}(t) = \frac{nt^2 - 1}{n-1},$$

$$G_3^{(n)}(t) = \frac{(n+2)t^3 - 3t}{n-1},$$

$$G_4^{(n)}(t) = \frac{(n+2)(n+4)t^4 - 6(n+2)t^2 + 3}{n^2 - 1}.$$

Now for given *n, a, b* we introduce polynomials $P_i(t)$, $i = 1,...,5$.

$i = 1: P_1(t) = (t-a)(t-b) = f_0^{(1)} + f_1^{(1)}t + f_2^{(1)}G_2^{(n)}(t).$

$i = 2: P_2(t) = (t-a)(t-b)(t+c) = f_0^{(2)} + f_1^{(2)}t + f_2^{(2)}G_2^{(n)}(t) + f_3^{(2)}G_3^{(n)}(t),$ where *c* is defined by the equation $f_1^{(2)} = 0$.

$i = 3: P_3(t) = (t-a)(t-b)(t+a+b) = f_0^{(3)} + f_1^{(3)}t + f_2^{(3)}G_2^{(n)}(t) + f_3^{(3)}G_3^{(n)}(t).$ Note that $f_2^{(3)} = 0$.

$i = 4: P_4(t) = (t-a)(t-b)(t^2 + ct + d) = \sum f_k^{(4)}G_k^{(n)}(t),$ where *c* and *d* are defined by the equations $f_1^{(4)} = f_2^{(4)} = 0$.

$i = 5: P_5(t) = (t-a)(t-b)(t^2 + ct + d) = \sum f_k^{(5)}G_k^{(n)}(t),$ where *c* and *d* are defined by the equations $f_2^{(5)} = f_3^{(5)} = 0$.

Each successive polynomial offers a more precise estimation of the upper bound. By simply plugging in the values of *n ,a, b,* after making sure that they follow the boundary rules, Delsarte's method can supply us the maximum cardinality of a spherical-two distance set |*S*|.

參、研究結果與討論

Results

Theorem 1. (Musin [12]) Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b. If $a + b \geq 0$, then $|S| \leq \frac{n(n+1)}{2}$.

Theorem 2. (Musin [12]) If $n \geq 7$, $\rho(n) = \frac{n(n+1)}{2}$

Theorem 3. (Delsarte, Goethals, Seidel [5]) Let T be a subset of the interval $[-1, 1]$. Let S be a set of unit vectors in R^n such that the set of inner products of distinct vectors of S lies in T. Suppose a polynomial f is a nonnegative linear combination of Gegenbauer polynomials $G_k^{(n)}(t)$, i.e.,

$f(t) = \sum_k f_k G_k^{(n)}(t)$, where $f_k \geq 0$.

If $f(t) \leq 0$ for all $t \in T$ and $f_0 > 0$, then

$$|S| \leq \left\lfloor \frac{f(1)}{f_0} \right\rfloor$$

肆、結論與應用

Discussion

We are on the verge of discovering a seemingly formulaic, consistent relationship between spherical isosceles sets and spherical two-distance sets. This may allow us to utilize a method similar to Delsarte's Method (linear programming) to unlock the upper bounds of the maximum cardinality of spherical isosceles sets across all dimensions, as has been done by Glazyrin and Yu [7] for spherical two-distance sets.

We considered the difference between two-distance sets and isosceles sets: In $\mathbb{R}^2$, the configuration with the maximum cardinality of two-distance sets is the five vertices of a regular pentagon. The configuration with the maximum cardinality of isosceles sets is composed of six points, the five vertices of a regular pentagon and its center point (Erdos and

Kelly [6]). In $\mathbb{R}^3$, the maximum two-distance set is the six vertices of an octahedron. The maximum isosceles set is an eight-point configuration with a regular pentagon on its equator, a center point, and a point at both the north and south pole. Through these examples, we realized the relationship between maximum two-distance sets and isosceles sets can be characterized as such: a maximum isosceles set is the maximum two-distance set of the same dimension plus its center point OR a maximum two-distance set of one dimension lower plus its center point, north pole, and south pole.

Let *I(n)* be the maximum cardinality of isosceles sets in $\mathbb{R}^n$ and *g(n)* be the maximum cardinality of spherical two-distance sets. We conjecture that

$$I(n) = \max\{g(n)+1, g(n-1)+3\} .$$

This is true for at least *n = 2,3,4*.

Kido has proved that $I(4) = 11 = \max\{g(4)+1, g(3)+3\}$ where *g(4) = 10*, and *g(3) = 6*, and *max{11,9} = 11*, so our conjecture holds true. If our conjecture is true, then *I(5) = max{g(5) + 1, g(4) + 3} = max{17,13}= 17*. The construction comes from half of a hypercube in $\mathbb{R}^5$ with *16* points and its center. The remaining question is how to find the upper bound of an isosceles set in $\mathbb{R}^5$, *17*. The proof is elusive, so we would like to offer a non-trivial construction of an isosceles set in *7*-dimensional space with *30* points.

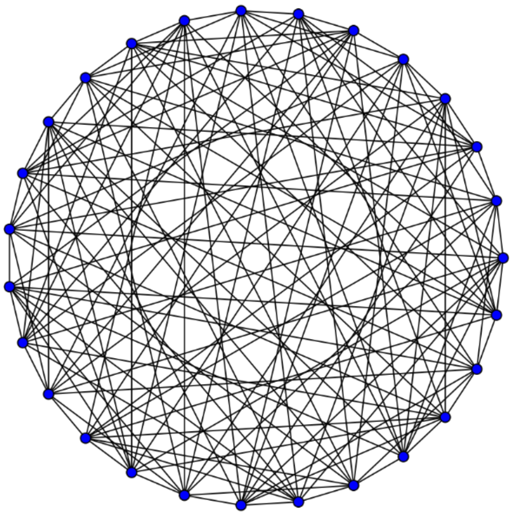

Figure 3

We can demonstrate how to find the lower bounds of isosceles sets by utilizing the upper bounds of two-distance sets. To give a specific example, in 6-dimensional Euclidean space, the upper bound for two-distance sets is 27 points. Then we can obtain a 7-dimensional 30-point configuration as the lower bound of the isosceles set. This configuration can be likened to a strongly regular graph with parameters (27, 10, 1, 5). A visual representation in 2D is shown in Figure 3. We denote this by *SRG (27,10,1,5)*, which is a strongly regular graph with 27 vertices and every vertex has 10 neighbors; every two adjacent vertices have 1 common neighbor and every two non-adjacent vertices have 5 common neighbors. This strongly regular graph can be interpreted to be a two-distance set through its adjacency matrix, A. $a_{i,j}$ is the *(i, j)* entries of A. $a_{i,j} = 1$ if *i* is adjacent to *j*. Otherwise, $a_{i,j} = 0$. If each row is treated like a vector that denotes the coordinates / location of each point in the configuration, we could take the inner products between any two row vectors and instantly realize that this is in fact a two-distance set. The inner product between any two row vectors is either 1 or 5, depending on whether the two row vectors are adjacent or nonadjacent, respectively. Now it is confirmed that the 27-point configuration is a two-distance set. The coordinates for the 27-point configuration in 6-dimensional Euclidean space obtained through spherical embedding of SRG (27,10,1,5) (Cameron [2]):

We can then add a center point, a top vertex, and a bottom vertex to this configuration to construct a non-trivial lower-bound 30-point configuration for isosceles sets in 7-dimensional Euclidean space. We show its coordinates in the appendix. We would like to prove soundly that 30 is the upper bound of isosceles sets in $\mathbb{R}^7$, but we do not know how to do it yet.

We also can offer a non-trivial construction of isosceles sets in $\mathbb{R}^{23}$ with 278 points. This construction can be obtained from the spherical embedding of SRG (275,112,30,56). We will not list the coordinates of the construction in the appendix, but we would like to offer this information since this is the largest example we know to attain $\frac{n(n+3)}{2}$ points as spherical two-distance sets in $\mathbb{R}^n$.

We know that if the size of a spherical two-distance set is large enough (in $\mathbb{R}^n$ with size larger than *2n+3*), then the Larman, Roger, Seidel Theorem applies, so the distances are quite restrictive. We wonder whether a similar theorem could be made for isosceles sets; namely, whether we can find the relationship between the distances of an isosceles set if its size is large enough.

Also, we wonder what other *s*-distance sets an isosceles set can be if it is not a two-distance set. Let us consider n = 2 first. The maximum isosceles set is a pentagon with a center point. Assume the regular pentagon lies on a unit circle. Then the distances from the center point to the vertices is 1. The other two distances should be $\sqrt{2-2\cos 72^\circ} = 1.1756$ and the previous number multiplied by the golden ratio, 1.9021. Thus, it is a three-distance set. Next, let us consider *n = 3*. It is simply the example in *n = 2* plus a north and south pole. Thus, it would be a 5-distance set: the previous three distances plus $\sqrt{2}$ and 2.

In short, we conjecture that an isosceles set is made up of a two-distance set and a collection points equidistant to all the original points in the two-distance set. This led us to the conclusion $I(n) = \max\{g(n)+1, g(n-1)+3\}$.

We shortly sketch the idea to approach this conjecture. We start with a spherical two-distance set of maximum cardinality in any dimension. We can add a center point to make it an isosceles set. That is where the *g(n)+1* formula comes from. We claim that any new point to be added to a spherical two-distance set must be equidistant to all the original points to make it an isosceles set. If not, then OA $\neq$ OB, where O is the new point and A, B are the original points in the spherical two-distance set. In this case, either OA or OB must equal AB by definition of an isosceles set. Then O must either be one of the original points in the spherical two-distance set or in its orthogonal complement. Without loss of generality, assume OA = AB. This means A must be on the perpendicular bisector of OB. This has to hold true for any two points of the original spherical two-distance set, which is quite unlikely to happen. Therefore, we conjecture that OA must equal OB; the new point must be equidistant to all points of the original spherical two-distance set. Thus, we can only add the center, north pole, and south pole, and that is where the *g(n-1)+3* formula comes from.

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Appendix

Here is the construction of the isosceles set with 30 points in $\mathbb{R}^7$

0.0373	0	0	0	-0.4635	-0.0772	0
0.2998	0.1730	0.0067	0.1452	0.2758	-0.0719	0
-0.1001	-0.0179	-0.0953	-0.0956	0.2872	-0.3334	0
-0.0354	-0.1124	0.3715	0.0711	0.2474	-0.0634	0
-0.3480	0.1660	-0.2063	-0.0440	-0.1294	-0.1109	0
0.1704	0.2742	-0.1361	-0.2413	-0.0747	-0.1888	0
-0.2833	0.0716	0.2605	0.1227	-0.1692	0.1591	0
-0.0166	-0.2446	-0.2672	-0.2679	-0.1067	-0.0872	0
0.1021	-0.0068	-0.3240	0.2198	-0.0732	-0.2304	0
0.0481	-0.3390	0.1996	-0.1012	-0.1464	0.1828	0
-0.0019	0.1124	-0.3715	-0.0711	0.2161	0.1406	0
-0.3372	-0.1730	-0.0067	-0.1452	0.1877	0.1492	0
0.0628	0.0179	0.0953	0.0956	0.1763	0.4107	0
-0.1502	0.3458	0.1244	-0.1187	0.2197	0.0476	0
0.1668	-0.1012	0.1428	0.3865	-0.1130	0.0396	0
0.1812	-0.0648	0.0635	-0.3425	0.2424	0.0713	0
-0.2185	0.0648	-0.0635	0.3425	0.2211	0.0059	0
0.2351	0.1798	0.3307	-0.0746	-0.1145	0.0813	0
0.1129	-0.3458	-0.1244	0.1187	0.2439	0.0296	0
-0.1649	-0.0111	0.2287	-0.3154	-0.1031	-0.1802	0
-0.0666	0.1192	-0.0476	-0.2909	-0.1742	0.2938	0
-0.0463	0.2267	0.1719	0.1723	-0.0697	-0.3235	0
-0.2332	-0.2922	0.0409	0.1457	-0.1017	-0.2219	0
0.2852	-0.1840	0.1111	-0.0516	-0.0469	-0.2998	0
0.3834	-0.0537	-0.1652	-0.0271	-0.1180	0.1743	0
-0.1349	-0.1619	-0.2354	0.1702	-0.1727	0.2521	0
0.0520	0.3570	-0.1044	0.1968	-0.1408	0.1506	0
0	0	0	0	0	0	0
0	0	0	0	0	0	1.0000
0	0	0	0	0	0	-1.0000

【評語】010011

作者研究的 Maximum isoceles sets 是離散幾何的一個重要問題，目前只有在維數較小的時候有正確的答案。本研究希望藉由有好研究成果的 Two distance sets 來回答上述在高維度的答案；經由一些已知的性質的確可以獲得一些結果，然而，整體而言進展不大，值得繼續努力。