細繩和圓柱間靜摩擦係數與張力分佈的研究

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熱愛世間一切的知識與學問，凡事總有學習的欲望，因此相較於 IMO 準國手弟弟的專才而言，我只算是個樣樣通但都不頂尖的人吧！高二資訊、高三物理的學科能力競賽均拿到雲嘉區第一名、全國 ARML(高一)和物理學科競賽二等獎，或許就像老師常不斷提醒我專精的必要性吧！想到九七年 IMO 金牌的安金鵬，不也是在物理一等、數學二等之後選擇單攻才獲此殊榮？我想大學的生涯規畫應審慎的修正自己的方向。

在國內覺得沒什麼，出國才知道國際地位的重要性，對於開幕連站到台面上的機會都沒有深感可惜，我們無法改變局勢，也只有盡力做好自己的角色、落實國民外交吧。

一個平凡的鄉下孩子，得有此幸參與此次國際科展的盛會，心中充滿著莫名的感動與興奮。這是我在申請上台大資工高中畢業後、進入大學前的一個人生標記點，一切為科展所做的努力與忙碌都是值得的，因為這是我一生中最難忘的回憶。

I、前 言：

眾所皆知，以較小的力量 T_1 施力於繩的一端，僅須將細繩在圓柱上繞幾圈就可以支持相當大的重物 T_2。對此一現象已在許多力學或大學物理的教科書中做過理論性的介紹[1]，但未見教科書中介紹過實驗上的驗證。十年前，(Bettis[2]以力桌實驗，細繩繞過力桌插栓且跨過滑輪，纏繞角固定，兩端掛砝碼，可減少因砝碼擺動時所產生的誤差)。Levin[3]利用纏繞在水平圓柱上的彈簧來測量動摩擦係數，他亦將一條長36.5cm，力常數3500dy/cm的彈簧纏繞水平圓柱上，兩端懸掛重量不等的重物，由每匝彈簧拉開的間隔可直接觀察張力變化的分佈。

在本報告中，我們以實驗證實 $T_2=T_1e^{\mu\theta}$ 的關係。又從 ℓnT_1 圖的截距得出細繩和圓柱間的靜摩擦係數。實驗上的作法，我們以力常數400dy/cm的彈簧掛於青銅圓柱上並利用倒置的投影機投影於屏。只要彈簧不軟到與圓柱接觸處呈現變形，那麼我們就能準確地測量彈簧上張力隨角度的變化，也能經由測量彈簧上每一段的張力得到直接的證實。

II、原 理：

細繩和固定不轉動的圓柱接觸，兩端張力T_1、T_2($T_1 < T_2$)，纏繞角θ，靜摩擦係數μ。假設圖一的曲線不光滑且細繩正好在AB方向上滑動。除了張力T和正向力N之外，另有摩擦力f_s作用於細繩運動的相反方向上。

$(T+dT)\cos(d\theta_2)=T\cos(d\theta_1)+f_s$……………(1a)

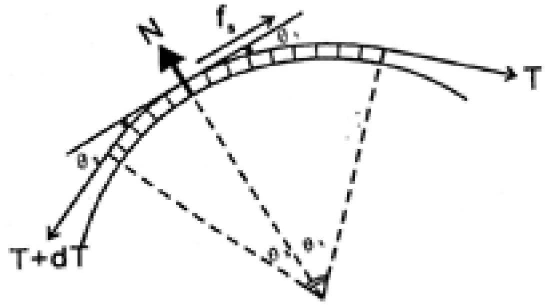
$(T+dT)\sin(d\theta_2)+T\sin(d\theta_1)=N$……………(1b)

$\because d\theta\to 0 \quad \therefore d\theta_1\to 0,\ d\theta_2\to 0,$

Thus

$T+dt=T+f_s$

$Td\theta_2+Td\theta_1=N$



圖一：細繩繞於圓柱之受力圖

$\because d\theta=d\theta_1+d\theta_2$

$\therefore f_s=dT$…………………………………………(2a)

$N=Td\theta$…………………………………………(2b)

以最大靜摩擦力 $f_s=\mu N$ 代入Eq.(2a)，並合併Eq.(2a)及(2b)，可得

$\therefore\ dT=\mu Td\theta$

即

$dT/T=\mu d\theta$

上式兩邊積之分

可得 $\ell n\ T_2/T_1=\mu\theta$…………………………………………(3a)

$\therefore T_2=T_1e^{\mu\theta}$…………………………………………(3b)

上式兩邊取對數。得

$\ell n\ T_2=\ell n\ T_1+\mu\theta$…………………………………………(3c)

Ⅰ、設 備：

1. 2號釣魚線
2. 天平
3. 力桌(附兩個滑輪)
4. 一些砝碼
5. 青銅圓柱(半徑1.90cm，長5cm)
6. 栓在支架上半徑5.13cm的青銅圓柱
7. 軟彈簧(截面直徑0.23mm鋼絲繞成的20cm長，力常數400dy/cm彈簧)
8. 鋼絲(截面直徑0.23mm)
9. 投影機
10. 量角器

Ⅳ、實 驗：

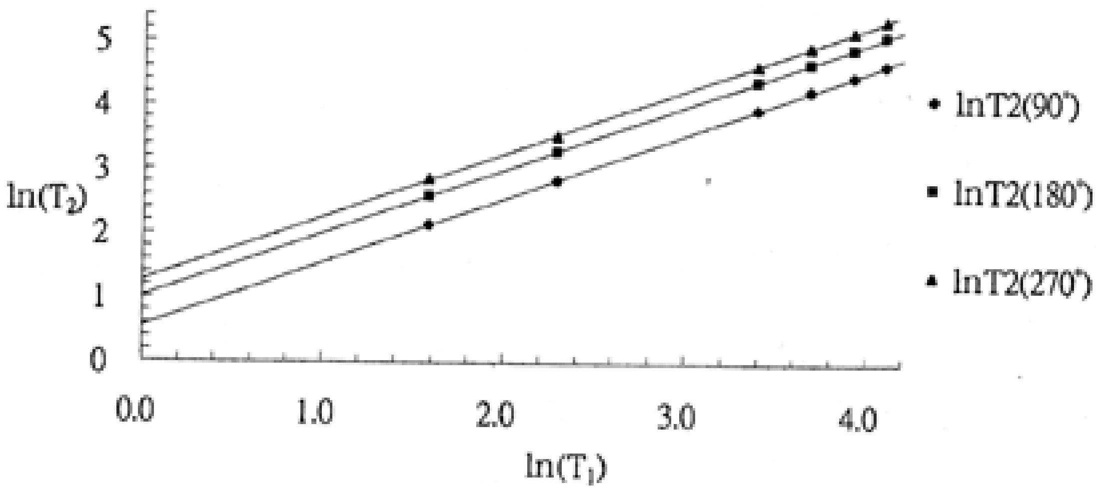
1. 如圖二所示，青銅圓柱栓於力桌插栓處，釣線繞過圓柱，接觸角 θ，兩端繞過滑輪後懸掛砝碼。一端加掛砝碼使張力 T_2 增加直到釣線恰滑動瞬間記錄 T_1、T_2、θ。
2. 以不同角度重複步驟1
3. 半徑5.13cm的空心青銅圓柱水平固定於支架上，圓柱上附有量角器。將軟彈簧繞於空心青銅圓柱上，兩端懸掛不等重的砝碼。在彈簧恰要滑動的瞬間，對彈簧做放大投影，如圖三所示。彈簧上每匝的變化在銀幕上清晰可見。彈簧每匝的伸長可觀察到其上每一點的張力。
4. 本實驗中，軟彈簧爲截面直徑0.23mm的鋼絲繞成。以此種鋼絲纏繞在半徑5.13cm的空心青銅圓柱上，纏繞角180° 。砝碼懸於鋼絲的兩側，如實驗步驟 1.。記錄 T_1 和 T_2。

V、結果及討論：

1. 實驗結果如表一所示，T_1爲控制變因，而不同纏繞角爲應變變因，得到不同的張力T_2。圖4爲ℓnT_1對ℓnT_2之圖形爲三條平行的直線，斜率爲1，與公式(3c)$\ell nT_2 = \ell nT_1 + \mu_s \theta$相一致。

表一：不同纏繞角之T_1、T_2

T_1(gw)	4.878	9.924	29.782	39.943	50.623	60.447
T_2(90°)	8.524	17.194	51.921	70.751	88.134	105.27
T_2(180°)	13.305	26.651	80.254	106.97	135.32	163.20
T_2(270°)	17.324	33.893	102.75	137.82	174.67	208.59



圖四：不同角度之 $\ell n\ T_2$ 對 ℓnT_1

(1)直線1截距爲0.56，$0.56 = \mu\theta = \mu_1 \times (\pi/2)$。則$\mu_1 = 0.357$。

(2)直線2截距爲1.01，$1.01 = \mu\theta = \mu_2 \times \pi$。則$\mu_2 = 0.321$。

(3)直線3截距爲1.27，$1.27 = \mu\theta = \mu_3 \times (3\pi/2)$。則$\mu_3 = 0.270$。

靜摩擦係數隨著$T_1 + T_2$的增加而減少($\mu_3 < \mu_2 < \mu_1$)。顯然地，因小的纏繞角和張力的關係，摩擦力相對的減小，靜摩擦係數則不規則地增加。摩擦係數爲定值只是巨觀的概略性描述罷了。

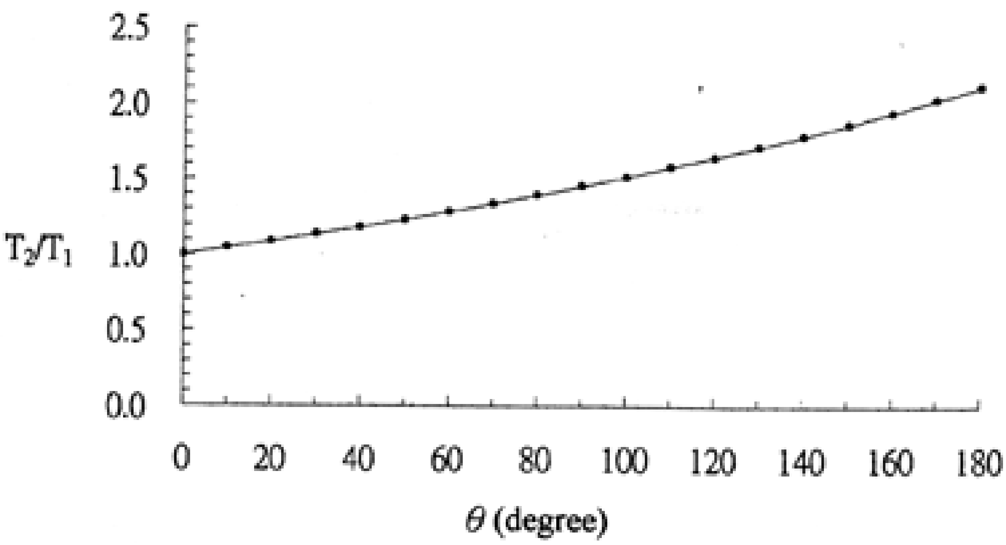
2. 表二的數據爲圖三彈簧每匝計算後的結果

表二：張力和θ之關係

θ(degree)	0	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
θ(rad)	0.000	0.175	0.349	0.524	0.698	0.873	1.047	1.222	1.396	1.571
T_2(gw)	4.952	5.162	5.382	5.610	5.848	6.097	6.356	6.626	6.907	7.200
T_2/T_1	1.000	1.042	1.087	1.133	1.181	1.231	1.284	1.338	1.395	1.454
InT_2/T_1	0.000	0.042	0.083	0.125	0.166	0.208	0.250	0.291	0.333	0.374

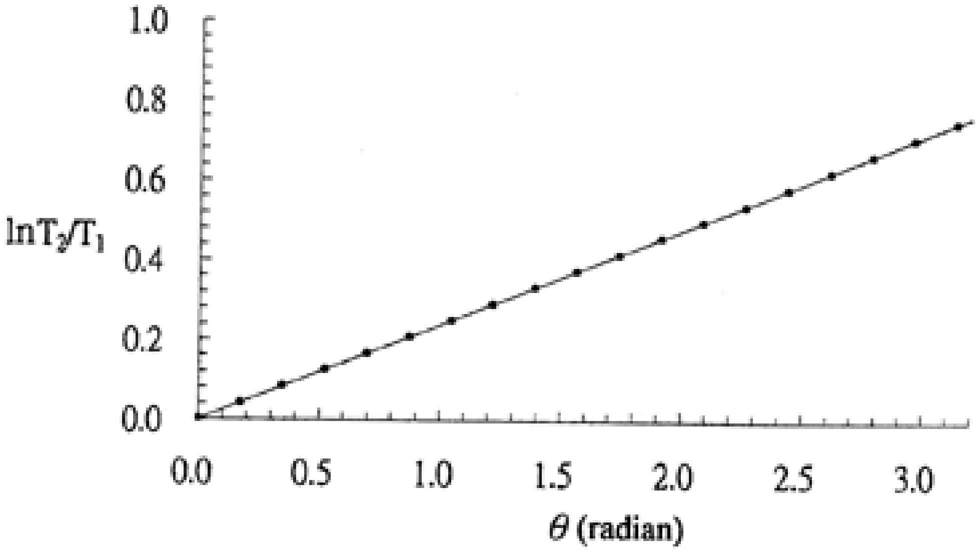
θ(degree)	100	110	120	130	140	150	160	170	180
θ(rad)	1.745	1.920	2.094	2.269	2.443	2.618	2.792	2.967	3.142
T_2(gw)	7.507	7.826	8.158	8.504	8.866	9.242	9.635	10.04	10.47
T_2/T_1	1.516	1.580	1.647	1.717	1.790	1.866	1.946	2.028	2.115
InT_2/T_1	0.416	0.458	0.499	0.541	0.582	0.624	0.666	0.707	0.749

(1)圖五爲T_2/T對θ的曲線圖，與公式(3b) T_2彈簧張力隨著θ呈指數函數遞增。



圖五：$\frac{T_2}{T_1}$和θ之關係圖

(2)$\ell n(T_2/T)$對θ圖則爲一通過原點的直線，斜率爲μ。我們得到鋼絲和青銅間的靜摩擦係數0.241。

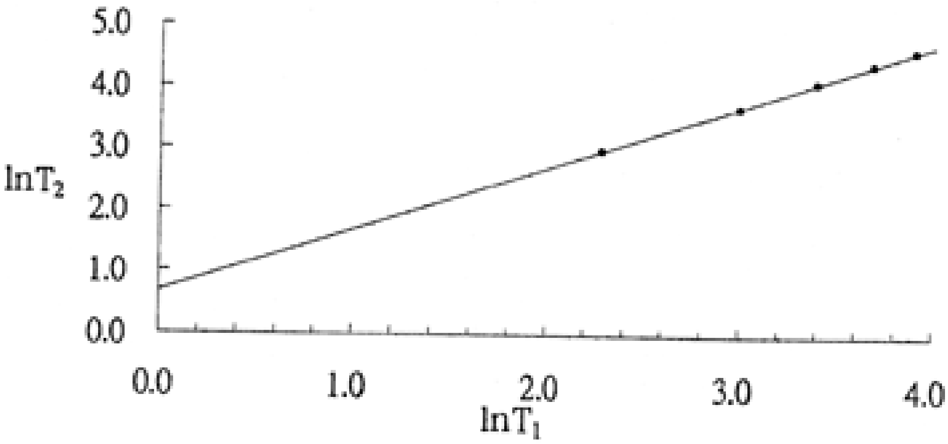


圖六：$\ell n(\frac{T_2}{T_1})$對θ之關係圖

3. 表三爲彈簧相同截面直徑的鋼絲之張力分佈圖。圖七的直線斜率爲1，截距爲0.670。$0.670=\mu\theta=\mu\pi$，所以$\mu=0.213$。與繞於青銅圓柱的彈簧相當。

表三：鋼絲之T_1、T_2 關係

T_1(gw)	9.844	20.086	29.867	40.018	49.649	59.728
T_2(gw)	19.756	39.855	60.549	80.585	100.74	121.33



圖七：鋼絲之ℓnT_2對ℓnT_1

VI、結 論：

1. 靜摩擦力的公式和細繩兩端的張力有關，爲$T_2 = T_1 e^{\mu\theta}$。我們可以從ℓnT_2對ℓnT_1圖的截距獲得μ。
2. 細繩上的張力佈可以軟彈簧纏繞在圓柱上，模擬出不同角度的張力變化$T = T_1 e^{\mu\theta}$關係。此爲本報告的重要結果。

VII、參考書目：

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3. E. Levin, "Friction experiment with a capstan," Am J. phys. 59, 80-84(1991).
4. S. M. Lea and J. R. Burke ,PHYSICS the nature of things, P. 139(1997) West publishing company, New York.

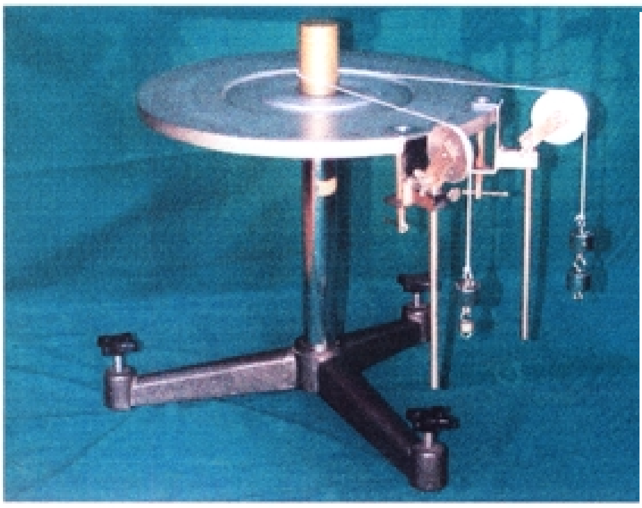

Fig.2 Force table modified for this experiment

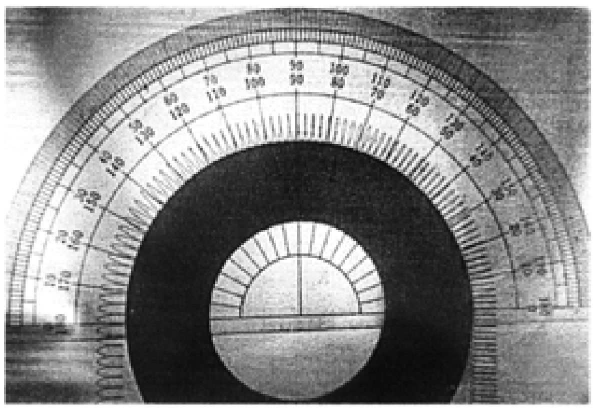

Fig.3 The variation of tension in spring wrapped around the cylinder

A Novel Measurement of the Distribution of the Tension of a cord in Contact with a Bronze Cylinder

ABSTRACT

A cord is wrapped around a bronze chlinder. At the moment the cord moved, the tensions at each side are T_1 and T_2. The relationship $T_2=T_1e^{\mu\theta}$ can be derived by calculus where μ is the coefficient of the static friction on a cord wrapped around the bronze cylinder, and θ is the wrap angle. Then μ can be calculated from the intercept according to the plotted figure $\ln T_1$ vs. $\ln T_2$, but the tension of each point on a cord can not be measured directly. In the present report, a new method is proposed to overcome this issue.

When a spring wrapped around the bronze cylinder is projected onto the screen, the variations of tension in different wrap angles are measured precisely. The relationship $T=T_1e^{\mu\theta}$ can be directly confirmed by measruing the tension of a cord at each point. Moreover, the coefficient of the static friction on a steel wire wrapped around the cylinder can be gotten from the slope on the basis of the plotted figure ℓnT vs. ℓnT_1.

I、INTRODUCTION

As everyone knows, a relatively small force T_1, applied to a rope, can often support a much larger force T_2 merely by wrapping the rope a few times about a post. A theoretical discussion on this phenomenon can be found in many textbooks of mechanics or general university physics[1]. But no experimental testfication of this phenomenon is introduced in textbooks in our knowledge. Ten years ago, Bettis[2] experimented with a force table. Masses were hung at the both ends of a string, and the bolt of the force table was wrapped around with the string which passed over the pulleys. So the wrap angle is fixed. Thus, the errors caused by swinging masses can be reduced. Levine[3] measured the coefficient of kinetic friction with a string wrapped on a horizontal cylinder. He also hung a spring, 36.5cm in length, with force constant 3500 dy/cm, on a horizontal cylinder with unequal weights suspended from its ends. The variation of tension in the spring was observed from the change of coil spring spaces.

In this report, a novel method of measuring the distribution of the tension of a cord in contact with a bronze cylinder is proposed. A spring with foce constant 400 dyne/cm wraps over a bronze cylinder. An overhead projector is used to project it on the screen. The tensions of the spring in different wrap angles are accurately measured. The relationship $T=T_1e^{\mu\theta}$ is directly confirmed by measuring tension of each point of the spring. The coefficient of static friction of a cord in contact with a bronze cylinder. can be gotton from the interception on the $\ell n\ T_2$ according to the plotted figure $\ell n\ T_2$ vs. $\ell n\ T_1$

、THEORY

1.A cord in contact with a clamped, nonrotating cylinder has forces T_1, T_2 at the free ends ($T_1<T_2$). The wrap angle is θ. μ is the coefficient of static friction. Suppose that the curve of Fig.1 is rough and that the cord is just on the point of slipping in the direction AB. That is, in addition to the tension T and the normal force N, there is a force of friction f_s acting on the cord in the opposite direction.

$(T+dT)\cos(d\theta_2)=T\cos(d\theta_1)+f_s$ ....(1a)

$(T+dT)\sin(d\theta_2)+T\sin(d\theta_1)=N$ ....(1b)

$\because d\theta\to 0 \quad \therefore d\theta_1\to 0,\ d\theta_2\to 0,$

Thus

$T+dT=T+f_s$

$Td\theta_2+Td\theta_1=N$

$\because d\theta=d\theta_1+d\theta_2$

$\therefore f_s = dT$ ··············(2a)

$N = Td\theta$ ··············(2b)

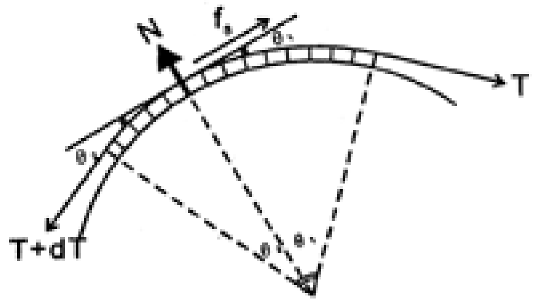

Fig.1 A cord is in contact with a cylinder

Because the maximum static friction force $f_s = \mu N$, then let us combine Eqs.(2a) and (2b).

We have $dT = \mu Td\theta$

or

$dT/T = \mu d\theta$

Integrate both side of the above equation, we have

$\ell n(T_2/T_1) = \mu\ \theta$ ··············(3a)

$\therefore T_2 = T_1 e^{\mu\theta}$ ··············(3b)

$\ell n\ T_2 = \ell n\ T_1 + \mu\ \theta$ ··············(3c)

III、EQUIPMENTS

1. Fishing thread：No.2
2. A balance
3. A force table with two pulleys
4. Some standard weights
5. A bronze cylinder replacing the bolt of force table (r_1=1.90cm)
6. A hollow bronze cylinder (r_2=5.13cm) clamped horizontally on a supporter
7. A soft spring (force constant 400 dy/cm, 20cm in length, made of an iron wire of diameter 0.23mm)
8. An overhead projector
9. A protractor

IV、METHOD

1. The bolt of a force table is replaced by a bronze cylinder. A fishing thread are wrapped around the cylinder in contact with an angle θ, and passes over the pulleys as shown in Fig.2. Masses are hung at the both ends of the fishing thread. The masses are added and the tension T_2 accordingly increases until the fishing thread starting to move. At this moment T_1, T_2 and θ are recorded.
2. Repeat step 1 at different angles.
3. A hollow bronze cylinder is clamped horizontally on a supporter. A protrastor is attached on the bronze cylinder. Now the cylinder is wrapped by the soft spring and unequal weights are suspended on both ends. At the monent of the spring starts to move, the spring was projected on the overhead projector. The individual spiral spring turns are quite visible on the screen as shown in Fig.3. The tension at different parts of the stretched spring may be obtained from the intercoil spring.
4. Now take an iron wire which has the same material with the soft spring in step 3. The bronze cylinder (r_2=5.13cm) is wrapped by the iron wire (the wrap angle is 180°). Masses are hung at both ends of the iron wire as did in Process 1. Then T_1, T_2 are recorded.

V、RESULTS AND DISCUSSION

1. T_1 is given, then the correspondng T_2 is measured for three different wrap angles, e.g. 90° ,180° and 270° In this experiment, T_1 is set at 4.878gw 9.924gw, 29.782gw39.843gw, 50.623gw and 60.447gw, The corresponding T_2(90°) T_2(180°), and T_2(270°) are measured, All measured datum are listed in Table1 . Fg4 shows the poltted figure ℓn T_2 vs. ln T_1, in which we obtain three parallel stiaight limes. The measured slop is one, in good agreement with Eq(3C)

Table 1:T_1,T_2 of different wrap angles

T_1(gw)	4.878	9.924	29.782	39.943	50.623	60.447
T_2(90°)	8.524	17.194	51.921	70.751	88.134	105.27
T_2(180°)	13.305	26.651	80.254	106.97	135.32	163.20
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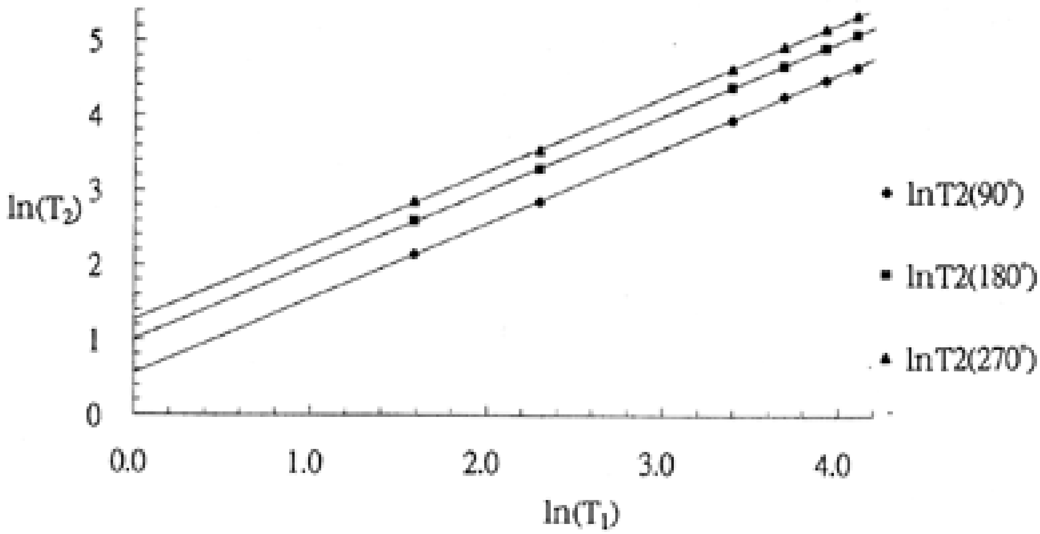

Fig.4 Graph of tension ℓnT_1, ℓnT_2 of different wrap angles

(1)Line 1: The intercept is $0.55=\mu\theta=\mu_1(\pi/2)$. Thus μ_1 is 0.357.

(2)Line 2: The intercept is $1.01=\mu\theta=\mu_2\pi$. Thus μ_2 is 0.321.

(3)Line 3: The intercept is $1.26=\mu\theta=\mu_3(3\pi/2)$. Thus μ_3 is 0.270.

The coefficient of static friction decreases ($\mu_3<\mu_2<\mu_1$) as the sum (T_1+T_2) increases. Apparently, due to a small wrap angle and low tension in cord, the friction force is relatively small, and the coefficient μ is anomalously large. Recalling that the linear relationships that describe macrospic friction are approximations[4].

2.The datum listed in Table 2 are the measured results of the individual spiral spring calculated from Fig.3.

Table 2:The tension distribution of the spring

θ(degree)	0	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
θ(rad)	0.000	0.175	0.349	0.524	0.698	0.873	1.047	1.222	1.396	1.571
T_2(gw)	4.952	5.162	5.382	5.610	5.848	6.097	6.356	6.626	6.907	7.200
T_2/T_1	1.000	1.042	1.087	1.133	1.181	1.231	1.284	1.338	1.395	1.454
$\ln T_2/T_1$	0.000	0.042	0.083	0.125	0.166	0.208	0.250	0.291	0.333	0.374
θ(degree)	100	110	120	130	140	150	160	170	180	
θ(rad)	1.745	1.920	2.094	2.269	2.443	2.618	2.792	2.967	3.142	
T_2(gw)	7.507	7.826	8.158	8.504	8.866	9.242	9.635	10.04	10.47	
T_2/T_1	1.516	1.580	1.647	1.717	1.790	1.866	1.946	2.028	2.115	
$\ln T_2/T_1$	0.416	0.458	0.499	0.541	0.582	0.624	0.666	0.707	0.749	

(1)The T_2/T vs. θ curve is plotted in Fig.5. It is in agreement with Eq(3b) $T_2=T_1 e^{\mu\theta}$ The tension of the spring wrapped on cylinder increase with θ exponentially.

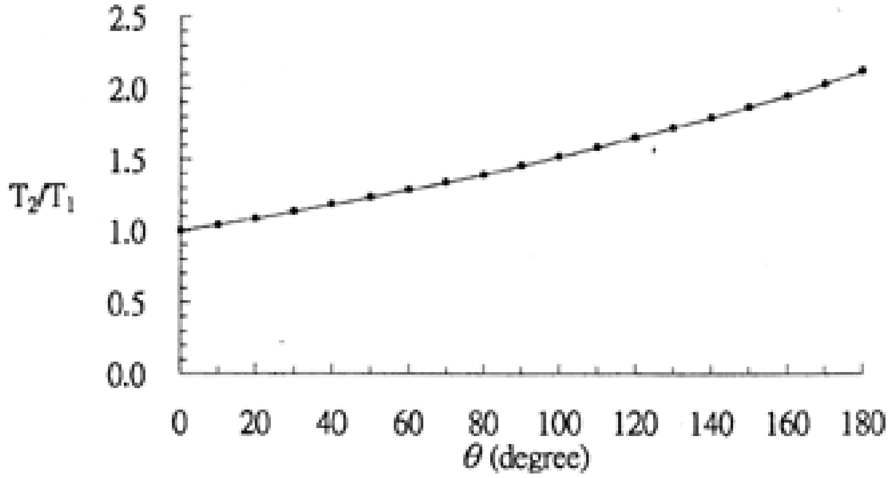

Fig.5 The tension distribution of the soft spring

(2)The $\ell n(T_2/T)$ vs. θ plot shows a straight line passing through the origin point. It's slope is μ. We get the coefficient of static friction of the spring with the bronze cylinder to be $\mu=0.241$.

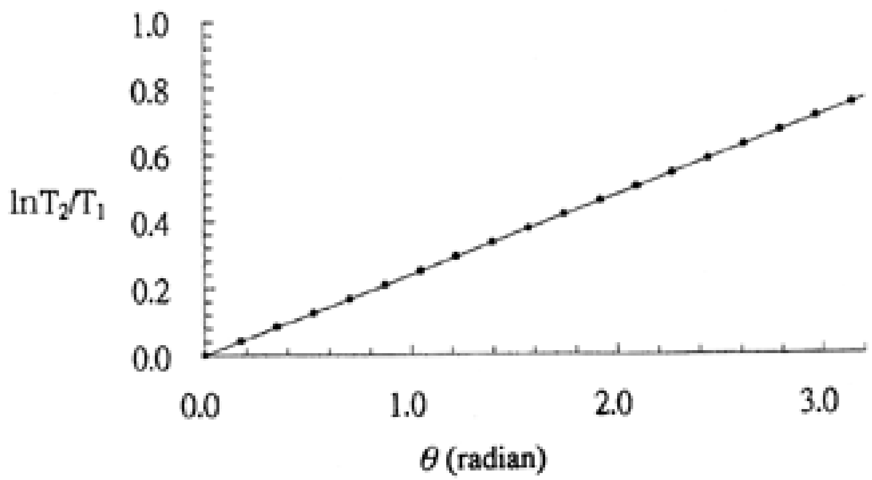

Fig.6 the plot of $\ell n(T_2/T_1)$ vs. angle

3.The datum in Table 3 are tensions of the iron wire which has the same diameter as the spring. The slope of Fig.7 is one, too. And the intercept on the vertical axis $\ell n T_2$ is $0.670 = \mu\theta = \mu\pi$. Thus μ is 0.213. Almost the same result is gotten as in the spring wrapped around the bronze cylinder.

Table3: T_1、T_2 of the iron wire on the cylinder

T_1(gw)	9.844	20.086	29.867	40.018	49.649	59.728
T_2(gw)	19.756	39.855	60.549	80.585	100.74	121.33

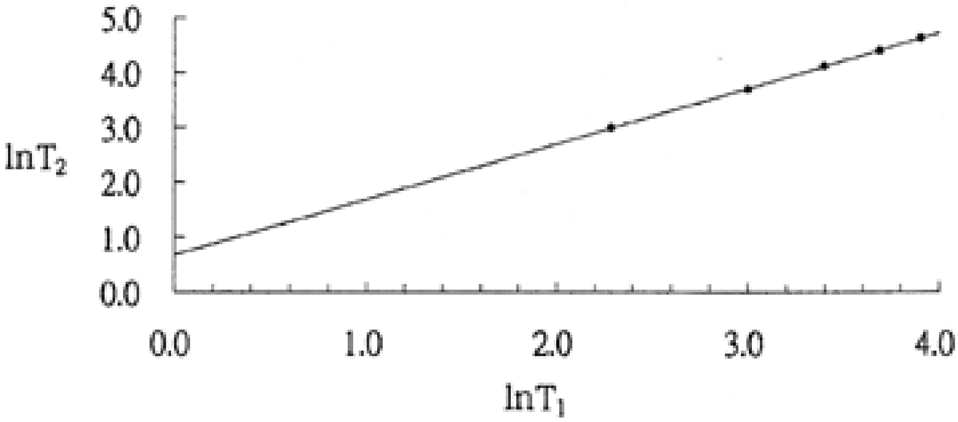

Fig.7 Graph of the iron line

VI、CONCLUSION

1.The equation for static friction is related with the tension at the ends of the cord $T_2 \leqq T_1 e^{\mu\theta}$. We can get μ from the intercept of $\ell n T_2$ vs. $\ell n T_1$ plot.

2.The tension distribution of a cord can be modified by a soft spring wrapped around a cylinder. The variations of tension in different wrap angles are in a very good agreement with $T = T_1 e^{\mu\theta}$.

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