On the Number of Special Subgraphs in an Edge 2-Colored or an Oriented Com plete Graph

Abstract Suppose there are n persons. We call three persons a monochromatic tri an gle if either any two of them know each other, or do not know each other. As a spe cial case of Ramsey Theorem, we know that if n≧6 then there exists a mono chro mat ic triangle. Naturally when n is large, there are many mono chro mat ic triangles. We shall determine the least number of mono chro mat ic ones among any n persons. Besides, suppose there are n persons and each two of them play a game on ta ble tennis. We say three persons A, B, C, form a directed triangle if A beats B, B beats C and C beats A. We say four persons A, B, C, D, form a di rect ed square if A beats B, B beats C, C beats D and D beats A. We shall de ter mine the larg est number of directed triangles and squares among any n persons. The solution for the largest number of directed tri an gles also gives an answer to the ques tion of least number of transitive subtournaments of or der 3 in an n-element tournament. We shall also give an upper bound for the least of transitive subtournaments of order 4 in an n-element tournament. 1.Introduction

According to Ramsey Theorem, for any integer k, there exists an integer m, such that for any $n \geq m$, if the edges of a complete graph K_n are colored by k colors then there is at least a monochromatic triangle. Naturally, the larger is the number n, the more monochromatic triangles there are. We ask the question "what is the least number of monochromatic triangles in an edge k-colored K_n?" In this study, I determine the least number of monochromatic triangles in an edge 2-colored K_n.

Another topic I studied is the largest number of directed triangles in an n-element tournament T_n. From the results I shall obtain the largest number of directed quadrilaterals in an n-element tournament T_n.

While obtaining the results of directed triangles, the least number of transitive triangles can be determined. So I tried to investigate the related problem, the least number of transitive sub-tournaments TT_m in an n-element tournament T_n.

Procedures

First, I use a double-counting method to obtain the minimum number of monochromatic triangles in an edge 2-colored complete graph K_n and the maximum number of directed ones in an n-element tournament T_n. From the results, I determine the largest number of directed quadrilaterals in a tournament T_n. Finally, I try to study the least number of transitive sub-tournaments TT_m.

A. Definitions and Notations

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges. Each edge is an unordered pair of distinct vertices. An oriented graph is obtained from a graph by assigning an orientation to each edge. Thus any edge of an oriented graph is an ordered pair of vertices. A complete graph with n vertices, denoted by K_n, is a graph with n vertices, and each pair of distinct vertices is joined by an edge. A tournament of order n, denoted by T_n, is an oriented graph obtained from K_n by assigning an orientation to each edge. Thus an n-element tournament has n vertices and edge pair of distinct vertices is joined by exactly one oriented edge. Usually the vertices of K_n or T_n are chosen to be 1, 2,…, n.

2.Procedures

In the first topic, suppose the edges of K_n are colored by two colors, blue and red. Denote by G_1 and G_2 the two subgraphs induced by blue edges and red ones respectively. We shall denote by $(d_1, d_2, \ldots, d_n)$ the degree sequence of G_1.

For an n-element tournament T_n, let $V_i = \{v_i | i \to v_i\} (i = 1, 2, \ldots n)$, and $d_i = |V_i|$.

Then $(d_1, d_2, \ldots, d_n)$ is called the out degree sequence of T_n.
We use (i, j, k) to denote the directed triangle i j k i. So (j, k, i) and (k, i, j) denote the same directed triangle. See Figure 1.
The form of a transitive triangle is "$i \to j \to k$" and "$i \to k$". See Figure 2. It is obvious that a triangle in a tournament is either a directed one or a transitive one.

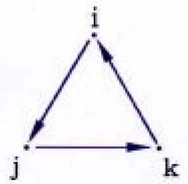

Figure 1

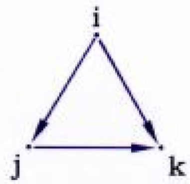

Figure 2

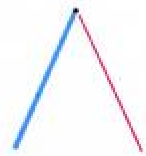
Figure 3

B. Counting Monochromatic Triangles in an Edge 2-Colored K_n

In this section, we consider the least number of monochromatic triangles in an edge 2-colored K_n. (Note that the maximum number of monochromatic triangles in an edge 2-colored K_n is trivially C_3^n, which occurs when all the edges are colored the same.)

Theorem 1: Suppose the edges of K_n are 2-colored and $(d_1, d_2, \ldots, d_n)$ is the degree sequence of the subgraph G_1 induced by blue edges. Then the

number of monochromatic triangles $= C_3^n - \frac{1}{2}\sum_{i=1}^{n} d_i(n-1-d_i)$.

Proof: Suppose X is the number of monochromatic triangles,

then $C_3^n - X$ is the number of non-monochromatic triangles.

We count the number of 2-colored paths of length 2 in two different ways. Firstly, for each vertex i the number of 2-colored paths of length 2 with i as the

central vertex is $d_i(n-1-d_i)$. Therefore the total is $\sum_{i=1}^{n} d_i(n-1-d_i)$.

On the other hand, each monochromatic triangle contains no 2-colored paths of length 2, while each non-monochromatic triangle contains two 2-colored paths of length 2. Therefore the total number of 2-colored paths of length 2 is

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$2(C_3^n - X)$. Thus, $2(C_3^n - X) = \sum_{i=1}^{n} d_i(n-1-d_i)$,

$$\Rightarrow X = C_3^n - \frac{1}{2}\sum_{i=1}^{n} d_i(n-1-d_i). \qquad \text{(Q.E.D.)}$$

Theorem 2:

(1) When $n = 4k+1, (n \in \mathrm{N})$, the least number X of monochromatic triangles in edge 2-colored K_n is $\frac{2}{3}k(k-1)(4k+1)$. This bound is attained if and only if the degree sequence of G_1 is $(2k, 2k, \ldots, 2k)$.

(2) When $n = 4k+3, (n \in \mathrm{N} \cup \{0\})$, the least number X of monochromatic triangles in an edge 2-colored K_n is $\frac{2}{3}k(k+1)(4k-1)$. This bound is attained if and only if the G_1 degree sequence (or G_2 degree sequence) can become $(\underbrace{2k+1, 2k+1, \ldots, 2k+1}_{4k+2}, 2k)$ by arranging the serial numbers.

(3) When $n = 2k, (k \in \mathrm{N})$, the least number X of monochromatic triangles in an edge 2-colored K_n is $\frac{1}{3}k(k-1)(k-2)$. This bound is attained if and only if $\sum_{i=1}^{n} d_i$ is even and $d_i \in \{k, k-1\}$.

By Theorem 1, to determine the minimum of X is equivalent to determine the maximum of $\sum_{i=1}^{n} d_i(n-1-d_i)$. It is related to the following lemmas:

Lemma 1: Suppose p, q, p', q' are positive integers such that $p+q = p'+q'$. If $|p-q| \le |p'-q'|$, then $pq \ge p'q'$.

Lemma 2: $2\left|\sum_{i=1}^{n} d_i\right.$.(It is because upon calculating the sum of d_i, we have counted each edge for two times. So the sum is even.)

Proof: (1) $n = 4k+1, (n \in \mathrm{N})$: By Lemma 1, the maximum of $\sum_{i=1}^{n} d_i(n-1-d_i)$ is achieved by letting $d_i = 2k$, and the sum of d_i is an even number.

Besides, a graph on n vertices with degree sequence $(2k,2k,\ldots,2k)$ can be constructed as follows: (i, j) is colored blue $\Leftrightarrow |i-j| \equiv 0$ or $1 \ (mod\ 4)$.

The condition for $k=2$ is as Figure 4-1. Therefore, we shall obtain:

- $min\ X = \frac{1}{6}(4k+1)(4k)(4k-1) - \frac{1}{2}(4k+1)(2k)^2 = \frac{2}{3}k(k-1)(4k+1)$
- Figure 4-2 shows that another graph (on nine vertices) has the minimum number of monochromatic triangles but different structures.

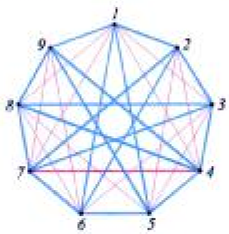

Figure 4-1

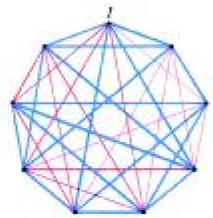
Figure 4-2

(2) $n = 4k+3, (n \in N)$: Originally, the maximum of $\sum_{i=1}^{n} d_i(n-1-d_i)$ is achieved by letting $d_i = 2k+1$. However, to ensure that the sum of d_i is an even number, one of the $max \sum_{i=1}^{n} d_i(n-1-d_i)$ is achieved by letting $d_i = 2k+1$ for $i = 1,2,\ldots,n-1$ and letting $d_n = 2k$. Besides, a graph on n vertices with G_1 degree sequence $(\underbrace{2k+1,2k+1,\ldots,2k+1}_{4k+2},2k)$ can be constructed on the previous condition $n = 4k+1$ as follows:

Add two vertices into a complete graph on $n-2$ vertices and with special G_1 degree sequence $\{k,k,\ldots,k\}$. Call these new two vertices $4k+2$, $4k+3$. Then, choose any $2k+1$ vertices connecting $4k+2$ with blue edges, and the other $2k$ vertices connecting $4k+3$ with blue edges also. The rest edges are colored red. By this construction, we obtain a graph with G_1 degree sequence $(\underbrace{2k+1,2k+1,\ldots,2k+1}_{4k+2},2k)$.

Figure 5-1 is for $k=1$. Therefore we shall obtain:

- $min\ X = C_3^n - \frac{1}{2}(n-1)(2k+1)^2 - \frac{1}{2}(2k)(2k+2)$

$$=\frac{1}{6}(4k+3)(4k+2)(4k+1)-\frac{1}{2}(4k+2)(2k+1)^2-k(2k+2)$$

$$=\frac{2}{3}k(k+1)(4k-1)$$

· It is easy to know, the graph is not unique the degree sequence. See Figure 5-2 for another graph of the same degree sequence.

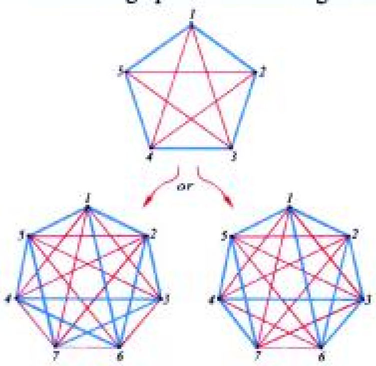

Figure 5-1　　Figure 5-2

(3) $n=2k,(k\in \mathrm{N})$: By Lemma 1, the maximum of $\sum_{i=1}^{n} d_i(n-i-d_i)$ is

achieved by letting $d_i=k$ or $k-1$, and the sum of d_i is an even number. Besides, a graph on n vertices with degree sequence $(k,k,\ldots,k)$ of G_1 can be constructed as follows: (i,j) is colored blue $\Leftrightarrow i\equiv j \pmod 2$ The construction would make all monochromatic triangles contained in the subgraph G_1. Figure 6 is for $k=4$. Then we can obtain:

· $min\ X=C_3^n-\frac{1}{2}nk(k-1)=\frac{1}{6}(2k)(2k-1)(2k-2)-\frac{1}{2}(2k)k(k-1)$

$$=\frac{1}{3}k(k-1)(k-2) \qquad \text{(Q.E.D.)}$$

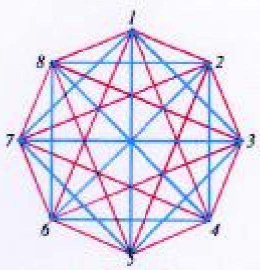

Figure 6

C. Counting Directed Triangles in an n-element Tournament T_n

This section investigates the largest number of directed triangles in an n-element tournament. First we observe that the minimum number of directed triangles in an n-element tournament is zero. Theorem 3 below characterizes tournaments that contain no directed triangles.

Theorem 3: A tournament T_n contains no directed triangle if and only if the out degree sequence can become $(0,1,2,..n-1)$.

Proof: $(\Leftarrow)$ Suppose $d_i = i-1$ $(i=1,2,...,n)$. Assume to the contrary that T_n contains a directed triangle, then this triangle does not contain the vertex n. Because n has out-degree $n-1$, there is no vertex directing to n. Similarly, the directed triangle does not contain $n-1$, as n is the only vertex directing to $n-1$. By induction, we prove that no vertex is contained in a directed triangle. $(\rightarrow\leftarrow)$ Above hints the triangles in this kind of graph are all transitive ones.

$(\Rightarrow)$ Suppose $d_1 \le d_2 \le ... \le d_n$, It suffices to prove: $\forall\ i \ne j \Rightarrow d_i \ne d_j$.

Suppose $\exists\ i \ne j,\ d_i = d_j$. By symmetry we may assume $i \rightarrow j$. Then from

$d_i - 1 < d_j$ we know: $\exists\ h \ni j \rightarrow h$ and $i \leftarrow h$. Therefore $i \rightarrow j \rightarrow h \rightarrow i$

i,e, (h,i,j) is a directed triangle. $(\rightarrow\leftarrow)$ (Q.E.D.)

Remark: We can know the structure of such graph is unique by induction.

Theorem 4: Suppose T_n is a tournament with out-degree sequence $(d_1, d_2, ..., d_n)$,

then the number of directed triangles $= \frac{1}{2}[(\sum_{i=1}^{n} d_i(n-1-d_i)) - C_3^n]$

Proof: Suppose Y the number of directed triangles, then $C_3^n - Y$ is the number of transitive ones. We count the number of directed paths of length 2 (Figure7) in T_n. Each directed triangle contains three directed paths of length 2, and each transitive triangle contains one directed path of length 2.

So the total number of directed paths of length 2 is $3Y + 1(C_3^n - Y)$.

Figure 7

On the other hand, consider the vertex i, there are $d_i(n-1-d_i)$ directed paths of length 2 with i as the central point. So the total number of directed paths of length 2 is $\sum_{i=1}^{n} d_i(n-1-d_i)$. Therefore, $2Y + C_3^n = \sum_{i=1}^{n} d_i(n-1-d_i)$

$$\Rightarrow Y = \frac{1}{2}[(\sum_{i=1}^{n} d_i(n-1-d_i)) - C_3^n] \qquad \text{(Q.E.D.)}$$

Theorem 5:

(1) When $n = 2k+1, (k \in \mathrm{N})$, the maximum number Y of directed triangles is $\frac{1}{6}k(k+1)(2k+1)$, the bound is attained if and only if the out degree sequence is $(k, k, \ldots, k)$.

(2) When $n = 2k, (k \in \mathrm{N})$, the maximum number Y of directed triangles is $\frac{1}{3}k(k-1)(k+1)$, the bound is attained if and only if the out degree sequence can become $(\underbrace{k, k, \ldots, k}_{k}, \underbrace{k-1, k-1, \ldots, k-1}_{k-1})$ by arranging the serial numbers.

By Theorem 4, to determine the maximum of Y is equivalent to determine the maximum of $\sum_{i=1}^{n} d_i(n-1-d_i)$. We shall combine Lemma 1 and the following lemma to prove Theorem 5.

Lemma 3: $\sum_{i=1}^{n} d_i = \frac{1}{2}n(n-1)$ (It is because each edge devotes once to the sum of d_i. So the sum equals the number of edges.)

Proof: (1) $n = 2k+1$: By Lemma 1, the maximum of $\sum_{i=1}^{n} d_i(n-1-d_i)$ is achieved by letting $d_i = k$. On the other hand, a graph on n vertices with out degree sequence $(k, k, \ldots, k)$ can be constructed as follows:
$\forall$ vertex $i: i \to v_i \Leftrightarrow v_i \in \{i+1, i+2, \ldots, i+k\}$, denote by v_i the vertex $v_i - (2k+1)$ when $v_i > 2k+1$. Figure 8-1 is for $k = 3$. We can obtain:

$$\cdot \max Y = \frac{1}{2}(nk^2 - C_3^n) = \frac{1}{2}[(2k+1)k^2 - \frac{1}{6}(2k+1)(2k)(2k-1)]$$

$$= \frac{1}{6}k(k+1)(2k+1)$$

· Figure 8-2 shows that another graph (on seven vertices) has the maximum of directed triangles but different structures.

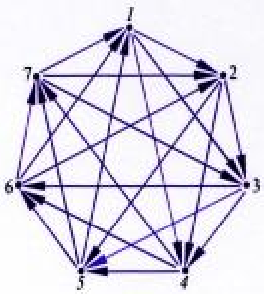

Figure 8-1

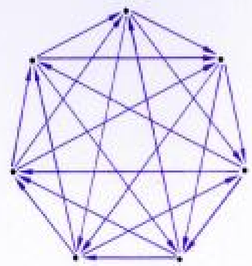

Figure 8-2

(2) When $n = 2k$, we choose d_i for $\frac{(n-1)-1}{2}$ or $\frac{(n-1)+1}{2}$ i.e. $k-1$ or k, in order that $d_i(n-1-d_i)$ is the maximum.

Suppose there are t $d_i = k-1$, $n-t$(i.e.$2k-t$) $d_i = k$. From Lemma 3, the coefficient t should fit: $(k-1)t + k(2k-t) = k(2k-1) \Rightarrow t = k$.

Without loss of generosity, we can assume $d_1 = d_2 = ... = d_k = k$, $d_{k+1} = d_{k+2} = ... = d_{2k} = k-1$. A feasible construction is as follows:

$\forall i \in \{1,2,...,k\}$: $i \to v_i \Leftrightarrow v_i \in \{i+1, i+2, ..., i+k\}$

$\forall_j \in \{k, k+1, ...2k\} \Leftrightarrow j \to v_j \Leftrightarrow v_j \in \{j+1, j+2, ..., j+k-1\}$

Denote $v_j = v_j - 2k$ when $v_j > 2k$.

The condition for $k = 4$ is as Figure 7-1. Then we can obtain:

· $max\ Y = \frac{1}{2}(nk(k-1) - C_3^n) = \frac{1}{2}[2k^2(k-1) - \frac{1}{6}(2k)(2k-1)(2k-2)]$

$= \frac{1}{3}k(k-1)(k+1)$

· Figure 7-2 shows that another graph (six vertices) has maximum number of directed triangles but different structure. (Q.E.D.)

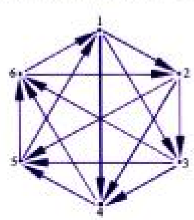

Figure 9-1

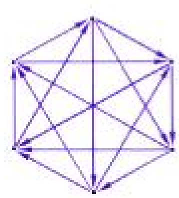

Figure 9-2

D. The Largest Number of Directed Quadrilaterals

A directed quadrilateral in a tournament T_n is a subgraph with 4 vertices and 4 edges, which form a directed cycle, i.e. the out degree of each vertex in it is 1.

Similarly we consider the minimum and the maximum number of directed quadrilaterals. It is easy to see that the least number of the directed quadrilaterals is zero! The construction in the proof of Theorem 3 gives an example of such tournaments, and there are other tournaments without directed quadrilaterals. So the main problem is the maximum number of directed quadrilaterals in a tournament T_n.

Theorem 6: The maximum number (M) of the quadrilaterals in T_n is

(1) $\frac{1}{6}k(k-1)(k+1)(2k+1)$ for $n=2k+1,(k\in \mathrm{N})$.

(2) $\frac{1}{6}k(k-1)(k+1)(2k-3)$ for $n=2k,(k\in \mathrm{N})$.

Proof: Suppose there are respectively x, y, z sub-graphs as Figure 10-1, 10-2, 10-3. Notice that the directed quadrilateral only occur in the Figure 8-1 and there is only one in each of it, so $M=\max x$; Suppose there are Y directed triangles. Notice that there are respectively two, zero, and one directed triangle in Figure 10-1, 10-2 and 10-3, respectively.

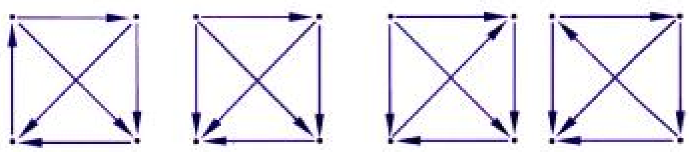

Figure 10-1 Figure 10-2 Figure 10-3

Besides, while calculating the number of the directed triangles by these three kinds of sub-graphs, each directed triangle have been counted for $(n-3)$ times. Thus $2x+z=(n-3)Y$, hence $2x\le(n-3)Y$.

Consider $n=2k+1$. Let $Y=\frac{1}{6}k(k+1)(2k+1)$, the inequality becomes

$2x\le(2k-2)[\frac{1}{6}k(k+1)(2k+1)]$, i.e. $x\le\frac{1}{6}k(k-1)(k+1)(2k+1)$.

To prove Theorem 6, we shall construct a graph for which the equality holds. Fortunately, the construction that we gave earlier for the directed triangle cases serve this purpose as well. The reason is according to the following lemma:

Lemma 4: In any graph constructed in the previous construction, each directed triangle with any other vertex can form a directed quadrilaterals. It is because as to any directed triangle (e,f,g), we choose h for the forth vertex. Then:

(1). If $h < e$ or $h > g$, we have $g \to h \to e$ (and $e \to f \to g$), so
$e \to f \to g \to h \to e$;
(2). If $e < h < f$, we have $e \to h \to f$ (and $f \to g \to e$), so
$e \to h \to f \to g \to e$;
(3). If $f < h < g$, we have $f \to h \to g$ (and $g \to e \to f$), so
$e \to f \to h \to g \to e$.

In each of the three cases above, the vertices make a directed quadrilateral! By this lemma, we have $2x \geq (n-3)Y$. As it was proved before that ($2x \leq (n-3)Y$), we conclude that $2x = (n-3)Y$. Thus,

$M=\frac{1}{6}k(k-1)(k+1)(2k+1)$ for $n = 2k+1, (k \in N)$. As to $n = 2k, (k \in N)$, the proof is the same. We omit the details. (Q.E.D.)

Remark: We may think of the question: Are there two graphs having the same out degree sequence but different numbers of quadrilaterals?
By Figure 6 and Figure 7 before, we know the answer is positive. (Take Figure 6 for example, there are respectively twelve and eight in the left and right graph. Thus we have an easy proof why the graphs are different.)
Thus, we cannot use the double counting method to determine x again.

E. Discussion on Transitive Sub-tournaments TT_m

In this section, we consider another generalized problem: the minimum number of transitive sub-tournaments TT_m in a graph T_n. First, we give a definition to the transitive sub-tournament: If any three vertices in a tournament form a transitive triangle, we call this a transitive one. Especially, if the graph is a subgraph of another, we call this a transitive sub-tournament. Denote by TT_m a transitive sub-tournament on m vertices. Obviously, TT_3 is just the transitive triangle. Theorem 3 gives that the maximum number of transitive subtournaments TT_m in a T_n is C_m^n, so we focus on the minimum. First, we prove the existence of TT_m, which is similar to what Ramsey Theorem shows in a large edge k-colored complete graph.

Theorem 7: If $n \geq 2^{m-1}$, then there exists a transitive subgraph TT_m in a graph T_n.

Proof: We prove it by induction. When $m = 3$, it is easy to see there are at least two transitive triangles in a tournament T_4. Now, suppose this theorem holds for $n = k$, then consider the condition $m = k+1$. To any vertex i in a graph T_{2^k}, either it directs to 2^{k-1} vertices or it is directed by 2^{k-1} ones at least. Without loss of generality, assume $i \to i_1, i_2, \ldots, i_{2^{k-1}}$. By the hypothesis of

$m = k$, we know there is at least a transitive subgraph TT_k in the tournament $T_{2^{k-1}}$ formed by the vertices $i_1, i_2, \ldots, i_{2^{k-1}}$ and edges among them. Combine one transitive subgraph TT_k with vertex i, we shall find a transitive sub-tournament TT_{k+1}. Therefore, the theorem holds for $m = k+1$. (Q.E.D.)

When we have proved the existence of the transitive sub-tournaments, we shall naturally consider the next problem, "What is the least number of transitive subgraphs TT_m in an n-element tournament T_n?" We have completely solved the condition for $m = 3$. It is because while the number of directed triangles is the largest, it is relative that the number of the transitive one is the least. By easy calculation, we shall obtain:

Theorem 8: The minimum of transitive triangles in an n-element tournament T_n is

(1) $C_3^n - \frac{1}{6}k(k+1)(2k+1) = \frac{1}{2}k(k-1)(2k+1)$, for $n = 2k+1, (k \in \mathrm{N})$.

(2) $C_3^n - \frac{1}{3}k(k-1)(k+1) = k(k-1)^2$, for $n = 2k, (k \in \mathrm{N})$.

For $m \geq 4$, we can obtain some upper and lower bounds by the previous results:

Theorem 9: Let *min S* be the least number of transitive sub-tournaments TT_m in a tournament T_n, then $min\ S < \frac{m}{2^{m-1}} C_m^n$;

$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathrm{N} : \forall n \geq n_\varepsilon,\ min\ S > (\frac{m!}{2^{m(m-1)/2}} - \varepsilon) C_m^n$.

Proof: The first inequality is obtained by counting the transitive sub-tournaments TT_m in the tournaments made by the constructions in the proof for Theorem 5. Besides, notice that the relation between the transitive sub-tournaments TT_m and TT_{m+1} used in the proof for Theorem 7. Then, we shall obtain the second one from Theorem 8 after some calculations.

Remark: It is not difficult to improve the upper bound- we just need a construction to replace that in Theorem 5, but I can't make the upper bound very close to the lower bound so far. The main problem is that I have not found a systematic method to count the number of TT_m for $m \geq 4$. If we consider the graph with vertices directing to the same ones relatively (Figure 9-1 is an example), the distance between upper and lower bounds may be difficult to shorten. Besides, the larger the number m is, the more difficulty we have. If we consider other kinds of graph, it is natural that we have much more trouble estimating the number of TT_m. Therefore, the direction for further study will be the deeper discussion on this topic.

Conclusion

In this project, I have completely determined

(1) the minimum number of monochromatic triangles in an edge 2-colored graph K_n (Theorem 2);

(2) the maximum number of directed triangles and the minimum number of transitive triangles in a tournament T_n (Theorem 5 & Theorem 8);

(3) the largest number of directed quadrilaterals in a tournament T_n (Theorem 6).

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3.Conclusion 4.References