Mathematics Game time, players select numbers be tween 1 and n. What are the natural numbers, "N", which will al low play er B to win?(3) Rules of the game is the same as (1) if the number selected at the tth time is at, and at +1 is not equal to f of at (mod n). What are the natu- ral numbers "N" which will allow player B to win? 3.Research Process (A)Verification: After testing the numbers with a computer, we ver i fied the following results. (B)Speculation:

n	The form of winning numbers	n	The form of winning numbers
3	I	21	II
5	II	23	III
7	II	25	II
9	III	27	I
11	I	29	II
13	II	31	II
15	I	33	III
17	II	35	I
19	I	37	II

Definition: 1.Introduction There is a very interesting game in the book "Arithmetics" written by Pro fes sor Chih-Nung Hsu. I find this game is somewhat similar to a game in Professor Yeong-Nan Yeh's research paper, "A Nim-like Game and Dynamic Recurrence Relations". Professor Yeh's research is the winning set of ei ther increasing or decreasing functions. However, the rules I have been working on differ from his, and my game is based on the function tak ing val ues in Res i due system modulo with some number.2.Research Goals My research tries to deal with the fol low ing problems: (1) Suppose players are A and B, A takes the first turn and has to abide by the following rules: players must alternately select a number be tween 1 and 5. Player A selects a number . Then play er B selects a number but it can not be the same as the number which has just been selected by the oth er player. Thus the players con tin ue in the end, add all of the numbers selected by both players. The first play er to reach the pre-desig- nated natural num ber "N" will be the winner. What are the nat u ral numbers, "N" which will al low play er B to win? (2) Rules of the game is the same as (1), but this

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n represents the largest selectable nature number,

N0 represents $N \cup (0)=(0,1,2,3,4,5\cdots\cdots)$: A set consists of non-negative whole numbers.

$Odd=(2n-1 \mid n \in N)=(1,3,5,7\cdots\cdots)$: A set consists of positive odd numbers.

$\langle w_m \rangle$ is the numbersfor which will allow player B to win the game

If $n=2^m \times k$ ($k \in Odd$), then $EP(n)=m$, $OP(n)=k$ are defined.

EP(n): the highest power to 2 of n

OP(n): the odd part of n

Type I : If $EP(n+1)=2k$ ($k \in N_0$), then $w_m=w_{m-1}+n+1$

$(w_0=0, m \in N)$

Type II : If $EP(n+1)=2k+1$, and $EP(OP(n+1)+n+2)=2t+1$ ($t \in N$) ,

then $w_m=w_{m-1}+n+1+\frac{1+(-1)^{m-1}}{2}$ $(W_0=0, m \in N)$

Type III : If $EP(n+1)=2k+1$, and $EP(OP(n+1)+n+2)=2t$

then $w_m=w_{m-1}+n+1+\left|3[\frac{m+\frac{5}{2}}{3}]-m-1\right|$ $(W_0=0, m \in N)$

(C)Proof: (a) Consider the following game: there are two players, A and B,A be the first player, both play ers take turns selecting any num- bers from 1 to n. Add all the numbers selected, un til the first per son who ex ceeds the tar get number shall lose the game. As far as B is concerned, what numbers do N have to be for play er B to win, regard- less of what num bers play er A has selected? 4.Research Regardless of what numbers player A has selected, the set of numbers with which player B will win, N, is called winning members, Wn. To prove Wn = {n+1,2(n+1),3(n+1),4(n+1).......} definition : i represents the ith number selected

Make N=k(n+1), no matter what number A had selected, B will win. When N=(k+1)(n+1), ifa1=q, and then b1=(n+1)-q,the res will be k(n+1) , so from the hy poth e si we know that no matter what number A ha selected, B will win. From MI, all of the elements of m(n+1), m being a whole number, no matter what number A had selected, B will win. In addition, if N is nc in the format of m(n+1), N=m(n+1)+p, an ≦p≦n, a1=p, and B will be left with m(n+1), the previous proves that no mat ter what number B had selected, A will win. Therefore, it is not a winning number. Therefore, the winning set is proved to be: Wn ={n+1,2(n+1),3(n+1),4(n+1).....} (b) Definition: wm: the mth winning number. Wn: the combination of winning numbers. left: the numbers which were left over. (c) The above proof and definition infer: Winning condition: C1: When player A selects the number, left =wm, then B will by player A, bi represents the ith number selected by play er B. When N=n+1, if $a_1=q$, then $b_1=(n+1)-q$, any num ber se lect ed by A will exceed N, Therefore, no mat ter what number A had selected, B will win.

win (by definition). C2: When player B selects the number, left =wm, then A will win (by definition). (d) (e) Winning condition:

Lemma 1.: $w_m+q \notin W_n, 1 \leqq q \leqq n$

Lemma 2.: If $w_m+(n+1) \in W_n$, then $a_1 = \frac{n+1}{2}$

Collorary1 .If n is an even number, then $w_{m+1} = w_m+($

Lemma 3 : If $w_m+(n+1) \notin W_n$, then $w_m+(n+2) \in W_n$

Collorary2: $w_{m+1} = w_m+(n+1)$ or $w_{m+1} = w_m+(n+2)$

If $w_m+(n+1) \notin W_n$, then singular set H_n :{ $w_m+(n+1) \notin W_n$ } is defined.

C_1:When player B selects the number, left $\in H_n$, and the number selected by player A is not $\frac{n+1}{2}$, then B will win.

C_2:When player A selects the number, left $\in H_n$, and the number selected by player B is not $\frac{n+1}{2}$, then A will win.

C_3:When player A selects the number, left $\in H_n$, and the number selected by player B is $\frac{n+1}{2}$, then B will win.

C_4:When player B selects the number, left $\in H_n$, and the number selected by player A is $\frac{n+1}{2}$, then A will win.

(f) Resistance : (g)Lemma 4. (h)Definition: 5. Conclusion (1) The special case n=5 : Suppose players are A and B, A takes the first turn and has to abide by the following rules: play ers must al ter nate ly select a number between 1 and 5. Player A selects a number . Then player B se lects a num ber but it can not be the same as the number which has just been selected by the other

R_1: If left= $w_m+2p=w_{m-1}+2i$ and the number selected must not be 2p ($1 \leqq 2p, i \leqq n$) , then the number selected must be p or i to keep the opposition from winning.

R_2: If left= $w_m+2p=w_{m-1}+(2i-1)$ and the number selected must not be 2p ($1 \leqq 2p, i \leqq n$) , then the number selected must be p to keep the opposition from winning.

R_3: If left= $w_m+(2p-1)=w_{m-1}+2i$ and the number selected must not be 2p ($1 \leqq 2p, i \leqq n$) , then the number selected must be i to keep the opposition from winning.

R_4: If left= $w_m+(2p-1)=w_{m-1}+(2i-1)$ and the number selected must not be 2p ($1 \leqq 2p, i \leqq n$) , then no number can keep the opposition from winning.

If $n+1=2^{2p}q$ ($q \in$ Odd) [being EP(n+1)=2p] then $w_1=n+1$

If $n+1=2^{2p-1}q$ ($q \in$ Odd) [being EP(n+1)=2p-1] then $w_2=n+2$

Collorary 3. If n is an even number, then $W_n \in$ I (From Collorary1 and Lemma4)

If W_n={(n+1),2(n+1),3(n+1), ⋯(n+1) ⋯}, $W_n \in$ I

If W_n={(n+2),(2n+3),(3n+5),2(2n+3) ⋯⋯}, $W_n \in$ II

If W_n={(n+2),(2n+4),(3n+5),(4n+7),(5n+9),2(3n+5), ⋯}, $W_n \in$ III

Theory 1.If EP(n+1)=2p , $p \in N$, then $W_n \in$ I

Theory 2.If EP(n+1)=2p-1 and EP[OP(n+1)+n+2]=2r+1,p, $r \in N$, then $W_n \in$ II

Theory 3.If EP(n+1)=2p-1 and EP[OP(n+1)+n+2]=2r ,p, $r \in N$, then $W_n \in$ III

player. Thus the players con tin ue in the end, add all of the numbers selected by both players. The first player to reach the pre-des ig nat ed natural num ber "N" will be the winner. What are the natural numbers, "N" which will allow play er B to win? Let us define sequence (0=W0<....Wm<....)to be the num bers for which will al low play er B to win the game. In this special case n=5, the num bers wm satisfies (2) The general case n: The results of the game is the same as in (1), but this time , players select number between 1 and n. We also define the sequence (0=W0<....Wm<.....)to be the num bers for which will al low player B to win the game. In the gen er al case, the sequence is more complicated. The re sults are given as following: 6.Discussion (1) Change the rules of the game, and find its win- ning combination. (2) The discussions were from the point of view of two players. I hope this game can developed into a multiple player game. 7.Reference

(1) Arithmetic, by Professor Zhi-Nong Hsu . (2) "A Nim-like Game and Dynamic Recurrence Relations", a dissertation by Professor Yeong -Nan Yeh. (3)

Type Ⅰ: If $EP(n+1)=2p$, where $p \in \{0,1,2,3\cdots\}$, then $W_n \in$ Ⅰ

Type Ⅱ: If $EP(n+1)=2p-1$ and $EP[OP(n+1)+n+2]=2r+1$, where $p \in \{1,2,3\cdots\}$ and $r \in \{1,2,3\cdots\}$, then $W_n \in$ Ⅱ

$w_0=0$

Type Ⅲ: If $EP(n+1)=2p-1$ and $EP[OP(n+1)+n+2]=2r$, where $p \in \{1,2,3\cdots\}$, and $r \in \{1,2,3\cdots\}$, then $W_n \in$ Ⅲ

Number Theory by George E Andrews. $w_m=w_{m-1}+6+[1+(-1)^{m-1}]/2$