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作品名稱　Crossing Number of Join Product of Some Graphs

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1 Introduction

A drawing of a graph G is a representation of G on a plane, with its vertices represented by distinct points, and its edges by arcs connecting the corresponding points. The crossing number of G is the minimum number of intersections between arcs across all possible drawings of G.

Finding the crossing number of a graph is known to be a difficult problem, with the exact values of crossing numbers known only for specific families of graphs. In particular, it has been conjectured by Zarankiewicz that the crossing number of the complete bipartite graph $K_{m,n}$ is $Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ [1], and has been proven for $min(m,n) \leq 6$[2], and cases $(m,n) = (7,7),(7,8),(7,9),(7,10),(8,8),(8,9),(8,10)$.[3] More recently, it has been shown that $\lim_{n\to\infty} \frac{cr(K_{m,n})}{Z(m,n)} \geq 0.83 \frac{m}{m-1}$. [4]

A natural extension is to investigate the crossing numbers of the join products of two graphs, which have a corresponding complete bipartite graph as a subgraph. The exact crossing numbers of $G+nK_1$, $G+P_n$ and $G+C_n$ for all graphs G of order 4 have been determined in [5], and for some graphs G of order 5 and 6, such as in [6, 7, 8, 9, 10, 11]. A more comprehensive review can be found in [12]. Many of these graphs are connected, and have a cycle going through 5 or 6 of the vertices. Notably, there have been several papers, including [7, 8, 9, 10, 11] using the idea of cyclic permutations to determine these crossing numbers.

In this report, we establish some bounds on the crossing number of the join product of two graphs, in particular, graphs G_1, G_2, where G_1 is $2C_3$ and G_2 is $2C_3$ with one edge between the cycles. We also use a counting argument to establish some inductive bounds (inducting on n) for join product of a general graph G and nK_1.

The choice of G_1 and G_2 was motivated by the result in [6] about the crossing number of $G+nK_1$, where G is $2C_3$, with two edges between the cycles (two edges are connected to distinct vertices). We noticed that removing one or both of these edges (thus getting G_1 and G_2) does not reduce the crossing number in the optimal drawing proposed. Furthermore, both G_1 and G_2 do not have a large cycle in them, and G_1 is disconnected, which is not commonly seen in the literature.

2 Definitions

The join product of two graphs G_1 and G_2, denoted by G_1+G_2, refers to the graph obtained from vertex disjoint copies of G_1 and G_2, and adding all edges between each vertex in G_1 and each vertex in G_2. In other words, $V(G_1+G_2) = V(G_1) \cup V(G_2)$, $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup K_{m,n}$, where $|V(G_1)| = m, |V(G_2)| = n$.

nK_1 is the graph of n isolated vertices with no edges.

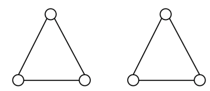
Consider some drawing D of a graph G. Let $cr_D(G)$ be the number of crossings between edges in G, and for any edge disjoint subgraphs H_1 and H_2 of G, let $cr_D(H_1, H_2)$ be the total number of crossings between an edge of H_1 and an edge of H_2.

We assume that in a drawing:
1. Each edge only passes through two vertices, namely its end points
2. No two edges touch each other (and do not cross)
3. No three edges cross at the same point

Note that in an optimal drawing of some graph G with minimum crossing number, we must also have:
1. No edge crosses itself
2. Any two edges cross at most once
3. Any two edges that share an end point do not cross

3 Graph G_1

The graph G_1 is is the union of two vertex disjoint C_3, with no edge between the cycles.

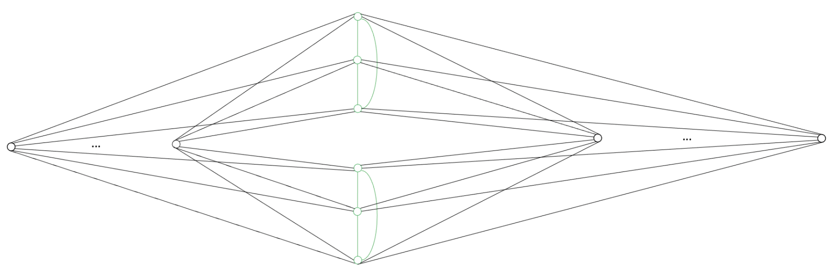


G_1

We use x_i to denote the vertices of G_1, and z_i to denote the vertices of nK_1. Let T_i be the subgraph of the six edges from vertex z_i to each vertex of G_1.

3.1 Upper bound

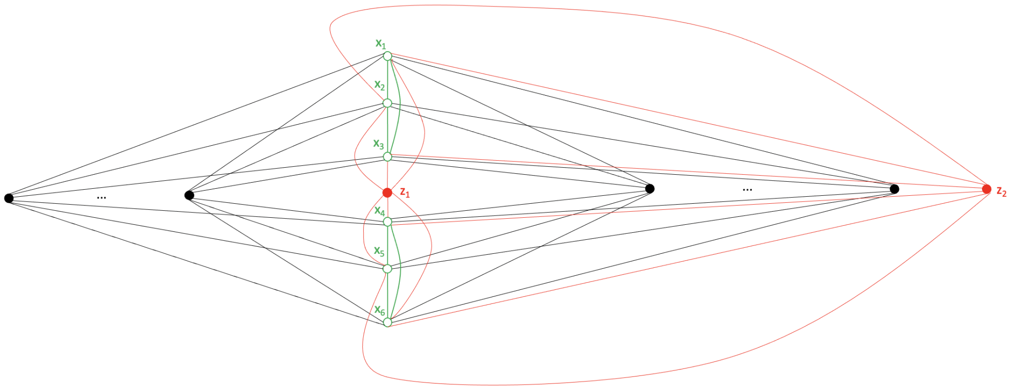
For all n, we show $cr(G_1 + nK_1) \leq Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.



Drawing of $G_1 + nK_1$

This is clear from the drawing above, where there are $\lfloor \frac{n}{2} \rfloor$ vertices of nK_1 on the right and $\lceil \frac{n}{2} \rceil$ vertices of nK_1 on the left. There are $Z(6,n)$ crossings from the bipartite graph $K_{6,n}$, and another $2\lfloor \frac{n}{2} \rfloor$ crossings on G_1.

In addition, for odd n, we show $cr(G_1 + nK_1) \leq Z(6,n) + 2\lfloor \frac{n}{2} \rfloor - 2$.



Drawing of $G_1 + nK_1$ for odd n

Consider the drawing above, where there are $\lfloor \frac{n-2}{2} \rfloor$ black vertices of nK_1 on the right and $\lceil \frac{n-2}{2} \rceil$ black vertices of nK_1 on the left. Let the vertices of G_1 from top to bottom be $x_1, \cdots, x_6$ respectively, and the red vertex in the centre and the right be z_1, z_2 respectively.

There are $Z(6, n-2) + 2\lfloor \frac{n-2}{2} \rfloor$ crossings between the edges of the black vertices.

The edges between z_1 and x_2, x_5 as well as z_2 and x_2, x_5 each cross one edge from each black vertex on the left, so they contribute $4\lceil \frac{n-2}{2} \rceil$ crossings in total.

The edges between z_1 and x_1, x_6 as well as z_2 and x_3, x_4 each cross two edges from each black vertex on the right, so they contribute $8\lfloor \frac{n-2}{2} \rfloor$ crossings in total.

The red edges cross each other twice, so for this drawing,

$$
\begin{aligned}
cr_D(G_1 + nK_1) &= Z(6, n-2) + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 4\left\lceil \frac{n-2}{2} \right\rceil + 8\left\lfloor \frac{n-2}{2} \right\rfloor + 2 \\
&= \left(6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6\left\lfloor \frac{n-2}{2} \right\rfloor\right) + \left(4\left\lfloor \frac{n-2}{2} \right\rfloor + 4\left\lceil \frac{n-2}{2} \right\rceil\right) + 2 \\
&= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-2) + 2 \\
&= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6\left\lfloor \frac{n-1}{2} \right\rfloor + n - 5 + 2 \text{ (when } n \text{ is odd)} \\
&= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor - 2 \\
&= Z(6,n) + 2\left\lfloor \frac{n}{2} \right\rfloor - 2
\end{aligned}
$$

Note that when n is even, number of crossings of this drawing is $Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$.

Thus, we propose the following crossing number.

Conjecture 3.1: $cr(G_1+nK_1) = Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor-2$ for odd n, and $cr(G_1+nK_1) = Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$ for even n.

3.2 Small cases

Lemma 3.2: $cr(G_1+2K_1)=2$

Proof:
We show $cr(G_1+2K_1)\geq 2$.

Consider one C_3 of G_1 and the two vertices z_1, z_2.

If the subgraph induced by z_1 and vertices of the C_3 has at least two crossings, we are done. Otherwise, the possible drawings of C_3 and the three edges from z_1 to the vertices of the cycle are shown below.

Drawings of one C_3 and z_i

Assume it is the first drawing, then if z_2 lies within the C_3, consider the other three vertices of G_1. For each of them, one of the two edges between them and z_1, z_2 will cross the C_3 in the drawing (depending on whether it is inside or outside the C_3), then there will be at least 3 crossings.

Otherwise, z_2 must lie in some other region, which all have at most two vertices of the C_3 on their boundary, thus the edge from z_2 to the vertex of C_3 not on the boundary will have at least 1 crossing.

If it is the second drawing, there will be at least 1 crossing in this subgraph.

Similarly, we can consider the other C_3 of G_1, and either we get $cr_D(G_1+2K_1)\geq 3$, or there is at least 1 crossing in that subgraph, so $cr_D(G_1+2K_1)\geq 2$.

Thus $cr(G_1+2K_1)\geq 2$.

From the construction earlier, $cr(G_1+2K_1)\leq 2$, and so $cr(G_1+2K_1)=2$. $\square$

Lemma 3.3: $cr(G_1 + 3K_1) = 6$

Proof:
Since $K(6,3)$ is a subgraph of G_1+3K_1, $cr(G_1+3K_1) \geq cr(K(6,3)) = 6$. From our construction above, $cr(G_1 + 3K_1) \leq 6$, and so $cr(G_1 + 3K_1) = 6$. □

We have also obtained lower bounds for $n = 4, 5, 6, 7, 8$ in Section 3.4, using the properties of cyclic permutations.

3.3 Results

We suppose $cr(G_1 + nK_1) < Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ for some even $n \geq 4$, and $cr(G_1 + nK_1) < Z(6,n) + 2\lfloor \frac{n}{2} \rfloor - 2$ for some odd $n \geq 5$.

Lemma 3.4: There exists i such that $cr_D(T_i, G_1) = 0$

Proof:
Suppose otherwise. Then $cr_D(T_i, G_1) \geq 1$ for all i.

$$\begin{aligned} cr_D\left(G + nK_1\right) &= cr_D\left(\bigcup_{i=1}^{n} T_i\right) + cr_D\left(G, \bigcup_{i=1}^{n} T_i\right) + cr_D\left(G\right) \\ &\geq Z(6,n) + n \\ &\geq Z(6,n) + 2\left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

which is a contradiction. □

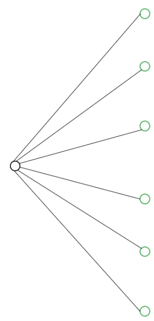
Lemma 3.5: $cr_D(G_1) = 0$

Proof:
Consider i such that $cr_D(T_i, G_1) = 0$.

We can draw T_i as below, and let the vertices of G_1 be $x_1, x_2, x_3, x_4, x_5, x_6$ from top to bottom respectively.

Consider each grouping of the 6 vertices of G_1 into two triples, with each triple of vertices forming one cycle.

For all drawings except $(1,2,3), (4,5,6)$ and $(1,2,6), (3,4,5)/(1,5,6), (2,3,4)$, we have $cr_D(G_1) \geq 1$, and each region has at most two vertices of G_1 on its boundary, so $cr_D(T_j, T_i \cup G_1) \geq 4$ for all $j \neq i$.



Drawing of T_i

$$\begin{aligned}
cr_D(G_1+nK_1) &= cr_D\left(G_1 \cup T_i\right) + cr_D\left(G_1 \cup T_i, \bigcup_{j\neq i} T_j\right) + cr_D\left(\bigcup_{j\neq i} T_j\right)\\
&\geq 1+4(n-1)+Z(6,n-1)\\
&= n+3(n-1)+Z(6,n-1)\\
&\geq Z(6,n)+n\\
&\geq Z(6,n)+2\left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}$$

which is a contradiction, so the drawing is either $(1,2,3),(4,5,6)$ or $(1,2,6),(3,4,5)/(1,5,6),(2,3,4)$. Notice these drawings are the same, so we can assume the drawing is $(1,2,3),(4,5,6)$, and as a result, $cr_D(G_1) = 0$. □

3.4 Cyclic permutations

We now use the properties of cyclic permutations, which have been used in [7] [8], to obtain some more results.

Let $rot_D(T_i)$ be the clockwise order in which the edges leave vertex z_i to the vertices of G_1. Cyclic permutations are considered to be the same, and so we assume all $rot_D(T_i)$ start from x_1.

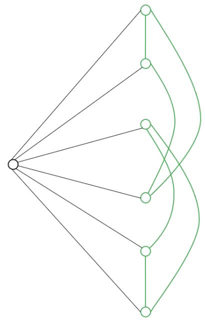
Define $\overline{rot_A}$ as the reverse permutation of rot_A, and $d(rot_A, rot_B)$ to be the minimum number of swaps between adjacent elements, to get from rot_A to rot_B.

It is known that $cr(T_i, T_j) \geq d(rot_D(T_i), \overline{rot_D(T_j)})$. [3]

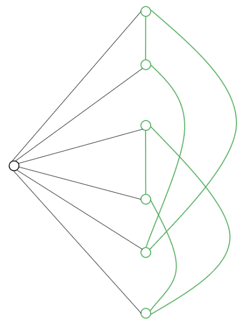
We now establish the possible permutations of T_i for i such that $cr(T_i, G_1) \leq 1$.

Assuming the drawing of G_1 above, z_i must be in the region with all 6 vertices of G_1 on its boundary (in view of the subdrawing of G_1), otherwise if it is inside one of the cycles, say $x_1x_2x_3$, then z_ix_4, z_ix_5, z_ix_6 will each cross the cycle $x_1x_2x_3$ at least once.

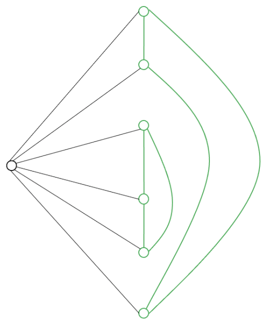
Thus $rot_D(T_i)$ with respect to each cycle is fixed (up to rotation).



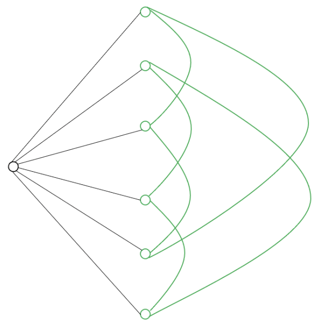
$(1, 2, 4), (3, 5, 6)$



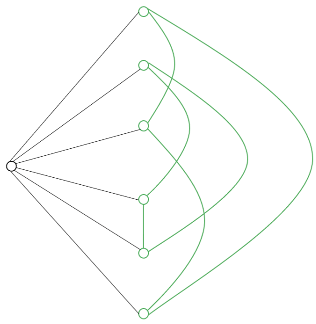
$(1, 2, 5), (3, 4, 6)$ and $(1, 3, 4), (2, 5, 6)$



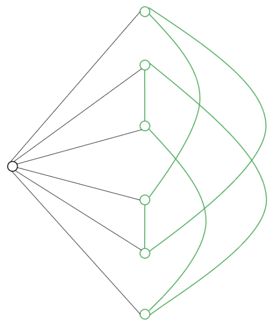
$(1, 2, 6), (3, 4, 5)$ and $(1, 5, 6), (2, 3, 4)$



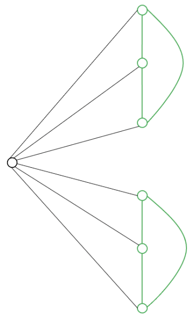
$(1, 3, 5), (2, 4, 6)$



$(1, 3, 6), (2, 4, 5)$ and $(1, 4, 6), (2, 3, 5)$



$(1, 4, 5), (2, 3, 6)$



$(1, 2, 3), (4, 5, 6)$

Figure 1: Drawings of $T_i \cup G_1$

Consider the subgraph induced by z_i and the cycle $x_1x_2x_3$.

If any 2 of x_4, x_5, x_6 are in different regions in view of this subgraph, the edge between them must cross some edge in this subgraph. From Lemma 3.5, it cannot cross an edge of G_1, so it must cross an edge between z_i and one of x_1, x_2, x_3.

If x_4, x_5, x_6 are not all in the same region, there must be at least 2 pairs of them in different regions, and so $cr(T_i, G_1) \geq 2$, which is a contradiction, thus they must all be in the same region.

For each $j = 4, 5, 6$, the edge between x_j and z_i must start in the same region as x_4, x_5, x_6. Otherwise, since the edge must leave the region it started in, and cannot cross any edge with endpoint z_i, it must cross one of the edges of the cycle $x_1x_2x_3$. This means the edge enters the region inside the cycle $x_1x_2x_3$, but x_j is outside this cycle, so the edge must cross the boundary of the region again, thus this edge crosses G_1 twice, which is a contradiction. (Note that by a similar argument, the entire edge must be contained in this region)

Similarly, each edge of the cycle $x_4x_5x_6$ must be fully contained within this region, otherwise since it must leave and enter the region, and it cannot cross an edge of G_1, it must cross the edges of T_i at least twice.

There are 3 ways to choose which region x_4, x_5, x_6 are in, and 3 ways to permute x_4, x_5, x_6. (rotation matters here, for example 123456 compared to 123564)

We have 9 possible values for $rot_D(T_i)$, namely $123456, 123564, 123645, 124563, 125643, 126453, 145623, 156423, 164523$. We label them $P_1, P_2, \cdots, P_9$ respectively.

Using a program (can be found in Appendix), we obtain the following table of values for each $d(P_i, \overline{P_j})$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
P_1	6	4	4	4	2	2	4	2	2
P_2	4	6	4	2	4	2	2	4	2
P_3	4	4	6	2	2	4	2	2	4
P_4	4	2	2	6	4	4	4	2	2
P_5	2	4	2	4	6	4	2	4	2
P_6	2	2	4	4	4	6	2	2	4
P_7	4	2	2	4	2	2	6	4	4
P_8	2	4	2	2	4	2	4	6	4
P_9	2	2	4	2	2	4	4	4	6

Lemma 3.6: $cr(G_1 + 4K_1) \geq 14$

Proof:

Case 1: For all $i = 1, 2, 3, 4$, $cr_D(T_i, G_1) \leq 1$.
Thus $rot_D(T_i)$ is one of P_j, and from table above, we can check that for any 4 P_j, the sum of their pairwise distances is at least 16, which means $cr_D(T_1 \cup T_2 \cup T_3 \cup T_4) \geq 16$.

Case 2: There exists i such that $cr_D(T_i, G_1) \geq 2$.
Then we have $cr(G_1 + 4K_1) \geq Z(6, 4) + 2 = 14$. □

Lemma 3.7: $cr(G_1 + 5K_1) = 26$

Proof:

Case 1: For all $i = 1, 2, 3, 4$, $cr_D(T_i, G_1) \leq 1$.
Thus $rot_D(T_i)$ is one of P_j, and from table above, we can check that for any 5 P_j, the sum of their pairwise distances is at least 28, which means $cr_D(T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5) \geq 28$.

Case 2: There exists i such that $cr_D(T_i, G_1) \geq 2$.
Then we have $cr(G_1 + 5K_1) \geq Z(6, 5) + 2 = 26$.

From the two cases, we have $cr(G_1 + 5K_1) \geq 26$, and from our construction in section 3.1, $cr(G_1 + 5K_1) \leq 26$, so $cr(G_1 + 5K_1) = 26$. □

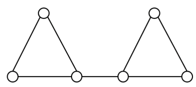
As a result of Lemma 3.7, we have the following result.

Lemma 3.8: $cr(G_1 + 6K_1) \geq 39, cr(G_1 + 7K_1) \geq 55, cr(G_1 + 8K_1) \geq 74$

Proof:
This follows directly from Lemma 3.7 and Lemma 5.1 below. □

4 Graph G_2

The graph G_2 is the union of two vertex disjoint C_3, and with one edge between the cycles.

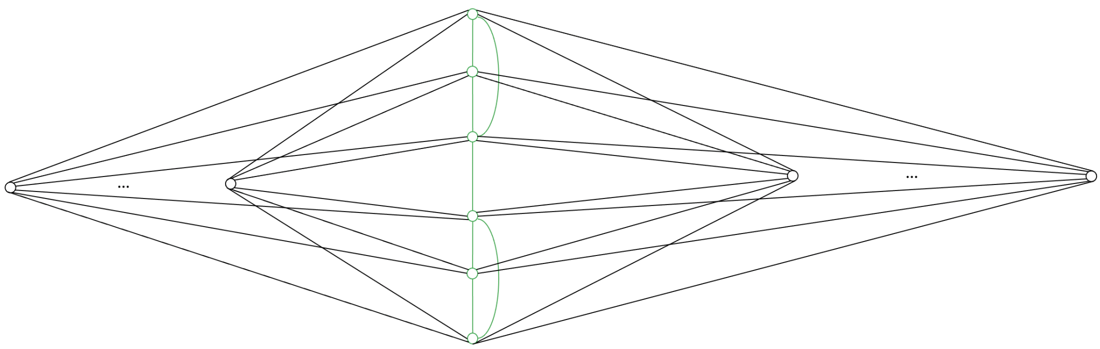


G_2

We denote the vertices similarly.

4.1 Upper bound

For all n, we show $cr(G_2 + nK_1) \leq Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.



Drawing of $G_2 + nK_1$

This is the same drawing as for $G_1 + nK_1$, but with one more edge in the centre that does not result in any additional crossings.

Since $G_1 + nK_1$ is a subgraph of $G_2 + nK_1$, we have $cr(G_2 + nK_1) \geq cr(G_1 + nK_1)$.

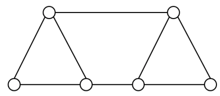
4.2 Small case

Lemma 4.1: $cr(G_2 + 2K_1) = 2$

Proof:
From the construction, $cr(G_2 + 2K_1) \leq 2$, and $cr(G_2 + 2K_1) \geq cr(G_1 + 2K_1) = 2$. □

4.3 Results

Consider the graph G_3, which is G_2 but with one additional edge between the cycles, forming a C_6.



G_3

It is known that $cr(G_3 + nK_1) = Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.[6]

Lemma 4.2: If $cr(G_2 + nK_1) < Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$, then the two cycles of G_2 do not intersect.

Proof:
This follows from Lemma 3.5, since G_1 is a subgraph of G_2, and we suppose $cr(G_2 + nK_1) < Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$ here as well. □

Theorem 4.3: $cr(G_2 + nK_1) \geq Z(6, n) + \lfloor \frac{n}{2} \rfloor$

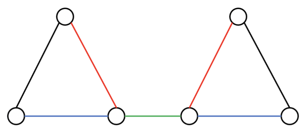
Proof:
Suppose otherwise.
WLOG let the edge between the cycles be between vertices x_3 and x_4, and let x_3 be adjacent to x_1, x_2 and x_4 be adjacent to x_5, x_6.

We add an additional edge, by starting from vertex x_1, and tracing along the edge x_1x_3, x_3x_4, then along the edge from x_4 to either x_5 or x_6, only crossing edges that these three edges cross. We thus have a drawing of the graph $G_3 + nK_1$.

Notice this new edge has at most as many crossings as G, and $cr_D(G) + cr_D(G, \cup T_i) < \lfloor \frac{n}{2} \rfloor$, so we have $cr_D(G_3+nK_1) < Z(6, n)+\lfloor \frac{n}{2} \rfloor+\lfloor \frac{n}{2} \rfloor = Z(6, n)+2\lfloor \frac{n}{2} \rfloor$, which is a contradiction. □

Lemma 4.4: If $cr(G_1 + nK_1) \geq Z(6, n) + x$, then $cr(G_2 + nK_1) \geq Z(6, n) + \lfloor \frac{n}{2} \rfloor + \frac{x}{4}$.

Proof:
Suppose otherwise.



Tracing edges of G_2

Similarly, let the edge between the cycles be between vertices x_3 and x_4, and let x_3 be adjacent to x_1, x_2 and x_4 be adjacent to x_5, x_6.

Let $rot_G(x_i)$ be the clockwise order in which the edges leave vertex x_i to the other vertices of G_2, and cyclic permutations are considered to be the same. Let $rot_G(x_3) = 124$ and $rot_G(x_4) = 356$.

Let the number of crossings on the edge between the two cycles in G_2 be a. $a \leq \lfloor \frac{n}{2} \rfloor + \frac{x}{4} - x$, otherwise we can remove this edge and get $cr_D(G_1 + nK_1) < Z(6, n) + x$.

After removing the edge between the cycles, there are at most $\lfloor \frac{n}{2} \rfloor + \frac{x}{4} - a$ crossings on G_2, and the remaining edges of G_2 do not cross each other from Lemma 4.2. Consider the edges x_1x_3, x_4x_6 and the edges x_2x_3, x_4x_6, which in the diagram are the blue/red edges. One of these pairs have at most $\frac{\lfloor \frac{n}{2} \rfloor + \frac{x}{4} - a}{2}$ edges on them, say x_1x_3, x_4x_6.

By adding an additional edge from x_1 to x_6, along this pair of edges and x_3x_4, we get at most

$$\begin{aligned} \frac{\lfloor \frac{n}{2} \rfloor + \frac{x}{4} - a}{2} + a &= \frac{\lfloor \frac{n}{2} \rfloor + \frac{x}{4} + a}{2} \\ &\leq \frac{2\lfloor \frac{n}{2} \rfloor - \frac{x}{2}}{2} \\ &= \lfloor \frac{n}{2} \rfloor - \frac{x}{4} \end{aligned}$$

more edges, so in total, $cr_D(G_3 + nK_1) < Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$, which is a contradiction. □

We now obtain some slightly improved lower bounds for small n.

Lemma 4.5: $cr(G_2 + nK_1) \geq Z(6, n) + \lfloor \frac{n}{2} \rfloor + 1$ for $n = 4, 5, 6, 7, 8$.

Proof:
This follows from Lemma 4.4, and the results in Section 3.4. □

4.4 Cyclic Permutations

Similar to Section 3.4 above, we establish possible permutations of $rot(T_i)$ for i such that $cr(T_i, G_2) \leq 1$, supposing that $cr(G_2 + nK_1) < Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.

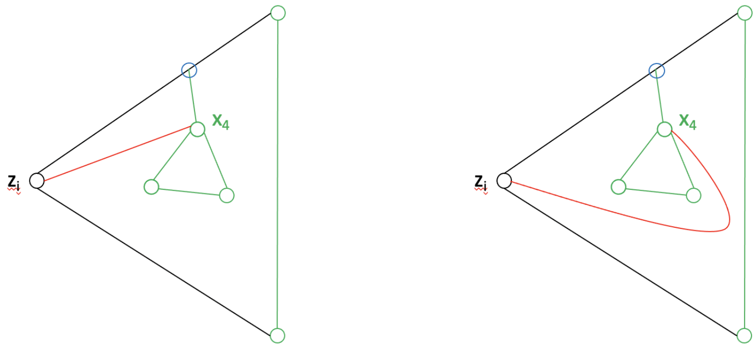
We follow the arguments of Section 3.4, since $G_1 + nK_1$ is a subgraph of $G_2 + nK_1$, and so we have also supposed here that $cr(G_1 + nK_1) < Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.

Consider the region, in the view of the subdrawing of the subgraph induced by z_i, x_1, x_2, x_3, where the vertices x_4, x_5, x_6 are.

WLOG assume that there is an edge between x_4 and x_1. By adding the edges x_4x_1 and z_ix_4, the region will be split into two regions. Note that the point where x_1x_4 crosses the boundary of the region (the point circled in blue in the diagrams) can be on any edge of the boundary, and also possibly the vertices of G_2 on the boundary.

We consider two cases.

Case 1: x_1x_4 does not cross any edge of the cycle $x_4x_5x_6$
If $cr_D(T_i, G_2) = 0$, x_5, x_6 must be in the same region (between the two regions created by the addition of x_4x_1 and z_ix_4), otherwise x_5x_6 either crosses x_4x_1 or z_ix_4.



Tracing edges of G_2

Depending on which region x_5, x_6 are in, there is only one possible permutation for the order in which the edges z_ix_4, z_ix_5, z_ix_6 leave z_i, namely 456 and 564 respectively.

Case 2: x_1x_4 crosses an edge of the cycle $x_4x_5x_6$
x_1x_4 cannot cross x_4x_5 or x_4x_6, so it must cross x_5x_6 (once).
If $cr_D(T_i, G_2) = 0$, x_5, x_6 must be in different regions (two regions which region that x_4, x_5, x_6 are in the view of subdrawing induced by z_i, x_1, x_2, x_3 has been divided into by the addition of x_4x_1 and z_ix_4), and so there is only one possible permutation for the order in which the edges z_ix_4, z_ix_5, z_ix_6 leave z_i, namely 645.

We attempt to obtain some restrictions on the drawing, if $cr(G_2 + 3K_1) < 8$.

We must have $cr_D(T_i, G_2) \leq 1$ for all i.

For case 2 above, if $cr_D(T_i, G_2) = 0$ for all i, then $rot_D(T_i)$ must be one of P_3, P_6, P_9 in the table above and we can check that for any 3 of them, the sum of their pairwise distances is at least 12 (code used can be found in Appendix). Thus $cr_D(T_1 \cup T_2 \cup T_3) \geq 12$.

Otherwise, there exists i with $cr_D(T_i, G_2) = 1$, and $cr_D(G_2) \geq 1$, so $cr_D(G_2 + 3K_1) \geq 8$.

For case 1 above, if $cr_D(T_i, G_2) = 0$ for all i, then $rot_D(T_i)$ must be one of $P_1, P_2, P_4, P_5, P_7, P_8$ in the table above and we can check that for any 3 of them, the sum of their pairwise distances is at least 8. Thus $cr_D(T_1 \cup T_2 \cup T_3) \geq 8$.

Otherwise, there exists i with $cr_D(T_i, G_2) = 1$, then $cr_D(G_2) = 0$, and so x_4 must be one of the vertices on the boundary of the region that x_4, x_5, x_6 is in (in the view of subdrawing induced by z_i, x_1, x_2, x_3), and the edge x_1x_4 is fully contained within this region.

$rot_D(T_i)$ cannot be one of $P_1, P_2, P_4, P_5, P_7, P_8$, otherwise we can follow a similar argument as above, and so $rot_D(T_i)$ is one of P_3, P_6, P_9.

5 Counting Argument

Consider some graph G of order 6, and suppose we know $cr(G + nK_1) \geq Z(6, n) + x$ for some n and $x > 0$.

Consider some drawing D of $G + (n + a)K_1$ where $a > 0$, with $cr_D(G) = m$. We want to find a lower bound for $cr_D(G + (n + a)K_1)$, so we let $cr_D(G, \bigcup_{i=1}^{n+a} T_i) = k$, and suppose $cr_D(G + (n + a)K_1) \leq Z(6, n + a) + m + b$, which means $k \leq b$.

We sum the crossings across all subgraphs $G + nK_1$, and the total is at least $\binom{n+a}{n}(Z(6, n) + x)$.

Each crossing between two edges of G are counted $\binom{n+a}{n}$ times. Each crossing between an edge of G and an edge of T_i is counted $\binom{n+a-1}{n-1}$ times. Each crossing between and edge of T_i and edge of T_j is counted $\binom{n+a-2}{n-2}$ times.

Thus we have

$$\begin{aligned} cr_D\left(G + (n + a)K_1\right) &= cr_D\left(\bigcup_{i=1}^{n+a} T_i\right) + cr_D\left(G, \bigcup_{i=1}^{n+a} T_i\right) + cr_D\left(G\right) \\ &= cr_D\left(\bigcup_{i=1}^{n+a} T_i\right) + k + m \\ &\geq \frac{1}{\binom{n+a-2}{n-2}}\left(\binom{n+a}{n}(Z(6, n) + x) - \binom{n+a}{n}m - \binom{n+a-1}{n-1}k\right) + k + m \end{aligned}$$

Lemma 5.1: If $cr(G_1 + nK_1) \geq Z(6, n) + x$, then $cr(G_1 + (n + 1)K_1) \geq Z(6, n) + x - 2$ when n is even, and $cr(G_1 + (n + 1)K_1) \geq Z(6, n) + x + 1$ when n is odd, assuming the crossing number is less than conjectured for even and odd n respectively.

Proof:
Suppose otherwise.

From Lemma 3.5, we have $cr(G_1) = 0$. Putting in $m = 0$ and $a = 1$, we get

$$\begin{aligned} cr_D\left(G_1 + (n + 1)K_1\right) &\geq \frac{1}{n-1}\left((n + 1)(Z(6, n) + x) - nk\right) + k \\ &= \frac{(n + 1)(Z(6, n) + x) - k}{n - 1} \end{aligned}$$

For even n, when $k \leq b < x - 2$ then

$$\begin{aligned}
cr_D\left(G_1 + (n+1)K_1\right) &\geq \frac{(n+1)(Z(6,n)+x)-k}{n-1} \\
&\geq \frac{(n+1)\left(\frac{3n(n-2)}{2}+x\right)+3-x}{n-1} \\
&= \frac{\frac{1}{2}(3n^3-3n^2-6n)+nx+x+3-x}{n-1} \\
&= \frac{3n^2}{2}+\frac{(x-3)n+3}{n-1} \\
&= \frac{3n^2}{2}+(x-3)+\frac{x-3+3}{n-1} \\
&> \frac{3n^2}{2}+(x-3) \text{ since } x>0 \\
&= Z(6,n+1)+x-3 \\
&\geq Z(6,n+1)+b
\end{aligned}$$

This is a contradiction, so $k \geq x-2$.

For odd n, when $k \leq b < x+1$ then

$$\begin{aligned}
cr_D\left(G_1 + (n+1)K_1\right) &\geq \frac{(n+1)(Z(6,n)+x)-k}{n-1} \\
&> \frac{(n+1)\left(\frac{3(n-1)^2}{2}+x\right)-x)}{n-1} \\
&= \frac{3(n+1)(n-1)}{2}+\frac{nx+x-x}{n-1} \\
&= \frac{3(n+1)(n-1)}{2}+\frac{nx}{n-1} \\
&= \frac{3(n+1)(n-1)}{2}+x+\frac{x}{n-1} \\
&> \frac{3(n+1)(n-1)}{2}+x \text{ since } x>0 \\
&= Z(6,n+1)+x \\
&\geq Z(6,n+1)+b
\end{aligned}$$

This is a contradiction, so $k \geq x+1$.

Note that this implies if Conjecture 3.1 holds for some n which is even, it holds for $n+1$.

6 Conclusion

We have used various methods to obtain lower and upper bounds for the crossing numbers of $G_1 + nK_1$ and $G_2 + nK_1$. Some of the methods could potentially be used for other families of graphs, particularly the double counting argument in Section 5, and the tracing argument in Lemma 4.4. Interestingly, we have also found two different optimal drawings for $G_1 + nK_1$ depending on the parity of n.

Appendix A Code for Cyclic Permutations

The following code was used to generate the table of distances for the permutations in Section 3.4 and Section 4.4, as well as to find the minimum pairwise sum of distances for some sets of permutations.

The adjacency matrix between permutations was found by iterating through each permutations and doing all possible swaps. The Floyd-Warshall algorithm is then used to find all pairs shortest paths.

```
#include <bits/stdc++.h>
using namespace std;

string reverse(string s){
  string ans = "1";
  return ans+s[5]+s[4]+s[3]+s[2]+s[1];
}

int main(){


  freopen("input.txt", "r", stdin);
  freopen("output.txt", "w", stdout);

  string s = "23456";
    sort(s.begin(), s.end());

    //permutations as strings
    string permutations [120];

    permutations[0] = "1"+s;

    int count = 1;

    while(next_permutation(s.begin(), s.end())){
      permutations[count] = "1"+s;
      count++;
    }

    //permutations index
    map<string, int> m;


  for(int i=0;i<120;i++)
    m[permutations[i]] = i;

  //initialise distances
  int distance[120][120];

  for(int i=0;i<120;i++){
    for(int j=0;j<120;j++){
      if(i!=j)distance[i][j] = 1000000;
      else distance[i][j] = 0;
    }
  }

  //find adjacency matrix
  for(int i=0;i<120;i++){
    for(int j=0;j<4;j++){
      s = permutations[i];
```

```
      swap(s[j+1],s[j+2]);
      distance[i][m[s]] = 1;
    }

    s = "1";
    s = s+permutations[i][2]+permutations[i][3]+permutations[i][4]+permutations[i][5]+permutations[i][1];
    distance[i][m[s]] = 1;

    s = "1";
    s = s+permutations[i][5]+permutations[i][1]+permutations[i][2]+permutations[i][3]+permutations[i][4];
    distance[i][m[s]] = 1;
  }

  //find all pair shortest path

  for(int k=0;k<120;k++){
    for(int i=0;i<120;i++){
      for(int j=0;j<120;j++){
        distance[i][j] = min(distance[i][j],distance[i][k]+distance[k][j]);

      }
    }
  }

  //possible permutations
  int index[9];
  index[0] = m["123456"];
  index[1] = m["123564"];
  index[2] = m["123645"];
  index[3] = m["124563"];
  index[4] = m["125643"];
  index[5] = m["126453"];
  index[6] = m["145623"];
  index[7] = m["156423"];
  index[8] = m["164523"];


  //print table
  int table[9][9];

  for(int i=0;i<9;i++){
    for(int j=0;j<9;j++){
      table[i][j] = distance[index[i]][m[reverse(permutations[index[j]])]];
      cout<<table[i][j]<<" ";
    }
    cout<<"\n";
  }


  //find minimum total of pairwise distance for 4 permutations
  int four = 1000000;


    for(int i=0;i<9;i++){
    for(int j=i;j<9;j++){
      for(int k=j;k<9;k++){
        for(int l=k;l<9;l++){
          for(int z=l;z<9;z++){
            four = min(four, table[i][j]+table[i][k]+table[i][l]+table[j][k]+table[j][l]+table[k][l]+table[z][i]+table[z][j]+table[z][k]+table[z][l]);
```

```
          }
        }
      }
    }
  }

  cout<<four<<"\n";


  //find minimum total of pairwise distance for 5 permutations
  int five = 1000000;


    for(int i=0;i<9;i++){
    for(int j=i;j<9;j++){
      for(int k=j;k<9;k++){
        for(int l=k;l<9;l++){
          for(int z=l;z<9;z++){
            for(int q=z;q<9;q++){
              five= min(five, table[i][j]+table[i][k]+table[i][l]+table[j][k]+table[j][l]+table[k][l]+table[z][i]+table[z][j]+table[z][k]+table[z][l]+table[q][i]+table[q][j]+table[q][k]+table[q][l]+table[q][z]);
            }
          }
        }
      }
    }
  }

  cout<<five<<"\n";


  //new index for G2

  int index1[9];
  index1[0] = m["123456"];
  index1[1] = m["123564"];
  index1[2] = m["124563"];
  index1[3] = m["125643"];
  index1[4] = m["145623"];
  index1[5] = m["156423"];
  index1[6] = m["123645"];
  index1[7] = m["126453"];
  index1[8] = m["164523"];


  //print table
  int table1[9][9];

  for(int i=0;i<9;i++){
    for(int j=0;j<9;j++){
      table1[i][j] = distance[index1[i]][m[reverse(permutations[index1[j]])]];
      cout<<table1[i][j]<<" ";
    }
    cout<<"\n";
  }


  //find minimum total of pairwise distance for 3 permutations among restricted set of 6 permutations
  int three1 = 1000000;

```

```
    for(int i=0;i<6;i++){
    for(int j=i;j<6;j++){
      for(int k=j;k<6;k++){
        three1 = min(three1, table1[i][j]+table1[i][k]+table1[j][k]);
      }
    }
  }

  cout<<three1<<"\n";




  //find minimum total of pairwise distance for 3 permutations among restricted
    set of 3 permutations
  three1 = 1000000;


    for(int i=6;i<9;i++){
    for(int j=i;j<9;j++){
      for(int k=j;k<9;k++){
        three1 = min(three1, table1[i][j]+table1[i][k]+table1[j][k]);
      }
    }
  }

  cout<<three1<<"\n";
}
```

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The author considers a problem in graph theory related to a conjecture by Zarankiewicz，i.e.，finding the crossing number for a family of special graphs. The author has an organized presentation style and knows the material inside out. The writing is clean. Several upper and lower bounds are obtained.