Decomposition Chain of k -power for In te gers th 1. Motivation In class, my teacher gives a problem, and he asks us to observe that Then, he asks us to see if any integer can be ex pressed in the form of , when . This expression aroused my interest in solving this problem. I also want to know if this expression can be generalized to k power.2.Procedure and Methods Definition:If a positive integral number n can be ex pressed as when , then, we say that this ex pres sion is the kth-power de com po si tion chain of n. (1) First and Second Order Decomposition Chain When we look at the sequence (2-1),(3-2),(4-3), we can have the first order sequence Property1-1: Theorem1-A: , the first

$1=1^2$

$2=-1^2-2^2-3^2+4^2$

$3=-1^2+2^2$

$4=-1^2-2^2+3^2$

$5=1^2+2^2$

de com po si tion $6=1^2-2^2+3^2$ $n=\varepsilon_1 1^2+\varepsilon_2 2^2+\varepsilon_3 3^2+\cdots+\varepsilon_m m^2$ $m,k\in N, \varepsilon_i\in\{1,-1\}, i=1,2,3,\cdots,m$ th- $n=\sum_{r=1}^{m}\varepsilon_i r^k$

x 1 2 3 4 5 6 7 8 9

1 1 1 1 1 1 1 1

$k\in N, \varepsilon_i\in\{1,-1\}, i=1,2,3,\cdots,m$ $\forall n\in N$, $(n+1)-n=1$ $\forall n\in N$ chain of n exists. And then according to the method of Property1 - 1, after we try some other figures, we can find its de com po si tion chain, although its ex pres sion is not al ways the one and the only one. When in tro duc ing in the notions of order, and writing out the first few terms, we eas i ly find the following relationships. If we take the first order sequence we can have an arithmetic progression . But such a result is not es sen tial to our ultimate goal, because by avoiding the repeat of terms in each expression, we still have the property. Therefore, we have to separate and; that is, we have to take out the even-numbered terms in the arithmetic progression and then we get an oth er arithmetic progression . We get such a new progression, we can have . From above, we have the identity , if we apply the notion of com mon dif fer ence by sub tract ing the former term with its fol low ing one. When ex pand ing it, we get . We will prove it as follows: Property1-2: If Property1-2 is further applied, we can easily find Property1-3. Property1-3: , if the second order de com po

x 1 2 3 4 5 6 7 8 9

x^2 1 4 9 16 25 36 49 64 81

3 5 7 9 11 13 15 17

2 2 2 2 2 2 2

$(2^2-1^2),(4^2-3^2),(6^2-5^2),(8^2-7^2),\cdots$ $[(x+3)^2-(x+2)^2]-[(x+1)^2-x^2]=4$ $[(x+3)^2-(x+2)^2]-[(x+1)^2-x^2]=4$

$\forall k\in N$， $(k+3)^2-(k+2)^2-(k+1)^2+k^2=4$ $\forall n\in N$

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si tion chain of can exist, then the second order de com po si tion chain of n+4 can exist, too. Proof:If Property1-4:The decomposition chains forn=1,2,3,4 exist. And so, we apply Property 1-3 to prove that oth er decomposition chains exist. Proof: Theorem1-B: the second order de com po si tion chain of exist. (2) Third Order Decomposition Chain According to the form of the second order moments, we can also find the form of the third order mo ments as follows: Furthermore we can have the following expression, A simple examination leads that Property 2-1: And then, similar to Property 1-3, ac cord ing to Prop er ty 1-2, we have Prop er ty 2-2. Property 2-2: if the third order

If $n=\varepsilon_1 1^2+\varepsilon_2 2^2+\varepsilon_3 3^2+\cdots+\varepsilon_m m^2, m\in N$, then

de com po si $n+4=\varepsilon_1 1^2+\varepsilon_2 2^2+\varepsilon_3 3^2+\cdots+\varepsilon_m m^2+(m+1)^2-(m+2)^2-(m+3)^2+(m+4)^2$

If $n=4k+1$, then $n=1^2+\sum_{j=1}^{2k}(-1)^{j+1}\left[(2j)^2-(2j+1)^2\right]$

If $n=4k+2$, then $n=-1^2-2^2-3^2+4^2+\sum_{j=2}^{2k+1}(-1)^{j}\left[(2j+1)^2-(2j+2)^2\right]$

If $n=4k+3$, then $n=-1^2+2^2+\sum_{j=1}^{2k}(-1)^{j+1}\left[(2j+1)^2-(2j+2)^2\right]$

If $n=4k+4$, then $n=-1^2-2^2+3^2+\sum_{j=2}^{2k+1}(-1)^{j+1}\left[(2j)^2-(2j+1)^2\right]$ $\forall n\in N$

x	1	2	3	4	5	6	7	8	9
x^3	1	8	27	64	125	216	343	512	729
	7	19	37	61	91	127	169	217	
	12	18	24	30	36	42	48		
	6	6	6	6	6	6			

$$\{[(x+7)^3-(x+6)^3]-[(x+5)^3-(x+4)^3]\}-\{[(x+3)^3-(x+2)^3]-[(x+1)^3-x^3]\}=48$$

$$(k+7)^3-(k+6)^3-(k+5)^3+(k+4)^3-(k+3)^3+(k+2)^3+(k+1)^3-k^3=48$$

$\forall k\in N$, $(k+7)^3-(k+6)^3-(k+5)^3+(k+4)^3-(k+3)^3+(k+2)^3+(k+1)^3-k^3=48$ $\forall n\in N$ tion chain of exist, then the third decomposition chain of exist, too. Now, we find it seems impossible to list, in the man u al methods, all the third or der de com po si tion chains among 1~47 ; what's worse, there could be some missing. Therefore, we may as well take chanc es to test it in Turbo C computer language on the computer. To our belief, we should suc cess ful ly list out all the third order decomposition chains. (Appendix 1) Property 2-3:The decomposition Chain of n=1,2,3,...,47 exists. According to Property2-3, and im i tat ing the way we get Property 1-4, we can therefore have The o rem 2. Theorem2: ,A the third order de com po si tion chain of n exists. (3) The k power Decomposition Chain First, we have to find out an equality like Property 1-2 and Property 2-1. We set to observing from the second order sequence and find that the listing or der of the positive and negative of the numbers, which are from small to large, is very regular, . When we further observe the difference, we also find a mysterious regularity, , in its listing order of the pos i tive and negative numbers, which are also listed from small ones to large ones. So, we assume the listing order of the positive and neg a tive numbers, from small ones to large one, should have the fol low ing relationships: The listing order of the kth order should be to re verse all the positive and neg a tive numbers of k- $\forall n\in N$ th-

1th order, and then to follow the original positive and negative order of the k-1th order, as follows. Example: First moment: -+ Second moment:+--+ Third moment:-++-+--+ Forth moment:+--+-++--++-+--+ According to the result of second moment and third moment, we assume that there should exist a sim i lar identity in the forth moment. When we actually op er ate with any 16 figures in a row, as we have expected, we can get a constant, which is always 1536. Here, we would like to de fine our assumed listing regulation as a new symbol, as follows: represents an expansion of a serial .And is the val ue of a series . In this expression, Let denote c consecutive series, where c is a constant. The listing regulations are as follows: If k=t , the series of is If k=t+1 , the series of is Our new notation shows: We can know b2=4,b3=48,b4=1536 Next, by means of observation, we also find a rath er regularity among these num bers 4,48,1536. Then, we set to fac tor ize it: We find that bk seems to exist the following mys te ri ous relationship: $\sum_{i=1}^{2^k}\varepsilon_i x_i^k$ B_x^k b_k B_x^k $x_1=1$, $x_i=x_1+(i-1)$ $k\ge 2$ $\varepsilon_i=(-1)^i\prod_{j=1}^{k-1}(-1)^{\left[\frac{i}{2^j}+1\right]}$ $cB_{x_1}^k$ B_x^t $\varepsilon_1,\varepsilon_2,\varepsilon_3,\cdots,\varepsilon_{2^t-1},\varepsilon_{2^t}$ B_x^{t+1}

$b_2=4=(2^2)$; $b_3=48=(2^4)\times 3$; $b_4=1536=(2^9)\times 3$,

$-\varepsilon_1,-\varepsilon_2,-\varepsilon_3,\cdots,-\varepsilon_{2^t-1},\varepsilon_{2^t},\varepsilon_1,\varepsilon_2,\varepsilon_3,\cdots,\varepsilon_{2^t-1},\varepsilon_{2^t}$ $\frac{48}{4}=12=2^2\times 3$; $\frac{1536}{48}=32=2^5=2^3\times 4$. So, we observe it and get: According to recursion, we successively multiple it and we have After a series of observations and as sump tions like above, we conclude it as Property 3-1. Property 3-1: (represents a progression , and bk rep re sents the val ue of the progression .) Known: When k = t, the signs of the pro gres sion , is When k = t + 1, the signs of the progression is According to the mathematical inductive, we can know , the value of the progression , , is always a constant. Also, and then we finally get the proof. With this relationship, by using the recursion, we can easily get the result, . And then we get the proof. bk exists,like Property 3-1, we have the following result. Property3-2 : If , let exist the k de com po si tion chain of n , and then the decomposition chain of n+bn also exists. According to the above theorem, we find the sim plest way: We classify the natural numbers into the form of , in it, When s = 0, we simply link t consecutive se quenc es together, we can list a de com po si tion chain. When $b_k = 2^{k-1} \cdot k \cdot b_{k-1}$ $b_k = \prod_{s=1}^{k-1} 2^s \cdot k! = 2^{\frac{k(k-1)}{2}} \cdot k!$

$b_k = 2^{\frac{k(k-1)}{2}} \cdot k!$ $\sum_{i=1}^{2^t} \varepsilon_i x_i^k$ B_x^k B_x^k B_x^t $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_{2^t-1}, \varepsilon_{2^t}$ B_x^{t+1} $-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, \cdots, -\varepsilon_{2^t-1}, \varepsilon_{2^t}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_{2^t-1}, \varepsilon_{2^t}$ $\forall k, x \in N, k \geq 2$ B_x^k, b_k,

$b_k = 2^{k-1} \cdot k \cdot b_{k-1}$ $b_k = \prod_{s=1}^{k-1} 2^s \cdot k! = 2^{\frac{k(k-1)}{2}} \cdot k!$. th $\exists n \in N$ $n = s + t \cdot b_k$ $s \in R_k, R_k = \{0, 1, 2, \ldots, b_k - 1\}$

$$\begin{cases} b_2 = 4 \\ b_3 = 48 = 4 \cdot 3 \cdot 4 = 2^2 \cdot 3 \cdot b_2 \\ b_4 = 1536 = 8 \cdot 4 \cdot 48 = 2^3 \cdot 4 \cdot b_3 \\ b_5 = 122880 = 16 \cdot 5 \cdot 1536 = 2^4 \cdot 5 \cdot b_4 \\ \vdots \end{cases}$$

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s = 1 to bk-1, we find the decomposition chain of n , so long as we can find the decomposition chain of s, link t consecutive sequences. (For convenience, we name as sequence.) For example, when k=2,b2=4 It may be pleasant when we apply the ab meth ods to prove second moment order or third mo ment order, but when we try to solve the high order mo ment decomposition chain, we find it undoubtedly pain-taking and time-consuming to look for the de com po si tion chains of 1 bk-1. Even when we try to solve it directly on computers, we still can't reach out original goal. Therefore, returning to the ve start, we can't but set to from the sequence Now we reverse to ob serve Property 3-2, and we find we can rewrite it into the following expression: If the decomposition chain of exists, then the de com po si tion chain of , n+t*bkexists, too. In it, . When t is smaller than zero, we can simply change the signs that are added at the end of the de com po si tion chain of n . example, when k=3 If we rewrite it in the modular form, then we can have: Property 3-3: , If the decomposition chain of n exists, and also, , then the decomposition chain of n' exists, too. $11 = 3 + 4 \times 2 = (-1^2+2^2) + (3^2-4^2-5^2+6^2) + (7^2-8^2-9^2+10^2)$ B_x^k $\forall n \in N$ $t \in Z$

$100 = 1^3+2^3+3^3+4^3$

則 $52 = 100 - 48 = 1^3 + 2^3 + 3^3 + 4^3 - (-5^3 + 6^3 + 7^3 - 8^3 + 9^3 - 10^3 - 11^3 + 12^3)$ $\forall n \in N$ $b_k' \equiv -2 \pmod{b_k}$ then ∩

$b_k' = -2\varepsilon_p \pmod{b_k}$, when $\varepsilon_p = -1$, $b_k' \equiv 2 \pmod{b_k}$

For example, $B_3^2 = 3^2 - 4^2 - 5^2 + 6^2 = 4 \equiv 0 \pmod{b_2}$.

(In this expression is the coefficient of) ε {1,-1} p k Xp That is, In this expression, $5 \equiv 1 \pmod{b_2}$

For example, $B_2^2 = 2^2 - 3^2 - 4^2 + 5^2 = 4 \equiv 0 \pmod{b_2}$, and in this expression, $5 \equiv 1 \pmod{b_2}$

$\Rightarrow B_2^2 - 2 \cdot 5^2 = 2^2 - 3^2 - 4^2 - 5^2 \equiv -2 \pmod{b_2}$

$\Rightarrow B_3^2 + 2 \cdot 5^2 = 3^2 - 4^2 + 5^2 + 6^2 = 4 \equiv 2 \pmod{b_2}$ $\Rightarrow -B_2^2 + 2 \cdot 5^2 = -2^2 + 3^2 + 4^2 + 5^2 \equiv 2 \pmod{b_2}$

[Ps: in the following, for the convenience to express $-B_2^2 + 2 \cdot 5^2 = -2^2 + 3^2 + 4^2 + 5^2$, it will be defined as the new expression, $\overline{B_2^2}$.]

That is, we set to from a certain existed de com po si tion chain of number n, to get its module bk . Now, what we have to do is to fin module is the de com po si tion chain of 1,2,3,.....,bk-1. When we observe sequence , we find if we change one element of positive and negative sign of , we can have a new chain and its value is . Of course, when we take , the best way is to take modular to be the number, 1; that is, . Then, when we alter the positive and negative signs of xp which we choose, we can ha the following result: When a p =1 , and ≡−2 (mod bk), then, we have to properly alter its signs so that we can create a seria Modular bk to be 2. However, not every sequence can have a number modular b2 to be 1. In fact, when b2 =4, any serial mu be the continual four figures. Therefore, there must exist a number modular b2 to be 1. But when b3=48, any serial must be t continual eight figures; therefore, there doesn't always exist a mod u lar b3 to be 1. So we have to link together 6 continual

若 $x_p \equiv n \pmod{b_k}$

則 $x_p^k \equiv n^k \pmod{b_k}$

$\Rightarrow \varepsilon_p x_p^k \equiv \varepsilon_p n^k \pmod{b_k}$ （其中 $\varepsilon_p = \pm 1$ 為 x_p^k 之係數）

$\Rightarrow b_k' = b_k - 2\varepsilon_p x_p^k$

$\equiv -2\varepsilon_p x_p^k \pmod{b_k}$

$\equiv -2\varepsilon_p n^k \pmod{b_k}$

x_p^k If 即 $b_k' \equiv -2\varepsilon_p n^k \pmod{b_k}$

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se ries and make it a continual 48 continual figures. Then, there must exist a number modular 48 to be 1. So, we can come to a conclusion that the length of a chain is . But, . That is, in every continual series, there must appear a mod u lar bk to the number 1. For example, we take a number, 3, in R3, 3 = 1+ 2x1 According to Property 3-3 , we can cre ate 3 de com po si tion chain. Through this step, we seem to have found a solution. Next, what we have to do is to give an ordinary proof. For the con ve nience to express the similar series, we again define the following symbol, If in the series doesn't exist a modular bk If to be 1, then , For example: Now, we can apply the previous method, we divide the elements in Rk={0,1,2,3,.....,bk-1} into 2t and 2t+1. It seems that we can directly find the se ries with t modular of bk to be 2. Then, we fill in of series in order to make it a con tin u al de com po si

When in the series, B_x^k , exists x_i, we make $x_i \equiv 1 (\mod b_k)$

$\overline{B_{x_i}^k} = -\varepsilon_i(B_{x_i}^k - 2\varepsilon_i x_i^k)$, then, $\overline{B_{x_i}^k} \equiv 2 \pmod{b_k}$

Proof: (1) $a_i = 1$, $\overline{B_{x_i}^k} = -1 \cdot (B_{x_i}^k - 2x_i^k)$

$\equiv -1 \cdot (B_{x_i}^k - 2)$

$\equiv 2 \pmod{b_k}$

(2) $a_i = -1$, $\overline{B_{x_i}^k} = (B_{x_i}^k + 2)$

$\equiv 2 \pmod{b_k}$

$b_k = \prod_{s=1}^{k-1} 2^s \bullet k! \geq 2^k$ 2^k $\frac{b_k}{2^k}$ $1^3 + (\frac{48}{8} - 1)B_2^3 - (B_{42}^3 - 2 \cdot 49^3) = 1^3 - 2^3 + 3^3 + 4^3 - 5^3 + 6^3 - 7^3 - 8^3 + 9^3 \cdots - (-42^3 + 43^3 + 44^3 - 45^3 + 46^3 - 47^3 - 48^3 - 49^3) \equiv 3 \pmod{b_3}$ $\overline{B_x^k}$. $\overline{B_x^k} = B_x^k$

(1) $B_5^2 = 5^2 - 6^2 - 7^2 + 8^2 = 4 \equiv 0 \pmod{b_2}$

$\Rightarrow -2 \cdot 5^2 + B_5^2 \equiv -2 \pmod{b_2}$

$\Rightarrow 2 \cdot 5^2 - B_5^2 \equiv 2 \pmod{b_2}$

$\Rightarrow \overline{B_5^2} = 2 \cdot 5^2 - B_5^2 = 5^2 + 6^2 + 7^2 - 8^2 \equiv 2 \pmod{b_2}$

(2) $\overline{B_2^3} = -2^3 + 3^3 + 4^3 - 5^3 + 6^3 - 7^3 - 8^3 + 9^3 \equiv 0 \pmod{b_3}$ $\frac{b_k}{2^k} - 1$ tion chain. Then, we can make up the de com po si tion of a mod u lar bk to be 2t or 2t+1. Following is the original proof: If r=2t According to Property 3-3 , we have that r ex ists in a decomposition. If r=2t+1 According to Property 3-1 , we have that r ex ists in a decomposition. From 1, and 2, we can have Property 3-4 : , the decomposition of r exists. When combining Property 3-2 and Prop er ty 3-4, we can also have the fol low ing theorem. Theorem 3: ,there exists the kth de com po si tion of . For example, (1)k=2 , if the remainder is 3, (2) k=3, if the remainder is 7, 5. The Development of Decomposition Chains After we successfully prove the existence of the K decomposition of any natural number, we will con tin ue to research and develop it. Observing the con tin u al integers,1,2,3,...., in the decomposition, $\forall r \in R_k$, $R_k = \{0,1,2,3,\ldots, b_k - 1\}$

$\overline{B_1^k} + (\frac{b_k}{2^k} - 1)B_{2^k+1}^k + \overline{B_{b_k+1}^k} + (\frac{b_k}{2^k} - 1)B_{b_k+2^k+1}^k + \ldots + \overline{B_{(t-1)b_k+1}^k}$

$\equiv 2+0+2+0+2+\cdots+2 \pmod{b_k}$ （其中有 t 個 $\overline{B_{x_i}^k}$ 有 t-1 個$(\frac{b_k}{2^k}-1)B_{x_i}^k$）

$\equiv 2t \pmod{b_k}$

$\equiv r \pmod{b_k}$

$1^k + (\frac{b_k}{2^k} - 1)B_2^k + \overline{B_{2+b_k-2^k}^k} + (\frac{b_k}{2^k} - 1)B_{2+b_k}^k + \overline{B_{2+2b_k-2^k}^k} + \ldots + \overline{B_{2+tb_k-2^k}^k}$

$\equiv 1+0+2+0+2+0+2+\cdots+2 \pmod{b_k}$

$\equiv 1+2t \pmod{b_k}$

$\equiv r \pmod{b_k}$

$\forall r \in R_k$, $R_k = \{0,1,2,3,\ldots,b_k - 1\}$ $\forall n,k \in N$

$B_2^2 = 2^2 - 3^2 - 4^2 + 5^2 = 4$

$1^2 + \overline{B_2^2} = 1^2 + (-2^2 + 3^2 + 4^2 + 5^2) = 47 \equiv 3 \pmod{b_2}$ $1^2 + 5B_2^3 + \overline{B_{42}^3} + 5B_{50}^3 + \overline{B_{90}^3} + 5B_{98}^3 + \overline{B_{138}^3} \equiv 7 \pmod{b_3}$ th

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Therefore, as to any natural integers, to change the common difference still exists the decomposition chain. Then, we have: Property 4-2: exists permanently. (3) In the case when we alter both the first term and the common difference at the same time. Next, we will assume if the de com po si tion chain exists as well when we alter both the first term and the common dif fer ence at the same time. Obviously, there must be some restrictions, be cause we can easily have a reversal example. When we take a=2 and d=2, if n is an odd number, then v(n,k,2,2) doesn't necessarily exist. Then, how can we establish it? Let the first term of an integer chain be and the common difference be d ; also, let a certain term of the integer chain be q=a+(p-1)d . According to the discussion in Property 4-1 and d are, both Property 4-2 and Property3-1 can be established. The only key to decide wheth er v(n,k,a,d) exists is to decide whether the re la tion ship of exists. That is, we have to see whether, in , exists. Let's try to look for the necessary re quire ments by means of the notion of the existence of integer solutions. If q=a+(p-1)d =bk(d)xt+1 That is: bk(d)xt-pxd=a-d-1 If the integer solutions of (p,t) exists, we should have that gcd(bk(d),d) divides (a-d-1). d(a-d-1) d(a-1) Let donate the x-axis, and t be the y-axis, then the straight line bk(d)xt-pxd=a-d-1 , Since the slope , the line must pass the first $\forall n,k,d \in N$, $v(n,k,1,d)$ $\overline{B_{x(d)}^k} \equiv 2 \pmod{b_k}$

$\overline{B_{x(d)}^k}$ $x \equiv x_1 \equiv (\mod b_k)$ $\frac{d}{b_{k(d)}} > 0$ quadrant. Therefore, the solution of t exists. It means that we can find a certain term d to make . Then, proposition 10 can be proved. Therefore, we can have: Theorem 4: If exists permanently. We can give some simple examples as follows: For example:3.Conclusion Through constant study and research, we have the completion from the initial sec ond order de com po si tion chain to third order decomposition chain. But when we expand it to forth order, fifth order or even kth order, we encounter great difficulties. Then by using the in tro duc tion of the notion of modular theory, we finally complete the kth decomposition chain for each integer Then, after repeated efforts, when we alter the com mon dif fer ence of the integer bigger than 1, like , we can also prove it to be established. And then, we again alter its common difference and the first term and finally n the get the

$2 = 1^2+3^2+5^2-7^2+9^2-11^2-13^2+15^2$

essential requirements. $p \in N$ $\overline{B^{t}_{s(d)}} \equiv 2 \pmod{b_k}$ $2 = 1^3+3^3+5^3-7^3+9^3-11^3-13^3+15^3+17^3-19^3-21^3+23^3-25^3+27^3+29^3-31^3$ n, $n = \varepsilon_1 1^t + \varepsilon_2 2^t + \varepsilon_3 3^t + \ldots + \varepsilon_m m^t$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$