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作品名稱 **Two Games on Graphs Extended from the Game of Squayles**

得獎獎項 四等獎

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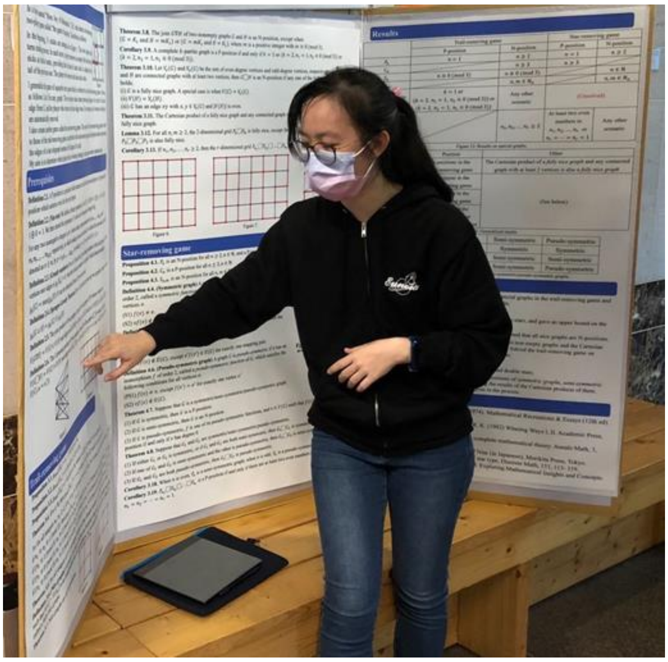
關鍵詞 <u>**Combinatorial game theory、Grundy numbers、graphs**</u>

# 作者簡介




我是游星閔，目前就讀北一女中三年級。我從小就對數學很有興趣，並以數學研究為志向；除了最愛的數學外，我也喜歡聽音樂、運動、看書，以及玩各種遊戲。前不久在書上看到一個很有意思的數學遊戲，我覺得非常有趣，便將其一般化為在任意圖上玩的遊戲，並自創變體規則，研究在不同圖上的必勝玩家，並有幸得到教授的幫忙、指點，讓這篇作品變得更加成熟。

感謝專題指導老師、教授、父母親，以及同學們的鼓勵、建議、教導，讓我的專題研究作品得以更加完善！

**摘要**

本研究是關於 nim 遊戲的兩種推廣（其中一種是一個稱為 the game of squayles 的遊戲的推廣），稱為 edge-removing game 和 star-removing game。此遊戲為兩人遊戲。在遊戲的一開始，有一個簡單圖 *G*。兩個玩家輪流刪除該圖的非空路徑或非空星子圖的邊。首先不能移動的一方輸掉遊戲。

在 edge-removing game 中，我成功計算出某些特殊圖的 Grundy numbers，並給出了一般 *k* 星的 Grundy numbers 上界。接著我定義了一種新的圖，稱為 nice graphs，並發現所有 nice graphs 都是 N-position。我由此給出了任意兩個非空圖的 join product 的解。至於圖的 Cartesian product，我給出了兩個滿足一定條件的非空圖的 Cartesian product 的解，並發現一個 fully nice graph 和任何至少有 2 個頂點的連通圖的 Cartesian product 也是 fully nice 的。使用這個性質，我給出了 *r*-dimensional grids 上的 edge-removing game 的解。

至於 star-removing game，我最大的突破是構思出對稱性這個概念。使用這個概念，我給出更一般化的結論，可以用來有效分析某些圖的 Cartesian product 上的的 star-removing game。使用這些結果，我給出了 *r*-dimensional grids 的解。

**Abstract**

This research studies two generalizations of the nim game, called the trail-removing game and the star-removing game. There are two players in the game. At the beginning, there is a simple graph *G*. The two players take turns removing the edges of a trail of at least one edge or a nonempty star subgraph of the graph. The one who cannot move loses the game and the other wins.

In the edge-removing game, I first compute the Grundy numbers of special graphs and give an upper bound on *k*-stars in general. I then define a new kind of graphs known as nice graphs and determine that all nice graphs are N-positions. Using this, I give a solution of the join product of any two non-empty graphs, solving the trail-removing game on complete *k*-partite graphs in the process.

As for the Cartesian product of graphs, I give a solution of the Cartesian product of two non-empty graphs that satisfy certain conditions and discover that the Cartesian product of a fully nice graph and any other connected graph with at least 2 vertices is also fully nice. By this, I am able to solve the trail-removing game on *r*-dimensional grids.

As for the star-removing game, my greatest achievement is the introduction of a concept known as symmetry. Using this concept, I am able to give more generalized results that can be used to analyze the star-removing game on Cartesian products of certain graphs effectively. Using these results, I am able to determine the winner on *r*-dimensional grids.

# 1 Study background

In a book named "More Joy of Mathematics," [6] there is an interesting two-playergame called "the game of squayles," described as follows.

At the beginning, 31 sticks are arranged as in Figure 1.

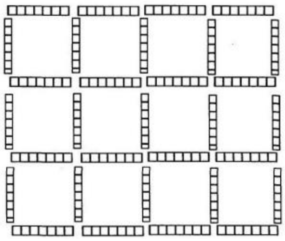


Figure 1: The arrangement of 31 sticks.

The game has two players: Alice and Bob. They take turns (Alice goes first) making moves as follows. In each move, a player removes as many (but at least one) sticks as she/he wants, providing that the head of one stick is adjacent to the tail of the previous one in sequence as numbered in the two examples in Figure 2; while the two examples in Figure 3 are illegal. The player unable to make a move loses the game, and the other wins. In other words, the player who removes the last stick wins.

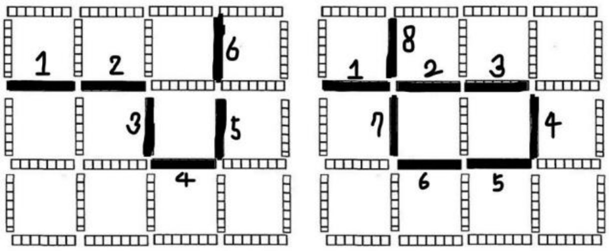



Figure 2: Two legal moves, where the sticks marked red are removed.

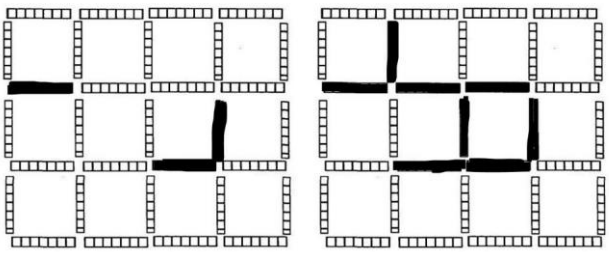


Figure 3: Two illegal moves, where the sticks marked red are removed.

This article generalizes the game of squayles into a game which is referred to as the trail-removing game described below.

Let $G$ be any graph. The two players, Alice and Bob, take turns removing the edges of a trail (a walk without repeating edges) of at least one edge from $G$. Also, for the sake of convenience, if a vertex has degree 0, this vertex is automatically removed. The player who takes away the last edge wins. Notice that the game of squayles is equivalent to the trail-removing game on the graph in Figure 4.

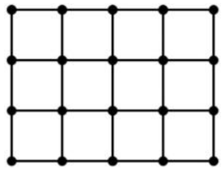


Figure 4: The graph on which the game of squayles is played.

This article also introduces another game called the star-removing game. The rules of the star-removing game are similar to those of the trail-removing game, except that in the star-removing game the players take turns removing the edges of a star subgraph instead of the edges of a trail.

The aim of this article is to determine which player has a winning strategy in these games under optimal play, i.e. when both players play the game perfectly.

# 2 Preliminaries

Combinatorial game theory is a branch of mathematics that studies certain games with perfect information. These games are typically two-player games that have a *position* the players take turns changing in defined ways or *moves* to achieve a defined winning condition.

We first give a quick review on some basic definitions, concepts and theorems in combinatorial game theory.

First, for the sake of convenience, a *game* in later paragraphs refers to a finite two-player game with perfect information, and any move available to one player must be available to the other as well. It is easy to see that the trail-removing game, the star-removing game and the matching-removing game are all such games.

Next are some terms in combinatorial game theory.

**Definition 2.1.** *A* P-position *is a position which secures a win for the previous player (the player who has just moved). An* N-position *is a position which secures a win for the next player (the player who is going to move). A* terminal position *isa position in which the following player has no legal moves.*

The nim game is one of the most classic combinatorial games in history. In the game, two players take turns removing stones from distinct piles of stones. On each turn, a player must remove *at least one stone*, and may remove any number of stones provided they all come from *the same pile*. The player who takes the last stone wins.

The nim game is specified by the numbers of stones in the piles. We use $N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})$ to denote the *position* with $r$ piles of $n^{(1)}, n^{(2)}, \ldots, n^{(r)}$ stones, where any number equal to 0 can be ignored. For convenience, $N(0)$ denotes the position of no stones.

The nim game is completely solved by Bouton [3] in 1901, who established the foundation on this line. To describe his result, we need some notations.

Fist, for any nonnegative integer $n$, we may write it as a *binary representation*

$$n = n_r 2^r + n_{r-1} 2^{r-1} + \ldots + n_0 2^0 \text{ which is denoted by } n_r n_{r-1} \ldots n_{0(2)} \text{ for short,}$$

where $r \geq 0$ and each $n_i \in \{0, 1\}$. In this notation, we allow leading zero(s). Forinstance, $5 = 101_{(2)} = 0101_{(2)} = 000101_{(2)}$.

Next, we define a binary operation on $\{0, 1\}$ by

$$0 \oplus 0 = 1 \oplus 1 = 0 \text{ and } 0 \oplus 1 = 1 \oplus 0 = 1.$$

Then extend it to the set of nonnegative integers $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ as follows. Let $n = n_r n_{r-1} \ldots n_{0(2)}$ and $m = m_r m_{r-1} \ldots m_{0(2)}$. Then the *nim sum* of $n$ and $m$, denoted as $n \oplus m$, is defined to be

$$p = p_r p_{r-1} \ldots p_{0(2)} \text{ with } p_i = n_i \oplus m_i \text{ for } r \geq i \geq 0.$$

Notice that $(\mathbb{N}_0, \oplus)$ is an abelian group. Namely the following conditions hold.

(1) The operation $\oplus$ is commutative, i.e., $n \oplus m = m \oplus n$ for $n, m \in \mathbb{N}_0$.
(2) The operation $\oplus$ is associative, i.e., $(n \oplus m) \oplus p = n \oplus (m \oplus p)$ for $n, m, p \in \mathbb{N}_0$.
(3) There is an identity, i.e., $n \oplus 0 = n$ for $n \in \mathbb{N}_0$.
(4) Each element $n \in \mathbb{N}_0$ has an inverse element, i.e., $n \oplus n = 0$.

With the above notation in mind, Bouton's result is as

follows.

**Theorem 2.2. (Bouton [3])** *In the nim game,* $N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})$ *is an N-position if and only if* $n^{(1)} \oplus n^{(2)} \oplus \cdots \oplus n^{(r)} \neq 0$.

From then on, variations of the nim game have been studied extensively. For a subset $S \subseteq \mathbb{N}_0$, define $\text{mex}(S)$ to be the smallest nonnegative integer not in $S$. For instance, $\text{mex}\{0, 1, 5, 6\} = 2$ and $\text{mex}\{1, 5, 6\} = 0$. An useful tool for studying the nim game is the *Grundy number* of a position $N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})$, which isdefined recursively as follows.

**Definition 2.3.** $g_{\text{nim}}(N(0)) = 0$. *As for any position* $N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})$ *not equivalent to* $N(0)$*,*

$$\begin{aligned} &g_{\text{nim}}(N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})) \\ &\quad = \text{mex}\{g_{\text{nim}}(N(m^{(1)}, m^{(2)}, \ldots, m^{(r)})) : N(m^{(1)}, m^{(2)}, \ldots, m^{(r)}) \\ &\qquad\quad \textit{can be obtained from } N(n^{(1)}, n^{(2)}, \ldots, n^{(r)}) \textit{ by one move}\}. \end{aligned}$$

An alternatively way to describe Bouton's result is that

$$g_{\text{nim}}(N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})) = n^{(1)} \oplus n^{(2)} \oplus \cdots \oplus n^{(r)}.$$

And so $N(n^{(1)}, n^{(2)}, \ldots, n^{(r)})$ is an N-position if and only if its Grundy number is not zero.

For x ∈ {trail, star, matching}, we may also define the Grundy number for the x-removing game at position $G$ as follows.

**Definition 2.4. (Grundy number of the x-removing game)** *The* Grundy number *of the null graph $K_0$ with no vertices nor edges is $g_x(K_0) = 0$. And the* Grundy number *of a graph G with at least one edge is*

$$g_x(G) = \text{mex}\{g_x(H) : H \text{ can be obtained from } G \text{ by one move}\}.$$

It is also the case that $G$ is an N-position if and only if $g_x(G) \neq 0$. Also, the same as Bouton's result, we have the following result.

**Theorem 2.5. (Sprague-Grundy Theorem)** *For* x ∈ {trail, star} *and any twographs G and H, we have $g_x(G \cup H) = g_x(G) \oplus g_x(H)$.*

Below are two lemmas.

**Lemma 2.6.** *For* x ∈ {trail, star, matching} *and any two graphs G and H, if H can be obtained by G in one move, then $|E(H)| < |E(G)|$.*

*Proof.* Since in any move a player removes at least one edge from the original graph,the above inequality is obviously true. □

Although very simple, Lemma 2.6 plays a very important role in many logicaldeductions we shall see in later parts. It also helps us prove the following lemma.

**Lemma 2.7.** *For* x ∈ {trail, star, matching} *and any graph G, we have $g_x(G) \leq |E(G)|$.*

*Proof.* We shall prove the lemma by mathematical induction on $|E(G)|$.

For the case of $|E(G)| = 0$, we have $g_x(G) = 0$ by definition, and so $g_x(G) \leq |E(G)|$.

Suppose $G$ has at least one edge and $g_x(G') \leq |E(G')|$ for any graph with $|E(G')| < |E(G)|$. For any subgraph $H$ obtained from $G$ by a move, by Lemma 2.6, $|E(H)| < |E(G)|$. By the induction hypothesis, $g_x(H) \leq |E(H)| < |E(G)|$. Then by definition, $g_x(G) \leq |E(G)|$.

Therefore, by mathematical induction, the lemma is true. □

# 3 Trail-removing game

The trail-removing game is the game generalized from the game of squayles, where the players take turns taking away the edges of a trail of at least one edge from the previous graph and the last player to make a move wins.

First, we discuss the Grundy numbers of some special graphs. We can quickly obtain the following propositions.

**Proposition 3.1. (Paths)** *If n is a positive integer, then $g_{\text{trail}}(P_n) = n - 1$.*

*Proof.* We shall prove the lemma by induction on $n$.

When $n = 1$, notice that $P_1$ consists of only one vertex with degree 0. There- fore, this vertex is automatically removed, and so by Definition 2.4, $g_{\text{trail}}(P_1) = 0 = n - 1$.

Suppose $n \geq 2$ and the lemma is true for $n' < n$. Notice that $P_n$ can be turned into $P_i$ in one move for all $1 \leq i \leq n - 1$. By the induction hypothesis, $g_{\text{trail}}(P_i) = i - 1$ for $1 \leq i \leq n - 1$ and so $g_{\text{trail}}(P_n) \geq n - 1$. Also, we have $g_{\text{trail}}(P_n) \leq |E(P_n)| = n - 1$. Therefore, $g_{\text{trail}}(P_n) = n - 1$.

The lemma then holds by induction. □

**Proposition 3.2. (Cycles)** *If $n \geq 3$ is an integer, then $g_{\text{trail}}(C_n) = n$.*

*Proof.* It is easy to see that by taking away the edges of $P_{k+1}(1 \leq k \leq n)$ from $C_n$ (which can be done in one single move), the graph turns into $P_{n-k+1}$. Therefore, we have $g_{\text{trail}}(C_n) \geq n$. Also, we have $g_{\text{trail}}(C_n) \leq |E(C_n)| = n$. Therefore, $g_{\text{trail}}(C_n) = n$ for all integers $n \geq 3$.

□

For any nonnegative integer $n$, the star graph $S_n$ is the graph with $n$ vertices $v_1, v_2, \ldots, v_n$ adjacent to a special vertex $v_0$, called the *center* of the star. TheGrundy numbers of stars are as follows.

**Proposition 3.3. (Stars)** *If $n$ is a nonnegative integer, then $g_{\text{trail}}(S_n) = r$, where $r \in \{0, 1, 2\}$ and $r \equiv n \pmod 3$.*

*Proof.* First, since the edges taken away must be the edges of an Eulerian graph, one can (and must) take 1 or 2 edges from $S_n$ in one move.

Also, note that no matter how many edges are taken away from a star graph, the remaining graph is still a star graph. Therefore, the only 2 graphs that can be obtained from $S_n$ by one move are $S_{n-1}$ and $S_{n-2}$.

We shall prove the proposition by induction on $n$.

When $n \leq 1$, $S_n = P_{n+1}$ and so $g_{\text{trail}}(S_n) = g_{\text{trail}}(P_{n+1}) = n$, where $n \in \{0, 1, 2\}$ and $n \equiv n \pmod 3$.

Therefore, the proposition is true when $n = 0$ and $n = 1$.

Suppose the proposition is true when $n = k$ and $n = k + 1$. Then when $n = k + 2$, the only 2 graphs that can be obtained from $S_{k+2}$ by one move are $S_{k+1}$and $S_k$. Therefore, $g_{\text{trail}}(S_{k+2}) = \text{mex}\{g_{\text{trail}}(S_{k+1}), g_{\text{trail}}(S_k)\}$.

Therefore, if $k \equiv 0 \pmod 3$, i.e., $k + 2 \equiv 2 \pmod 3$, then $g_{\text{trail}}(S_{k+2}) = \text{mex}\{0, 1\} = 2$;

if $k \equiv 1 \pmod 3$, i.e., $k + 2 \equiv 0 \pmod 3$, then $g_{\text{trail}}(S_{k+2}) = \text{mex}\{1, 2\} = 0$;

if $k \equiv 2 \pmod 3$, i.e., $k + 2 \equiv 1 \pmod 3$, then $g_{\text{trail}}(S_{k+2}) = \text{mex}\{2, 0\} =$

1.Hence the proposition is true by induction. □

**Definition 3.4. (Double stars)** *We define a* double star $S_{n,m}$ *as the union of two stars $S_n$ and $S_m$ ($n$, $m \in \mathrm{N}_0$), with their centers joined together by one additional edge.*

Notice that $S_{n,0} = S_{n+1}$ for all $n \in \mathrm{N}_0$ whose Grundy numbers are given above. We now consider other double stars in two cases.

**Lemma 3.5.** *If $n$ is a positive integer, then $g_{\text{trail}}(S_{n,1}) = r + 2$, where $r \in \{1, 2, 3\}$ and $r \equiv n \pmod 3$.*

*Proof.* Consider the set $M = \{S_{n,0}, S_{n-1,1}, S_{n-2,1}, S_n \cup S_1, S_n \cup S_0, S_{n-1} \cup S_1, S_{n-1} \cup S_0\}$. It is easy to see that these the graphs in $M$ are exactly the only ones $S_{n,1}$ can be turned into in one move. Also, by Proposition 3.1, we have $g_{\text{trail}}(S_{0,1}) = g_{\text{trail}}(P_3)2$.

When $n = 1$, by Proposition 3.1, $g_{\text{trail}}(S_{1,1}) = g_{\text{trail}}(P_4) = 3$.

When $n = 2$, $g_{\text{trail}}(S_{2,1}) = \text{mex}\{g_{\text{trail}}(S_{2,0}), g_{\text{trail}}(S_{1,1}), g_{\text{trail}}(S_{0,1}), g_{\text{trail}}(S_2 \cup S_1), g_{\text{trail}}(S_2 \cup S_0), g_{\text{trail}}(S_1 \cup S_1), g_{\text{trail}}(S_1 \cup S_0)\} = \text{mex}\{0, 3, 2, 3, 2, 0, 1\} = 4$.

Suppose $n \geq 3$, and the lemma is true for $n - 1$ and $n - 2$. It is easy to see that we have $g_{\text{trail}}(S_{n,1}) = \text{mex}\{g_{\text{trail}}(S_{n,0}), g_{\text{trail}}(S_{n-1,1}), g_{\text{trail}}(S_{n-2,1}), g_{\text{trail}}(S_n \cup S_1), g_{\text{trail}}(S_n \cup S_0), g_{\text{trail}}(S_{n-1} \cup S_1), g_{\text{trail}}(S_{n-1} \cup S_0)\}$.

Therefore, if $n \equiv 0 \pmod 3$, $g_{\text{trail}}(S_{n,1}) = \text{mex}\{g_{\text{trail}}(S_{n,0}), g_{\text{trail}}(S_{n-1,1}), g_{\text{trail}}(S_{n-2,1}), g_{\text{trail}}(S_n \cup S_1), g_{\text{trail}}(S_n \cup S_0), g_{\text{trail}}(S_{n-1} \cup S_1), g_{\text{trail}}(S_{n-1} \cup S_0)\}$

$= \text{mex}\{1, 4, 3, 1, 0, 3, 2\} = 5$;

if $n \equiv 1 \pmod 3$, $g_{\text{trail}}(S_{n,1}) = \text{mex}\{g_{\text{trail}}(S_{n,0}), g_{\text{trail}}(S_{n-1,1}), g_{\text{trail}}(S_{n-2,1}), g_{\text{trail}}(S_n \cup S_1), g_{\text{trail}}(S_n \cup S_0), g_{\text{trail}}(S_{n-1} \cup S_1), g_{\text{trail}}(S_{n-1} \cup S_0)\}$

$= \text{mex}\{2, 5, 4, 0, 1, 1, 0\} = 3$;

if $n \equiv 2 \pmod 3$, $g_{\text{trail}}(S_{n,1}) = \text{mex}\{g_{\text{trail}}(S_{n,0}), g_{\text{trail}}(S_{n-1,1}), g_{\text{trail}}(S_{n-2,1}), g_{\text{trail}}(S_n \cup S_1), g_{\text{trail}}(S_n \cup S_0), g_{\text{trail}}(S_{n-1} \cup S_1), g_{\text{trail}}(S_{n-1} \cup S_0)\}$

$= \text{mex}\{0, 3, 5, 3, 2, 0, 1\} = 4$.

The lemma then holds by induction. □

After we have proven the lemma, here is the result on double stars in general.

**Theorem 3.6. (Double stars)** *For all integers $n_1, n_2 \geq 2$,*

*if $n_1 \equiv n_2 \equiv 0 \pmod 3$ or $n_1 \equiv n_2 \equiv 2 \pmod 3$, then $g_{\text{trail}}(S_{n_1,n_2}) = 1$;*

*if $n_1 \equiv n_2 \equiv 1 \pmod 3$, then $g_{\text{trail}}(S_{n_1,n_2}) = 2$;*

*if ($n_1 \equiv 1 \pmod 3$, $n_2 \equiv 2 \pmod 3$) or ($n_1 \equiv 2 \pmod 3$, $n_2 \equiv 1 \pmod 3$),then $g_{\text{trail}}(S_{n_1,n_2}) = 4$;*

*if ($n_1 \equiv 0 \pmod 3$, $n_2 \equiv 1 \pmod 3$) or ($n_1 \equiv 1 \pmod 3$, $n_2 \equiv 0 \pmod 3$),then $g_{\text{trail}}(S_{n_1,n_2}) = 5$;*

*if ($n_1 \equiv 0 \pmod 3$, $n_2 \equiv 2 \pmod 3$) or ($n_1 \equiv 2 \pmod 3$, $n_2 \equiv 0 \pmod 3$),then $g_{\text{trail}}(S_{n_1,n_2}) = 6$.*

*Proof.* Consider the set $M = \{S_{n_1-1,n_2}, S_{n_1,n_2-1}, S_{n_1-2,n_2}, S_{n_1,n_2-2}, S_{n_1} \cup S_{n_2}, S_{n_1-1} \cup S_{n_2}, S_{n_1} \cup S_{n_2-1}, S_{n_1-1} \cup S_{n_2-1}\}$. It is easy to see that these the graphs in $M$ are exactly the only ones $S_{n_1,n_2}$ can be turned into in one move.

Notice that $g_{\text{trail}}(S_{n_1,n_2}) = g_{\text{trail}}(S_{n_2,n_1})$ for all $n_1$ and $n_2$. We then only need to verify the theorem by the following cases. Suppose $g_{\text{trail}}(S_{n'_1,n'_2})$ is known for $n'_1 + n'_2 < n_1 + n_2$.

**Case 1.** $2 \leq n_1, n_2 \leq 4$:

$g_{\text{trail}}(S_{2,2}) = \text{mex}\{4, 4, 0, 0, 0, 3, 3, 0\} = 1$;

$g_{\text{trail}}(S_{2,3}) = \text{mex}\{5, 1, 4, 1, 2, 1, 0, 3\} = 6;$
$g_{\text{trail}}(S_{2,4}) = \text{mex}\{3, 6, 2, 0, 3, 0, 2, 1\} = 4;$
$g_{\text{trail}}(S_{3,3}) = \text{mex}\{6, 6, 5, 5, 0, 2, 2, 0\} = 1;$
$g_{\text{trail}}(S_{3,4}) = \text{mex}\{4, 1, 3, 6, 1, 3, 0, 2\} = 5;$
$g_{\text{trail}}(S_{4,4}) = \text{mex}\{5, 5, 4, 4, 0, 1, 1, 0\} = 2.$

**Case 2.** $n_1 > 4$ with $r_1 = 0$ and $2 \leq n_2 \leq 4$:
$g_{\text{trail}}(S_{n_1,2}) = \text{mex}\{1, 5, 4, 1, 2, 0, 1, 3\} = 6;$
$g_{\text{trail}}(S_{n_1,3}) = \text{mex}\{6, 6, 5, 5, 0, 2, 2, 0\} = 1;$
$g_{\text{trail}}(S_{n_1,4}) = \text{mex}\{4, 1, 2, 6, 1, 3, 0, 2\} = 5.$

**Case 3.** $n_1 > 4$ with $r_1 = 1$ and $2 \leq n_2 \leq 4$:
$g_{\text{trail}}(S_{n_1,2}) = \{6, 3, 1, 2, 3, 2, 0, 1\} = 4;$
$g_{\text{trail}}(S_{n_1,3}) = \{1, 4, 6, 3, 1, 0, 3, 2\} = 5;$
$g_{\text{trail}}(S_{n_1,4}) = \{5, 5, 4, 4, 0, 1, 1, 0\} = 2.$

**Case 4.** $n_1 > 4$ with $r_1 = 2$ and $2 \leq n_2 \leq 4$:
$g_{\text{trail}}(S_{n_1,2}) = \{4, 4, 0, 0, 0, 3, 3, 0\} = 1;$
$g_{\text{trail}}(S_{n_1,3}) = \{5, 1, 4, 1, 2, 1, 0, 3\} = 6;$
$g_{\text{trail}}(S_{n_1,4}) = \{3, 6, 2, 0, 3, 0, 2, 1\} = 4.$

**Case 5.** $n_1, n_2 > 4$:
if $r_1 = r_2 = 2$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{4, 4, 6, 6, 0, 3, 3, 0\} = 1;$
if $r_1 = 2$ and $r_2 = 0$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{5, 1, 1, 4, 2, 1, 0, 3\} = 6;$
if $r_1 = 2$ and $r_2 = 1$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{2, 6, 5, 1, 3, 0, 2, 1\} = 4;$
if $r_1 = r_2 = 0$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{6, 6, 5, 5, 0, 2, 2, 0\} = 1;$
if $r_1 = 0$ and $r_2 = 1$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{4, 1, 2, 6, 1, 3, 0, 2\} = 5;$
if $r_1 = r_2 = 1$, then $g_{\text{trail}}(S_{n_1,n_2}) = \{5, 5, 4, 4, 0, 1, 1, 0\} = 2.$

By induction, the theorem then holds for all integers $n_1, n_2 \geq 2$. □

Afterwards, let us introduce the concept of a special kind of graphs we now call *k-stars*.

**Definition 3.7.** **($k$-stars)** *For nonnegative integers $n_1, n_2, \ldots, n_k$, the $k$-*star *$S_{n_1,n_2,\ldots,n_k}$ is the graph obtained from the union of k stars $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ with centers $v_1, v_2, \ldots, v_k$ by adding $k-1$ edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$.*

From Proposition 3.3, Lemma 3.5, and Theorem 3.6, we can see that the upper bound of the Grundy number of a star is 2, while that of a double star is 6. What about the Grundy number of $k$-stars in general? Before continuing, here is a remark about certain properties the operation $\oplus$ has.

**Remark 3.8.** *The following properties hold for all nonnegative integers a and b.*

*(1) If $a \geq b$ and $2^{k-1} \leq a \leq 2^k - 1$, then $a \oplus b \leq 2^k - 1$.*

*(2) $a \oplus b \leq a + b$.*

With this remark in mind, we now have the following upper bound for the Grundy number of $k$-stars as follows.

**Theorem 3.9. (Grundy numbers of $k$-stars)** *In the trail-removing game, for all integers $k \geq 3$, an upper bound of the Grundy number of a $k$-star is $k^2 + 4k - 7$. (Notice that this upper bound is likely not tight for some $k$.)*

*Proof.* First, for convenience, let the *tight* upper bound of a $m$-star be $a_m$. Thenwe know that $a_1 = 2$ and $a_2 = 6$.

When $k = 3$, a 3-star can be turned into the following graphs in one move: another 3-star, the union of a 2-star and a star, and the union of 3 stars. Then by Remark 3.8, the Grundy number of the union of 3 stars has maximum value 3, and the Grundy number of the union of a 2-star and a star has maximum value 7. Also, notice that the number of distinct possible 3-stars the original 3-star can be turned into in one move is at most $3 \times 2 = 6$. Therefore, $a_3 \leq 7 + 6 + 1 = 14$.

When $k \geq 4$, let the upper bound hold for all integers less than $k$. It is easy to see that a $k$-star can be turned into another $k$-star, or the union of an $h_1$-star, an $h_2$-star, and $(k - h_1 - h_2)$ stars (where $h_1, h_2 \geq 1$ and $h_1 + h_2 \leq k$) in one move. Notice that the number of distinct possible $k$-stars the original $k$-star can be turned into in one move is at most $2k$. I shall now prove that the Grundy number of the union of an $h_1$-star, an $h_2$-star, and $(k - h_1 - h_2)$ stars has maximum value$k^2 + 2k - 8$.

Notice that if $h_1 = h_2 = 1$, the Grundy number of the union of multiple starshas maximum value 3;

if $h_1 = 1, h_2 = k - 1$ or $h_1 = k - 1, h_2 = 1$, since the upper bound holds for all integers less than $k$, the Grundy number in this case has maximum value$2 + (k-1)^2 + 4(k-1) - 7 = k^2 + 2k - 8$;

otherwise, we must have $h_1 + h_2 \leq k - 4$. Then the Grundy number of the union of an $h_1$-star, an $h_2$-star, and $(k - h_1 - h_2)$ stars has maximum value

$$3 + h_1^2 + 4h_1 - 7 + h_2^2 + 4h_2 - 7 = 11 + 4(h_1 + h_2) + h_1^2 + h_2^2$$
$$\leq 11 + 4(k-4) + (k-4)^2 = 11 + 4k - 16 + k^2 - 8k + 16 = k^2.$$

Then since $k \geq 4$, we have $19 < 6k$, and thus $k^2 - 4k + 11 < k^2 + 2k - 8$.

Therefore, $a_k \leq (k^2 + 2k - 8) + 2k + 1 = k^2 - 4k + 11$. The theorem then holds by induction. □

**Definition 3.10.** *A graph $G$ is* nice *if it has an Eulerian subgraph $G_0$ with at least one edge such that $V(G) - V(G_0)$ is an independent set. If, in addition, $G_0$ is a spanning subgraph of $G$, then $G$ is called a* fully nice graph*.*

**Theorem 3.11.** *In the trail-removing game, nice graphs are all N-positions.*

*Proof.* Suppose $G$ is a nice graph with an Eulerian subgraph $G_0$ such that $V(G) - V(G_0)$ is an independent set. For any vertex $w \in V(G_0)$, the edge set $E(G_0)$ formsa $w$-$w$ trail $T_0$ in $G$. Consider the graph $G - E(T_0)$.

For the case when $G - E(T_0)$ is a P-position, the first player can remove the edges of $T_0$, making the remaining graph $G - E(T_0)$ a P-position. Therefore, $G$ isan N-position.

For the case when $G - E(T_0)$ is an N-position, $G - E(T_0)$ has a $u$-$v$ trail $T$ of at leat one edge such that $(G - E(T_0)) - E(T)$ is a P-position. Since $V(G) - V(G_0)$ is independent, $T$ has at least one vertex $w$ in $G_0$. As $E(T) \cap E(T_0) = \emptyset$, we have that $T' := T(u \text{ to } w)T_0T(w \text{ to } v)$ is a trail from $u$ to $v$. Thus, the first player can remove the edges of $T'$, making the remaining graph $G - E(T')$ a P-position. Therefore, $G$ is an N-position.

□

**Remark 3.12.** *If $H$ is a subgraph of $G$, $V(H) = V(G)$, and $H$ is a fully nice graph, then $G$ is also a fully nice graph. This is obvious because if $H_0$ is a fully nice subgraph of $H$, then it must also be a subgraph of $G$. Then since $V(H_0) = V(H) = V(G)$ and $H_0$ is an Eulerian graph with 0 odd vertices, $H_0$ is also a fully nice subgraph ofG, and therefore $G$ is also a fully nice graph.*

*In other words, if a graph $G$ satisfies that after deleting some edges in $G$, the remaining graph is a fully nice graph, then $G$ must also be a fully nice graph.*

Below is an operation done on two graphs that results in another graph, called the *join product* of two graphs.

**Definition 3.13. (Join product of two graphs)** *The* join product *of two graphs $G$ and $H$ is the graph $G\nabla H$ whose*

*vertex set $V(G\nabla H) = V(G) \cup V(H)$ and*

*edge set $E(G\nabla H) = \{vu : v \in V(G) \text{ and } u \in V(H)\} \cup E(G) \cup E(H)$.*

Here is the result for the join of two graphs.

**Theorem 3.14. (Trail-removing game on $G\nabla H$)** *The join $G\nabla H$ of two nonempty graphs $G$ and $H$ is an N-position, except when ($G = K_1$ and $H = mK_1$) or ($G = mK_1$ and $H = K_1$), where $m$ is a positive integer with $m \equiv 0 \pmod 3$.*

*Proof.* Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_m\}$. Without loss of generality, we may assume that $1 \le n \le m$.

If $G = K_1$ and $H = mK_1$, then $G\nabla H = S_m$. By Proposition 3.3, $G\nabla H$ is a P-position if and only if $m \equiv 0 \pmod 3$.

We then consider the remaining case when either $n = 1$ with $H$ having at leat one edge or else $2 \le n \le m$.

For the case when $n = 1$ with $H$ having at least one edge, choose a maximal matching $M = \{x_1x_2, x_3x_4, \dots, x_{2r-1}x_{2r}\}$ of $H$, where $r \ge 1$. Then $M \cup \{v_1x_i : 1 \le i \le 2r\}$ induces an Eulerian subgraph $G_0$ of $G\nabla H$ such that $V(G\nabla H) - V(G_0)$ is independent. By Theorem 3.11, $G\nabla H$ is an N-position.

For the case when $2 \le n \le m$, choose a maximal matching $M = \{x_1x_2, x_3x_4, \dots, x_{2r-1}x_{2r}\}$ of $H - \{u_1, u_2, \dots, u_n\}$, where $r \ge 0$. Then $\{v_1u_1, u_1v_2, v_2u_2, u_2v_3, \dots,$

$v_nu_n, u_nv_1\} \cup M \cup \{v_1x_i : 1 \le i \le 2r\}$ induces an Eulerian subgraph $G_0$ of $G\nabla H$ such that $V(G\nabla H) - V(G_0)$ is independent. Again, by Theorem 3.11, $G\nabla H$ is an N-position.

□

Another special kind of graphs is called *complete k-partite graphs*, defined as follows.

**Definition 3.15. (Complete $k$-partite graphs)** *A graph G is* complete $k$-partite *if there exists a partition* $V(G) = V_1(G) \cup V_2(G) \cup \ldots \cup V_kG)$ *such that for all* $u, v \in V(G)$, *u and v are not adjacent* if and only if *u and v are in the same* $V_i(G)$ *for some* $1 \le i \le k$.

*Let* $|V_i(G)| = n_i$ *for all integers* $1 \le i \le n$. *We then denote G as* $K_{n_1,n_2,\ldots,n_k}$.

It is easy to see that $K_{n_1,n_2,\ldots,n_k}$ can also be described as the join product of the graphs $N_{n_1}\nabla N_{n_2}\nabla \ldots \nabla N_{n_k}$, where the graph $N_n$, called the *edgeless graph*, has $n$ vertices and no edges. By this observation and Theorem 3.14, the following corollary immediately follows.

**Corollary 3.16. (Complete $k$-partite graphs)** *A complete k-partite graph is a P-position if and only if* $k = 1$ *or* $(k = 2, n_1 = 1, n_2 \equiv 0 \pmod 3)$ *or* $(k = 2, n_2 = 1, n_1 \equiv 0 \pmod 3)$.

Next, we define another operation on graphs, called the *Cartesian product* oftwo graphs, as follows.

**Definition 3.17. (Cartesian product of two graphs)** *The* Cartesian product *of two graphs G and H is the graph $G\square H$ whose*

vertex set $V(G\square H) = V(G) \times V(H)$ and

edge set $E(G\square H) = \{(v, u)(v', u') : (v = v', uu' \in E(H))$ or $(vv' \in E(G), u = u')\}$.

The following is an example of the Cartesian product of two graphs.

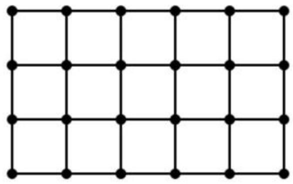


Figure 5: The Cartesian product $P_4\square P_6$.

We introduce a notation first.

**Definition 3.18. ($v$-rows and $u$-columns)** *Let G and H be graphs. For every vertex* $v \in V(G)$, *the v-row of* $G\square H$, *denoted by* $R_v(G\square H)$, *is the subgraph induced by* $\{(v, u) : u \in V(H)\}$. *Notice that the v-row* $R_v(G\square H)$ *is isomorphic to H.*

*Similarly, for every vertex* $u \in V(H)$, *the u-column of* $G\square H$, *denoted by* $C_u(G\square H)$, *is the subgraph induced by* $\{(v, u) : v \in V(G)\}$. *Notice that the u-column* $C_u(G\square H)$ *is isomorphic to G.*

In the following, for any graph $G$, let $V_e(G)$ and $V_o(G)$ be the sets of even-degree vertices and odd-degree vertices, respectively, in $G$.

**Theorem 3.19.** *If $G$ and $H$ are connected graphs with at least two vertices, then $G\square H$ is an* N*-position in the trail-removing game if any one of the following conditions holds.*

(1) *$G$ is a fully nice graph. A special case is when $V(G) = V_e(G)$.*
(2) *$V(H) = V_o(H)$.*
(3) *$G$ has an edge $xy$ with $x, y \in V_e(G)$ and $|V(H)|$ is even.*

*Proof.* (1) Suppose $G$ is a fully nice graph with a spanning Eulerian subgraph $G_0$. (For the special case of $V(G) = V_e(G)$, choose $G_0 = G$.) In this case, $G_0$ has an edge $xy$ such that $G_0 - xy$ is connected. And so the subgraph $G'_u$ isomorphic to $G_0 - xy$ in the column $C_u(G\square H)$ is connected.

Consider the subgraph $I$ of $G\square H$ which is the union of the connected subgraphs $R_x(G\square H)$, $R_y(G\square H)$, and $G'_u$ for $u \in V(H)$. Then $I$ is a connected spanning subgraph of $G\square H$. Also, every vertex $(v, u)$ of $I$ is of even degree:

$$\deg_I(v,u) = \begin{cases} \deg_{G_0}(v), & \text{if } v \notin \{x,y\}; \\ \deg_{G_0}(v) + \deg_H(u), & \text{if } v \in \{x,y\} \text{ and } u \in V_e(H); \\ \deg_{G_0}(v) + \deg_H(u) - 1, & \text{if } v \in \{x,y\} \text{ and } u \in V_o(H). \end{cases}$$

Hence $I$ is a spanning Eulerian subgraph of $G\square H$. By Theorem 3.11, $G\square H$ is an N-position.

(2) Consider the case when $V(H) = V_o(H)$. By the result in (1), we may assume that $V(G) \neq V_e(G)$, i.e., $V_o(G) \neq \emptyset$.

Let $I$ be the subgraph of $G\square H$ which is the union of rows $R_v(G\square H)$ for $v \in V_o(G)$ and columns $C_u(G\square H)$ for $u \in V(H)$. Then $I$ is connected spanning subgraph of $G\square H$. Also, every vertex $(v, u)$ of $I$ is of even degree:

$$\deg_I(v,u) = \begin{cases} \deg_G(v), & \text{if } v \in V_e(G); \\ \deg_G(v) + \deg_H(u), & \text{if } v \in V_o(G). \end{cases}$$

Hence $I$ is a spanning Eulerian subgraph of $G\square H$. By Theorem 3.11, $G\square H$ is anN-position.

(3) Consider the case when $G$ has an edge $xy$ with $x, y \in V_e(G)$ and $|V(H)|$ is even. By the result in (1), we may assume that $V(G)\ \ V_e(G)$, i.e., $V_o(G) {/=} \emptyset$; and by the result in (2), we may assume that $V(H) {/=} V_o(H)$, i.e., $V_e(H) {/=} \emptyset$.

Assume that $V_e(H) = \{u_1, u_2, \ldots, u_{2m}\}$. For $1 \leq i \leq m$, choose a path $P_i$ between $u_i$ and $u_{m+i}$. If some $P_i$ and $P_j$ have common edges, we may replace them by two edge-disjoint paths between pairs of vertices of the four end vertices of these two paths. Repeating the process if necessary, we then may assume paths $P_1, P_2, \ldots, P_m$ are edge-disjoint. Let $T$ be the subgraph of $H$ induced by the union of these $m$ paths. Then $H - E(T)$ is a graph in which all vertices are of odd degree.

Let $I$ be the subgraph of $G\square H$ which is the union of the subgraph of $R_v(G\square H)$ isomorphic to $H - E(T)$ for $v \in V_o(G)$, rows $R_v(G\square H)$ for $v \in \{x, y\}$, and columns $C_u(G\square H)$ for $u \in V(H)$ but deleting edges $(x, u)(y, u)$ for $u \in V_o(H)$. To see that $I$

is connected, we consider the following steps. First rows $R_x(G\square H)$ and $R_y(G\square H)$ are connected, and they are connected by the edges $(x, u)(y, u)$ for $u \in V_e(H)$. Secondly, any edge-deleted-column $C_u(G\square H) - (x,u)(y, u)$, if not connected, has two connected components, each with a common vertex $(x, u)$ with row $R_x(G\square H)$ or $(y, u)$ with row $R_y(G\square H)$. And these rows and edge-deleted-columns cover all vertices of $G\square H$. Hence $I$ is connected spanning subgraph of $G\square H$. Also, every vertex $(v, u)$ of $I$ is of even degree:

$$\deg(v,u) = \begin{cases} \deg_G(v), & \text{if } v \in V_e(G) - \{x,y\}; \\ \deg_G(v) + \deg_H(u), & \text{if } v \in \{x,y\} \text{ and } u \in V_e(H); \\ \deg_G(v) + \deg_H(u) - 1, & \text{if } v \in \{x,y\} \text{ and } u \in V_o(H); \\ \deg_G(v) + \deg_{H-E(T)}(u), & \text{if } v \in V_o(G). \end{cases}$$

Hence $I$ is a spanning Eulerian subgraph of $G\square H$. By Theorem 3.11, $G\square H$ is an N-position.

□

**Remark 3.20.** *In fact, from the first part of the proof of Theorem 3.19, we can see that the Cartesian product of a fully nice graph and any connected graph with at least* 2 *vertices must also be a fully nice graph.*

We now use the approach of nice graphs to solve the trail-removing game on $r$-dimensional grids $P_{n_1}\square P_{n_2}\square\ldots\square P_{n_r}$. For a path $P_r$, let its vertex set $V(P_r) = \{1, 2, \ldots, r\}$ and edge set $E(P_r) = \{12, 23, \ldots, (r-1)r\}$.

**Lemma 3.21.** *For $n, m \geq 2$, the 2-dimensional grid $P_n\square P_m$ is fully nice, except for $P_3\square P_3$, which is only nice.*

*Proof.* For the case when $n=m=3$, the vertex set $\{(1, 1), (3, 3)\}$ is independent in $P_3\square P_3$. And the other vertices induce an Eulerian subgraph of $P_3\square P_3$. Hence $P_3\square P_3$ is a nice graph.

For the case when $n$ or $m$ is even, we may assume that $n$ is even. Then the following form a spanning Eulerain subgraph of $P_n\square P_m$:

row $R_1(P_n\square P_m)$,

one-edge-deleted-row $R_i(P_n\square P_m) - (i, 1)(i, 2)$ for $2 \leq i \leq n-1$,row $R_n(P_n\square P_m)$,

column $C_1(P_n\square P_m)$,

edges $(i, 2)(i+1, 2)$ for even $i$ with $2 \leq i \leq n-2$,

edges $(i, m)(i+1, m)$ for odd $i$ with $1 \leq i \leq n-1$.

Therefor, $P_n\square P_m$ is fully nice.

For the case when $n$ and $m$ are both odd, we may assume that $n \geq 5$. Thenthe following form a spanning Eulerain subgraph of $P_n\square P_m$:

edge $(1, 1)(1, 2)$ and edges $(1, j)(1, j+1)$ for even $j$ with $2 \leq j \leq m-1$,

edges $(2, j)(2, j+1)$ for odd $j$ with $1 \leq j \leq m-2$,

row $R_3(P_n\square P_m)$,

one-edge-deleted-row $R_i(P_n\square P_m) - (i, 1)(i, 2)$ for $4 \leq i \leq n-1$,

row $R_n(P_n \square P_m)$,
one-edge-deleted-column $C_1(P_n \square P_m) - (2, 1)(3, 1)$,
edges $(1, j)(2, j)$ for $3 \leq j \leq m - 1$,
edge $(n, 1)(n, 2)$ and edges $(n, j)(n, j + 1)$ for even $j$ with $2 \leq i \leq m - 1$.

Therefore, $P_n \square P_m$ is fully nice. □

**Lemma 3.22.** *The graph $P_3 \square P_3 \square P_3$ is a fully nice graph.*

*Proof.* The subgraph induced by the following edges is a spanning Eulerian sub-graph: $(1, 1, k)(1, 2, k)$, $(1, 2, k)(2, 2, k)$, $(2, 2, k)(2, 1, k)$ for $k = 1, 2$; $(1, 1, 1)(1, 1, 2)$;
$(2, 1, 1)(2, 1, 2)$. □

From Theorems 3.11 and 3.19, and Lemmas 3.21 and 3.22, and Remark 3.20, we have the following result for $r$-dimensional grids.

**Corollary 3.23.** *If $n_1, n_2, \ldots, n_r \geq 2$, then the r-dimensional grid $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$ is an N-position.*

# 4 Star-removing game

The star-removing game is a game where the players take turns taking away the edges of a star graph from the previous graph and the last player to make a move wins. I later found several results other people have gotten before on the star- removing game (a more common name for it is "Graph Nim") [1] [2] [4] [5].

Similar to the procedure of the trail-removing game, I tried to calculate the Grundy numbers of special graphs like paths and cycles first, but found it quite difficult to find a pattern.

**Remark 4.1.** *In fact, the Grundy numbers of paths, stars and double stars (undera number of special cases) have already been computed by others before.*

I shall not go into details here (Interested readers are directed to [1] [2] [4] [5]), but note that the pattern of Grundy numbers of paths and double stars are quite complicated.

Although unable to compute the Grundy numbers of said graphs, I was able to determine the player who has a winning strategy, as shown below.

**Proposition 4.2. (Paths)** *In the star-removing game, $P_n$ is an N-position for all $n \geq 2, n \in \mathbb{N}$, and a P-position when $n = 1$.*

*Proof.* First, let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n - 1\}$.

Then, we consider the following three cases.

**Case 1.** $n = 1$. In this case, since $P_1$ consists of only 1 vertex of degree 0, itis automatically removed and the first player loses.

**Case 2.** $n$ is odd and $n > 1$. In this case, let $n = 2k + 1$. Then the firstplayer can take away the edges $v_k v_{k+1}$ and $v_{k+1} v_{k+2}$ (notice that this can always be done since these edges share a common vertex $v_{k+1}$), making the remaining graph $2P_{k-1}$, securing a win for the first player.

**Case 3.** $n$ is even. In this case, let $n = 2k$. Then the first player can take away the edge $v_k v_{k+1}$, making the remaining graph $2P_{k-1}$, securing a win for the first player.

Therefore, $P_n$ is an N-position for all $n \geq 2, n \in \mathbb{N}$, and a P-position when $n = 1$. □

**Proposition 4.3. (Cycles)** *In the star-removing game, $C_n$ is a P-position for all integers $n \geq 3$.*

*Proof.* It is easy to see that the first player can (and must) take away 1 edge or 2 edges that share a common vertex from $C_n$ in their first turn, making the remaining graph $P_n$ or $P_{n-1}$. Then, by Proposition 4.2, the second player has a winning strategy.

Therefore, $C_n$ is a P-position for all integers $n \geq 3$. □

**Proposition 4.4. (Double stars)** *In the star-removing game, $S_{n,m}$ (where $n \geq m$) is an N-position for all $n, m \in \mathbb{N}_0$.*

*Proof.* The first player can take away $n - m$ edges from the sub-graph $S_n$ and the edge connecting the centers of the two stars.

Then, the remaining graph becomes $2S_m$, securing a win for the first player. □

An *isomorphism* of a graph $G$ is a bijection $f: V(G) \to V(G)$ that preserves adjacency. That is, $vu \in E(G)$ implies $f(v)f(u) \in E(G)$.

A bijection $f: V \to V$ is of *order* 2 if $f \circ f$ is the identity function but $f$ itself is not unless $|V| = 1$. Notice that for a bijection $f: V \to V$ of order 2, it is allowed that $f(v) = v$ for some element $v$ in $V$, but not for all elements unless $|V| = 1$. Also, $f(v) = u$ if and only if $f(u) = v$; that is, $v$ and $u = f(v)$ form a *mapping pair*.

**Definition 4.5. (Symmetric graph)** *A graph is* symmetric *if it has an isomorphism $f$ of order 2, called a* symmetric function *of $G$, which satisfies the followingconditions for all vertices $v$.*

(S1) $f(v) \quad v$.
(S2) $vf(v) \notin E(G)$.

Notice that a symmetric graph has an even number of vertices.
Examples ofsymmetric graphs include $C_{2n}$, $K_{2n_1, 2n_2, \ldots, 2n_r}$ and $2G$.

**Definition 4.6. (Semi-symmetric graph)** *A graph is* semi-symmetric *if it has an isomorphism $f$ of order 2, called a* semi-symmetric function *of $G$, which satisfies the following conditions for all vertices $v$.*

(S1) $f(v) /= v$.
(SS2) $vf(v) \notin E(G)$, *except $v^* f(v^*) \in E(G)$ for exactly one mapping pair.*

Notice that a semi-symmetric graph has an even number of vertices. Examples of semi-symmetric graphs including $P_{2n}$ and $DS_{n,n}$, and more generally $2G$ addingand edge between two corresponding vertices.

**Definition 4.7. (Pseudo-symmetric graph)** *A graph is* pseudo-symmetric *if it has an isomorphism f of order* 2, *called a* pseudo-symmetric function *of G, which satisfies the following conditions for all vertices v.*

(PS1) $f(v) \neq v$, *except* $f(v^*) = v^*$ *for exactly one vertex* $v^*$.
(S2) $vf(v) \notin E(G)$.

Notice that a pseudo-symmetric graph has an odd number of vertices. Examples of pseudo-symmetric graphs including $P_{2n+1}$ and $K_{1+2n_1,2n_2,\ldots,2n_r}$, and more generally the graph obtained from $2G$ by identifying two corresponding vertices.

Next, I am going to discuss the properties of these special graphs introduced above. We have the theorem below:

**Theorem 4.8.** *Suppose that G is symmetric/semi-symmetric/pseudo-symmetric. Then, in the star-removing game, the following properties hold.*

*(1) If G is symmetric, then G is a P-position.*
*(2) If G is semi-symmetric, then G is an N-position.*
*(3) If G is pseudo-symmetric, f is one of its pseudo-symmetric functions, and* $v \in V(G)$ *such that* $f(v) = v$, *then G is a P-position if and only if v has degree 0.*

*Proof.* (1) Let $f$ be a symmetric function of $G$. I shall now propose a strategy for the second player:

The second player takes away $f(v_1)f(v_2)$ in their turn if and only if the first player took away $v_1v_2$ in the previous turn.

Now, I am going to prove that this strategy can be done under any circumstances, and that it secures a win for the second player.

In the first move the first player makes, let $v$ denote the center of the star sub-graph whose sides they remove. Assume that the first player takes away the sides $vv_1, vv_2, \ldots, vv_h$. Then we have

(i) $v$ is adjacent to $v_i$ for all $1 \leq i \leq h$. Therefore, $f(v)$ is adjacent to $f(v_i)$ for all $1 \leq i \leq h$.

(ii) $v$ is not adjacent to $f(v)$. Then,

$$\{vv_1, vv_2, \ldots, vv_h\} \cap \{f(v)f(v_1), f(v)f(v_2), \ldots, f(v)f(v_h)\} = \emptyset.$$

Therefore, the second player is able to take away the edges $f(v)f(v_1)$, $f(v)f(v_2)$, . . ., $f(v)f(v_h)$. Also, it is obvious that the remaining graph is still symmetric (in fact,the restriction of $f$ on it is still its symmetry function).

Thus, as long as the first player has a move, the second player also has a move, which means that the second player cannot lose. And since the game is finite, the second player will eventually win.

(2) Let $f$ be a semi-symmetric function of $G$. According to Definition 4.6, exactly two vertices $v_1, v_2 \in V(G)$ exist such that $f(v_1) = v_2$, and $v_1$ and $v_2$ are adjacent.

Then the first player can take away the edge $v_1v_2$. Therefore, in the remaining graph $G'$, no two vertices $u_1, u_2 \in V(G')$ exist such that $f(u_1) = u_2$, and $u_1$ and $u_2$are adjacent.

Notice that the restriction of $f$ on $G'$ is a symmetric function of $G'$. Therefore, $G'$ is symmetric and thus a P-position (see the case above).

Therefore, $G$ is an N-position.

(3) If $v$ has degree 0, it is easy to see that the restriction of $f$ on $G - v$ is a symmetric function of $G - v$. Therefore, $G - v$ is a symmetric graph and thus a P-position. Also, since $v$ has degree 0, it will be automatically removed, and therefore $G$ is a P-position.

Otherwise, the first player can remove all the edges with $v$ as one of its vertices (notice that since $v$ has degree greater than 0, the first player can at least remove one edge). Notice that this move makes $deg(v) = 0$, causing $v$ to be automatically removed. Therefore, in the remaining graph $G'$, $f(u) \neq u$ for all $u \in V(G')$.

Therefore, the restriction of $f$ on $G'$ is a symmetric function of $G'$. $G'$ is thensymmetric and thus a P-position, and therefore $G$ is an N-position. □

**Theorem 4.9.** *Suppose the graphs $G_1$ and $G_2$ are symmetric/semi-symmetric/pseudo-symmetric.*

*(1) If either $G_1$ or $G_2$ is symmetric, or if $G_1$ and $G_2$ are both semi-symmetric,then $G_1 \square G_2$ is symmetric.*
*(2) If one of $G_1$ and $G_2$ is semi-symmetric and the other is pseudo-symmetric,then $G_1 \square G_2$ is semi-symmetric.*
*(3) If $G_1$ and $G_2$ are both pseudo-symmetric, then $G_1 \square G_2$ is pseudo-symmetric.*

*Proof.* Suppose for $i = 1, 2$, the bijection $f_i : V(G_i) \to V(G_i)$ is a symmetric/semi-symmetric/pseudo-symmetric function of $G_i$. Define

$$f : V(G_1 \square G_2) \to V(G_1 \square G_2) \text{ by } f(v_1, v_2) = (f_1(v_1), f_2(v_2)).$$

Since $f_1$ and $f_2$ are bijections, so is $f$.

For any $(v_1, v_2)(u_1, u_2) \in E(G_1 \square G_2)$, either $v_1 = u_1$ with $v_2u_2 \in E(G_2)$ or else $v_1u_1 \in E(G_1)$ with $v_2 = u_2$. For the former case, $f_1(v_1) = f_1(u_1)$ with $f_2(v_2)f_2(u_2) \in E(G_2)$ and so $f(v_1, v_2)f(u_1, u_2) \in E(G_1 \square G_2)$. For the latter case, $f_1(v_1)f_1(u_1) \in E(G_1)$ with $f_2(v_2) = f_2(u_2)$ and so $f(v_1, v_2)f(u_1, u_2) \in E(G_1 \square G_2)$. Therefore $f$ preserves adjacency and so is an isomorphism of $G_1 \square G_2$.

For any vertex $(v_1, v_2)$ in the graph $G_1 \square G_2$, since $f_1$ and $f_2$ are of order 2, we have $(f_1 \circ f_1)(v_1) = v_1$ and $(f_2 \circ f_2)(v_2) = v_2$, i.e., $f_1(f_1(v_1)) = v_1$ and $f_2(f_2(v_2)) = v_2$. Therefore,

$$(f \circ f)(v_1, v_2) = f(f_1(v_1), f_2(v_2)) = (f_1(f_1(v_1)), f_2(f_2(v_2))) = (v_1, v_2).$$

Also, if $G_1 \square G_2$ has more than one vertex, then either $G_1$ or $G_2$ has more than one vertex; thus, either $f_1$ or $f_2$ is not an identity function and so is $f \circ f$. Hence $f$ is of order 2.

Finally, we check the last two conditions (S1)-(S2) / (S1)-(SS2) / (PS1)-(S2) for $f$. Consider all vertices $(v_1, v_2)$ of $G_1 \square G_2$.

First consider (S1) and (PS1). Notice that $f(v_1, v_2) = (v_1, v_2)$ is the same as $f_1(v_1) = v_1$ and $f_2(v_2) = v_2$. This is possibly only when $G_1$ and $G_2$ are both pseudo- symmetric, and $v_1$ is the only exceptional vertex $v_1^*$ in (PS1) for $G_1$ and $v_2$ is the only exceptional vertex $v_2^*$ in (PS1) for $G_2$. Therefore, (PS1) holds for $G_1 \square G_2$ with the only exceptional vertex $(v_1^*, v_2^*)$ when $G_1$ and $G_2$ are both pseudo-symmetric. And for the other cases (S1) holds for $G_1 \square G_2$.

Next consider (S2) and (SS2). Notice that $(v_1, v_2)f(v_1, v_2) \in E(G_1 \square G_2)$ is the same as either $v_1 = f_1(v_1)$ with $v_2 f_2(v_2) \in E(G_2)$ or else $v_1 f_1(v_1) \in E(G_1)$ with $v_2 = f_2(v_2)$. The former case is possibly only when $G_1$ is pseudo-symmetric with $v_1$ is the only exceptional vertex $v_1^*$ in (PS1) for $G_1$, and $G_2$ is semi-symmetric with $v_2^*, f_2(v_2^*)$ is the only mapping pair in (SS2) for $G_2$. Therefore, (SS2) holds for $G_1 \square G_2$ with the only exceptional mapping pair $(v_1^*, v_2^*), f(v_1^*, v_2^*)$ when $G_1$ is pseudo-symmetric and $G_2$ is semi-symmetric. Similarly, for the latter case, (SS2) holds for $G_1 \square G_2$ with the only exceptional mapping pair $(v_1^*, v_2^*), f(v_1^*, v_2^*)$ when $G_1$ is semi-symmetric and $G_2$ is pseudo-symmetric. And for the other cases (S2) holds for $G_1 \square G_2$.

These complete the proof of the theorem. □

Now, using the results above, we are going to solve the star-removing game on $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$, namely the $r$-dimensional grid. But before that, we have totake notice of the following properties of $P_n$.

**Corollary 4.10.** *$P_n$ is* semi-symmetric *when $n$ is even, and* pseudo-symmetric *when $n$ is odd.*

*Proof.* Consider the function $f : V(P_n) \longrightarrow V(P_n)$ defined as follows: $f(v_i) = v_{n-i}$, where $1 \leq i \leq n$.

Then, when $n$ is even, it is easy to see that $f$ is a semi-symmetric function of $P_n$, and $P_n$ is therefore semi-symmetric.

When $n$ is odd, it is easy to see that $f$ is a pseudo-symmetric function of $P_n$,and $P_n$ is therefore pseudo-symmetric. □

Then, using Corollary 4.10, I shall now solve the star-removing game on the $r$-dimensional grid, as follows.

**Corollary 4.11. ($r$-dimensional grid)** *In the* star-*game, $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$ isan N-position if and only if at most one of $n_1, n_2, \ldots, n_r$ is even, or if $n_1 = n_2$ =
$\ldots = n_r = 1$.*

*Proof.* By Corollary 4.10, if at least two of $n_1, n_2, \ldots, n_r$ is even, then $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$ is symmetric (see Theorem 4.9) and therefore a P-position (see Theorem 4.8).

On the other hand, if at most one of $n_1, n_2, \dots, n_r$ is even, then $P_{n_1} \square P_{n_2} \square \dots \square P_{n_r}$ is either pseudo-symmetric or semi-symmetric (see Theorem 4.9). Then since the only pseudo-symmetric graphs that are P-positions are those without edges, and the rest of pseudo-symmetric graphs and semi-symmetric graphs are N-position, $P_{n_1} \square P_{n_2} \square \dots \square P_{n_r}$ is a P-position if $n_1 = n_2 = \dots = n_r = 1$, and otherwise an N-position.

□

# 5 Results

In this section, I shall summarize the results I have above.

Here are the results on the trail-removing game.

(1) $g_{\text{trail}}(P_n) = n - 1$ for $n \in \mathbb{N}$. (Proposition 3.1)

(2) $g_{\text{trail}}(C_n) = n$ for $n \geq 3$ and $n \in \mathbb{N}$. (Proposition 3.2)

(3) $g_{\text{trail}}(S_n) = r$ for all $n \in \mathbb{N}$, where $r \in \{0, 1, 2\}$ and $r \equiv n \pmod 3$. (Proposition 3.3)

(4) If $n$ is a positive integer, then $g_{\text{trail}}(S_{n,1}) = r + 2$, where $r \in \{1, 2, 3\}$ and $r \equiv n \pmod 3$.

As for all integers $n_1, n_2 \geq 2$,

if $n_1 \equiv n_2 \equiv 0 \pmod 3$ or $n_1 \equiv n_2 \equiv 2 \pmod 3$, then $g_{\text{trail}}(S_{n_1,n_2}) = 1$;

if $n_1 \equiv n_2 \equiv 1 \pmod 3$, then $g_{\text{trail}}(S_{n_1,n_2}) = 2$;

if ($n_1 \equiv 1 \pmod 3$, $n_2 \equiv 2 \pmod 3$) or ($n_1 \equiv 2 \pmod 3$, $n_2 \equiv 1 \pmod 3$), then $g_{\text{trail}}(S_{n_1,n_2}) = 4$;

if ($n_1 \equiv 0 \pmod 3$, $n_2 \equiv 1 \pmod 3$) or ($n_1 \equiv 1 \pmod 3$, $n_2 \equiv 0 \pmod 3$), then $g_{\text{trail}}(S_{n_1,n_2}) = 5$;

if ($n_1 \equiv 0 \pmod 3$, $n_2 \equiv 2 \pmod 3$) or ($n_1 \equiv 2 \pmod 3$, $n_2 \equiv 0 \pmod 3$), then $g_{\text{trail}}(S_{n_1,n_2}) = 6$. (Lemma 3.5 and Theorem 3.6)

(5) For all integers $k \geq 3$, an upper bound of the Grundy number of a $k$-staris $k^2 + 4k - 7$. (Theorem 3.9)

(6) Nice graphs are all N-positions. (Theorem 3.11)

(7) The join $G \nabla H$ of two nonempty graphs $G$ and $H$ is an N-position, except when ($G = K_1$ and $H = mK_1$) or ($G = mK_1$ and $H = K_1$), where $m$ is a positive integer with $m \equiv 0 \pmod 3$. (Theorem 3.14)

(8) A complete $k$-partite graph is a P-position if and only if $k = 1$ or ($k = 2, n_1 = 1, n_2 \equiv 0 \pmod 3$) or ($k = 2, n_2 = 1, n_1 \equiv 0 \pmod 3$). (Corollary 3.16)

(9) If $G$ and $H$ are connected graphs with at least two vertices, then $G \square H$ is an N-position if any one of the following conditions holds.

(i) $G$ is a fully nice graph. A special case is when $V(G) = V_e(G)$.

(ii) $V(H) = V_o(H)$.

(iii) $G$ has an edge $xy$ with $x, y \in V_e(G)$ and $|V(H)|$ is even.

(Theorem 3.19)

(10) The Cartesian product of a fully nice graph and any connected graphwith at least 2 vertices must also be a fully nice graph. (Theorem 3.19)

(11) For $n, m \geq 2$, the 2-dimensional grid $P_n \Box P_m$ are fully nice, except for $P_3 \Box P_3$, which is only nice. (Lemma 3.21)

(12) $P_3 \Box P_3 \Box P_3$ is a fully nice graph. (Lemma 3.22)

(13) If $n_1, n_2, \ldots, n_r \geq 2$, then the $r$-dimensional grid $P_{n_1} \Box P_{n_1} \Box \ldots \Box P_{n_r}$ isan N-position. (Corollary 3.23)

Notice that by (11) I have solved the original game of squayles, which is a trail-removing game on $P_4 \Box P_5$.

Instead of computing the Grundy numbers of graphs in the star-removing game, I proposed winning strategies in order to determine the player with a winning strategy. My results are as follows.

(1) $P_n$ is an N-position for all integers $n \geq 2$ and a P-position when $n = 1$.(Proposition 4.2)

(2) $C_n$ is a P-position for all integers $n \geq 3$. (Proposition 4.3)

(3) $S_{n,m}$ is an N-position for all $n, m \in \mathbb{N}_0$. (Proposition 4.4)

(4) *symmetric* graphs are *P-positions*, *semi-symmetric* graphs are *N-positions*, and *pseudo-symmetric* graphs are *N-positions* except for the ones without edges. (Theorem 4.8)

(5) The Cartesian product of symmetric graphs, semi-symmetric graphs and pseudo-symmetric graphs with one another follow a special rule, which is summarized in the table below.

| □ | Symmetric | Semi-symmetric | Pseudo-symmetric |
|---|---|---|---|
| Symmetric | Symmetric | Symmetric | Symmetric |
| Semi-symmetric | Symmetric | Symmetric | Semi-symmetric |
| Pseudo-symmetric | Symmetric | Semi-symmetric | Pseudo-symmetric |

Figure 6: The Cartesian product of special graphs.

(6) $P_n$ is semi-symmetric when $n$ is even, and pseudo-symmetric when $n$ is odd. (Corollary 4.10)

(7) The $r$-dimensional grid $P_{n_1} \Box P_{n_2} \Box \ldots \Box P_{n_r}$ is an N-position if and only ifat most one of $n_1, n_2, \ldots, n_r$ is even or if $n_1 = n_2 = \ldots = n_r = 1$. (Corollary 4.11)

# 6 Conclusions

In this research, I start by giving two basic but important lemmas; afterwards, I analyze special graphs in the trail-removing game and the star-removing game, and aim to give a more generalized result.

In the trail-removing game, I successfully compute the Grundy numbers of special graphs such as paths, cycles, stars, and double stars, I also give an upper bound on $k$-stars in general. Afterwards, I define a new kind of graphs, known as *nice graphs*. Using a non-constructive proof, I am able to determine that all nice graphs are N-positions. Using this, I give a solution of the join product of any two non-empty graphs, solving the trail-removing game on complete $k$-partite graphs in the process. As for the Cartesian product of graphs, I give a solution of the Cartesian product of two non-empty graphs that satisfy certain conditions, and discover that the Cartesian product of a fully nice graph and any other connected graph with at least 2 vertices is also fully nice. By this, I am able to completely solve the trail-removing game on the $r$-dimensional grid $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$.

As for the star-removing game, my greatest achievement is, without a doubt, the results I get by introducing a concept known as *symmetry*. Using this concept, I am able to give more generalized results that can be used to analyze the star- removing game on Cartesian products of certain graphs effectively. Using these results, I successfully determine the winner on the $r$-dimensional grid $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$.

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## 【評語】010042

本作品主要是探討在圖形上的兩種 nim game (捻)。一種是兩個玩家輪流在一個簡單圖上,移走一個路徑(trail);另一種是兩個玩家輪流在簡單圖上移走星子圖(star),取走最後一個邊的玩家得勝。作者在這些圖上定義並計算某些簡單例子的 Grundy number。在這樣兩種特殊取法的設定之下,Grundy number 的計算,基本上都可以採用數學歸納法去完成(Prop. 3.1 & Theorem 3.9)。至於作者新定義的 nice graphs,會使得 nice graph 拿掉後變成 independent set,所以 Theorem 3.11 也是一個不難預見的結果。第四節基本上和之前的作品結果一樣,只是給了完整的證明。本作品數學不難,但是作品說明書寫得不錯,是一個有趣的題目。