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作品名稱　Forming Polygons with Broken Pick-up Chocolate Bars and Spaghetti Noodles

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作者簡介

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是一名不久前考完學測的高三生，目前就讀台北市立建國高級中學，如果學分不夠，明年也會繼續就讀台北市立建國高級中學。在能應屆畢業的前提下，楊承義決定在高中的最後一年奮發向上，以優秀的李承翰同學作為楷模，積極往科展和競賽努力，讓自己的數學旅程不留遺憾。

「巧克力棒的極限，是義大利麵，也是我對你的思念。」楊承義，2023

李承翰

是一名不久前考完學測的高三生，目前就讀台北市立建國高級中學。從小對數學抱持著巨大的熱忱，也終於在數奧的領域中攻下一席之地。研究數學的過程雖然往往是孤獨的，但獲得的反饋猶如明燈讓我在這條道路上緊握初衷，努力地以一筆一畫刻畫出我的夢想，在這個方向上發光發熱。

「義大利麵的極限，是我對你的欣羨。」李承翰，2023

Abstract

”The broken pick-up sticks problem” is proposed by T. Kyle Petersen and Bridget Eileen Tenner in 2020. We solve the problem by considering the discrete version using random variables, and the limit behaviour of the discrete version gives us a combinatorial solution to the original problem. We also evaluate the probabilities of the triangles formed by the broken/pick-up sticks satisfying some specific geometric conditions with various techniques, including calculus and elementary number theory.

1 Introduction

Below is the problem proposed by T. Kyle Petersen and Bridget Eileen Tenner in their article.

Problem 1 (The broken pick-up sticks problem)**.** Stick (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m sticks chosen from a uniform distribution of stick lengths. For each i, break the ith stick into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. What is the probability that the resulting k pieces form a k-gon?

The stick in the problem is also sometimes referred to as ”spaghetti”. In this article, we call the problem as ”spaghetti problem”. In order to solve the spaghetti problem, we follow the way of the two authors and take a discrete approach to the problem, which brings us to 1-dimensional chocolate bars, which is also referred to as "brick". For a chocolate bar with the length of $l \in \mathbb{N}$, the $l-1$ integer points are the only available cut points.

Problem 2 (The broken pick-up chocolate bars problem)**.** Chocolate bar (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m chocolate bars chosen from a uniform distribution of bar lengths on $\{1, 2, \ldots, n\}$. For each i, break the ith bar into λ_i pieces by choosing $\lambda_i - 1$ cut points (integer points) independently at random. What is the probability that the resulting k pieces form a k-gon?

In order to obtain the result of spaghetti, we wanted to consider the limit as $n \to \infty$ of the result of chocolate bar. However, when the chocolate bar with length of 1 is picked, it cannot be further divided since there are no integer points in it, and that makes it different from spaghetti noodle. Therefore we introduce our new product *Premium Chocolate Bar*. For a *Premium Chocolate Bar* of length l, $1/(2s), 2/(2s), \ldots, 2l - 1/(2s)$ are the only available cut points, i.e. there are $2sl - 1$ available cut points evenly distributed on it, where s is a sufficiently large integer.

Problem 3 (The broken pick-up premium chocolate bars problem)**.** Premium chocolate bar (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m premium chocolate bars chosen from a uniform distribution of bar lengths on $\{1, 2, \ldots, n\}$. For each i, break the ith bar into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. What is the probability that the resulting k pieces form a k-gon?

We call the solutions to these three problems $P(\lambda)$, $P_n(\lambda)$, and $P_{s,n}(\lambda)$, respectively.

2 Prerequisites

Proposition 1. $\sum_{i=0}^{k}\binom{n+i}{r}=\binom{n+k+1}{r+1}-\binom{n}{r+1}$ for $r\leq n$.

Proposition 2. $\sum_{i=a}^{n-b}\binom{i}{a}\binom{n-i}{b}=\binom{n+1}{a+b+1}$ for $a+b\leq n$.

Proposition 3. Fix a positive integer $k\geq 3$ and a k-element multiset S of positive numbers. There exists a (convex) polygon whose side lengths are the elements of S if and only if $x<\|S\|-x$ for each $x\in S$. ($\|S\|$ denotes the sum of the elements of S.)

Proposition 4. Given $n,k,p\in\mathbb{N}$, where $n\geq k$ and $n-(k-1)\geq p\geq n/2$, then there are $k\binom{n-p}{k-1}$ natural solution to

$$n=a_1+a_2+\cdots+a_k \quad \text{and } \exists\ i \text{ s.t. } a_i\geq p.$$

Proposition 5. For a super big m and a given j, $\binom{m}{j}$ can be approximated by $\frac{1}{j!}m^j$.

Proposition 6. On a plane, a point P with barycentric coordinate (a,b,c) with respect to the triangle ABC where $A=(0,\sqrt{3}),B=(-1,0),C=(1,0)$ has Cartesian coordinate

$$\begin{cases}(\frac{(c+b)^2}{c-b},\sqrt{3}a) & \text{if } b\neq c,\\ (0,\sqrt{3}a) & \text{if } b=c.\end{cases}$$

3 A simple case of chocolate bars problem

First we take a look at the "chocolate bars problem". We start off our adventure by picking up 2 chocolate bars, then we will break them into $2, 1$ pieces, respectively. Then we will see if we can get something out of it.

Let X and Y be the random variables representing the lengths of the two bars that we pick, respectively. We choose a uniform random integer point on the first bar where we break the bar into two pieces. Let X_1 and X_2 be the random variables representing the lengths of the two pieces, so $X = X_1 + X_2$. Notice that X must be greater than 1 or it cannot be broken into two pieces with integer lengths, and there is at most one piece among X_1, X_2, Y whose length is greater or equal to $(X+Y)/2$. By one of our propositions,

$$P_n(2,1) = 1 - \mathbb{P}(X=1) - \mathbb{P}\left((X_1 \geq \frac{X+Y}{2} \text{ or } X_2 \geq \frac{X+Y}{2} \text{ or } Y \geq \frac{X+Y}{2}) \text{ and } X \geq 2 \right).$$

The probability that $X_1 \geq (X+Y)/2$ under the condition that $X \neq 1$ is

$$\begin{aligned}\mathbb{P}(X_1 \geq \frac{X+Y}{2} \text{ and } X \geq 2) &= \sum_{\substack{2 \leq a \leq n \\ 1 \leq b \leq n}} \mathbb{P}(X_1 \geq \frac{X+Y}{2} \text{ and } X = a, Y = b) \\ &= \sum_{\substack{2 \leq a \leq n \\ 1 \leq b \leq n}} \left(\mathbb{P}(X_1 \geq \frac{a+b}{2} \mid X = a, Y = b) \cdot \mathbb{P}(X = a, Y = b) \right).\end{aligned}$$

We evaluate $\mathbb{P}(X_1 \geq (X+Y)/2 \text{ and } X \geq 2)$. Since we are choosing bars from an uniform distribution on $\{1, 2, \ldots, n\}$, for all $a \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}(X = a) = \frac{1}{n},$$

also for all $a, b \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}(X = a, Y = b) = \frac{1}{n^2}.$$

We choose an integer point X_1 on the first bar, uniformly at random. For all $a_1 \in \{1, 2, \ldots, a - 1\}$,

$$\mathbb{P}(X_1 = a_1 \mid X = a \neq 1) = \frac{1}{a-1}.$$

Subsequently, given $h \in \mathbb{N}$,

$$\mathbb{P}(X_1 \geq h \mid X = a \neq 1) = \begin{cases} \dfrac{a-h}{a-1} & \text{if } h < a, \\ 0 & \text{if } h \geq a. \end{cases}$$

Notice that means $a < 3$ or $b > a-2$ implies $\mathbb{P}(X_1 \geq \frac{a+b}{2} \mid X = a \neq 1) = 0$, since

$$\frac{a+1}{2} \leq \frac{a+b}{2} \; (\leq \lceil \frac{a+b}{2} \rceil \leq X_1) \leq a-1 \iff 3 \leq a \leq n \text{ and } 1 \leq b \leq a-2.$$

Therefore

$$\begin{aligned}
&\sum_{\substack{2\leq a\leq n\\1\leq b\leq n}} \mathbb{P}(X_1 \geq \frac{a+b}{2} \text{ and } X = a, Y = b)\\
&\sum_{\substack{2\leq a\leq n\\1\leq b\leq n}} \left(\mathbb{P}(X_1 \geq \frac{a+b}{2} \mid X = a, Y = b) \cdot \mathbb{P}(X = a, Y = b)\right)\\
=&\frac{1}{n^2} \sum_{3\leq a\leq n} \sum_{1\leq b\leq a-2} \mathbb{P}(X_1 \geq \lceil \frac{a+b}{2} \rceil \mid X = a, Y = b)\\
=&\frac{1}{n^2} \sum_{3\leq a\leq n} \sum_{1\leq b\leq a-2} \frac{a - \lceil (a+b)/2 \rceil}{a-1}\\
=&\frac{1}{n^2} \sum_{3\leq a\leq n} \sum_{1\leq b\leq a-2} \frac{\lfloor (a-b)/2 \rfloor}{a-1} = \frac{1}{n^2} \sum_{3\leq a\leq n} \sum_{1\leq b\leq a-1} \frac{\lfloor (a-b)/2 \rfloor}{a-1}\\
=&\frac{1}{n^2} \sum_{a=3}^{n} \left(\sum_{b=1}^{a-1} \frac{(a-b)/2}{a-1} - \sum_{\substack{b=1,\\2\nmid a-b}}^{a-1} \frac{1/2}{a-1} \right)\\
=&\frac{1}{n^2} \left(\frac{1}{8}n^2 - \frac{1}{8}n - \frac{1}{8}(H_{n-1} + H'_{n-1}) \right)\\
=&\frac{1}{8} - \frac{1}{8n} - \frac{1}{8n^2}(H_{n-1} + H'_{n-1}),
\end{aligned}$$

since

$$\begin{aligned}
\sum_{a=3}^{n} \sum_{b=1}^{a-1} \frac{(a-b)/2}{a-1} &= \sum_{a=3}^{n} \frac{a}{4} = \frac{1}{8}(n^2 + n - 6),\\
\sum_{a=3}^{n} \sum_{\substack{b=1,\\2\nmid a-b}}^{a-1} \frac{1/2}{a-1} &= \frac{n}{4} + \frac{H_{n-1} + H'_{n-1}}{8} - \frac{3}{4}, \text{ where } H_{n-1} = \sum_{i=1}^{n-1} \frac{1}{i}, H'_{n-1} = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i}.
\end{aligned}$$

Similarly,

$$\sum_{\substack{2\le a\le n\\1\le b\le n}} \mathbb{P}(X_2 \ge \frac{a+b}{2} \text{ and } X=a, Y=b)$$
$$= \frac{1}{n^2}\sum_{3\le a\le n}\sum_{1\le b\le a-2}\frac{\lfloor (a-b)/2\rfloor}{a-1} = \frac{1}{8}-\frac{1}{8n}-\frac{1}{8n^2}(H_{n-1}+H'_{n-1}),$$
$$\sum_{\substack{2\le a\le n\\1\le b\le n}} \mathbb{P}(Y \ge \frac{a+b}{2} \text{ and } X=a, Y=b) = \frac{1}{n^2}\sum_{2\le b\le n}\sum_{2\le a\le b} 1 = \frac{1}{2}-\frac{1}{2n}.$$

To sum up,

$$P_n(2,1) = 1-\mathbb{P}(X=1)-\mathbb{P}\left((X_1 \ge \frac{X+Y}{2} \text{ or } X_2 \ge \frac{X+Y}{2} \text{ or } Y \ge \frac{X+Y}{2}) \text{ and } X\ge 2\right)$$
$$= \frac{1}{4}-\frac{1}{4n}+\frac{1}{4n^2}(H_{n-1}+H'_{n-1}).$$

4 Solving the problems in general

Let's solve the chocolate bars problem in general!

Theorem 1. Chocolate bar (λ). *Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m chocolate bars chosen from a uniform distribution on $\{1, 2, \ldots, n\}$ of chocolate lengths. For each i, break the ith bar into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. Then the probability that the resulting k pieces form a k-gon is*

$$P_n(\lambda) = \frac{\prod_{i=1}^{m}(n - \lambda_i + 1)}{n^m} - \sum_{i=1}^{m} \frac{\lambda_i}{n^m} \sum_{a=k+\lambda_i-2}^{n} \sum_{t=k-\lambda_i}^{a-2\lambda_i+2} \frac{\binom{\lfloor (a-t)/2 \rfloor}{\lambda_i - 1}}{\binom{a-1}{\lambda_i - 1}} \binom{t-k+\lambda_i+m-2}{m-2}.$$

Proof. For $i = 1, 2, \ldots, m$, let X_i be the random variable representing the length of the ith chocolate bar that we pick. Notice that for $i = 1, 2, \ldots, m$, X_i must be greater or equal λ_i in order to be broken λ_i pieces with integer lengths. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \lambda_m$, let $X_{i,j}$ be the random variable representing the length of the jth pieces broken from the ith chocolate bar, so $X_i = X_{i,1} + X_{i,2} + \cdots + X_{i,\lambda_i}$.

$$\begin{aligned}
P_n(\lambda) &= 1 - \mathbb{P}(\exists\ i \text{ s.t. } X_i < \lambda_i) - \sum_{i=1}^{m} \sum_{j=1}^{\lambda_i} \mathbb{P}(X_{i,j} \geq \frac{1}{2} \sum_{l=1}^{n} X_l \text{ and } \forall\ u = 1, 2, \ldots, m, X_u \geq \lambda_u) \\
&= \frac{\prod_{i=1}^{m}(n - \lambda_i + 1)}{n^m} - \sum_{i=1}^{m} \sum_{j=1}^{\lambda_i} \mathbb{P}(X_{i,j} \geq \frac{1}{2} \sum_{l=1}^{n} X_l \text{ and } \forall\ u = 1, 2, \ldots, m, X_u \geq \lambda_u).
\end{aligned}$$

Given $a_1, a_2, \ldots, a_m$ and $p = a_1 + \cdots + a_m$. WLOG, let's take $\mathbb{P}(X_{1,1} \geq \frac{1}{2} \sum_{l=1}^{n} X_l \text{ and } \forall\ u = 1, 2, \ldots, m, X_u \geq \lambda_u)$ for example.

$$\begin{aligned}
&\mathbb{P}(X_{1,1} \geq \frac{1}{2} \sum_{l=1}^{n} X_l \text{ and } \forall\ u = 1, 2, \ldots, m, X_u \geq \lambda_u) \\
&= \sum_{\lambda_i \leq a_i \leq n\ \forall\ i} \mathbb{P}(X_{1,1} \geq \frac{p}{2} \text{ and } \forall\ j = 1, 2, \ldots, m, X_j = a_j).
\end{aligned}$$

Let $t_1 = a_2 + a_3 + \cdots + a_m$. Notice that $a_1 < k + \lambda_1 - 2$ or $t > a_1 - 2\lambda_1 + 2$ implies $X_{1,1} < p/2$

since

$$\frac{a_1+\lambda_2+\dots\lambda_m}{2} \le \frac{a_1+a_2+\cdots+a_m}{2} (\le \lceil \frac{a_1+a_2+\cdots+a_m}{2} \rceil \le X_{1,1}) \le a_1-(\lambda_1-1)$$

$$\iff a_1 \ge k+\lambda_1-2 \text{ and } t \le a_1-2\lambda_1+2.$$

And we are choosing bars from an uniform distribution on $\{1,2,\dots,n\}$, so given $a_1 \in \{1,2,\dots,n\}$ and $t_1 \in \{k-\lambda_1, k-\lambda_1, \dots, a_1-2\lambda_1+2\}$,

$$\mathbb{P}(X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1) = \frac{1}{n^m}\binom{t_1-k+\lambda_1+m-2}{m-2}.$$

We randomly choose $\lambda_1 - 1$ integer point on X_1, given $a_{1,1} \in \{1,2,\dots,a_1-(\lambda-1)\}$,

$$\mathbb{P}(X_{1,1} = a_{1,1} \mid X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1) = \binom{a_1-a_{1,1}-1}{\lambda_1-2} / \binom{a_1-1}{\lambda_1-1}.$$

Subsequently, given $h \in \mathbb{N}$,

$$\mathbb{P}(X_{1,1} \ge h \mid X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1 \ge \lambda_1) = \begin{cases} \binom{a_1-h}{\lambda_1-1} / \binom{a_1-1}{\lambda_1-1} & \text{if } h \le a_1-(\lambda_1-1), \\ 0 & \text{if } h > a_1-(\lambda_1-1). \end{cases}$$

Now we evaluate

$$
\begin{aligned}
&\sum_{\lambda_i \le a_i \le n \ \forall \ i} \mathbb{P}(X_{1,1} \ge \frac{1}{2}\sum_{l=1}^{n} X_l \text{ and } X_j = a_j \ \forall \ j = 1, 2, \ldots, m) \\
&= \sum_{\substack{\lambda_1 \le a_1 \le n \\ k-\lambda_1 \le t_1 \le (m-1)n}} \mathbb{P}\left(X_{1,1} \ge \frac{p}{2} \text{ and } (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)\right) \\
&= \sum_{\substack{\lambda_1 \le a_1 \le n \\ k-\lambda_1 \le t_1 \le (m-1)n}} \left(\mathbb{P}(X_{1,1} \ge \frac{p}{2} \mid (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)) \cdot \mathbb{P}(X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)\right) \\
&= \sum_{a_1 = k+\lambda_1 - 2}^{n} \sum_{t_1 = k - \lambda_1}^{a_1 - 2\lambda_1 + 2} \left(\mathbb{P}(X_{1,1} \ge \frac{p}{2} \mid (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)) \cdot \frac{1}{n^m} \binom{t_1 - k + \lambda_1 + m - 2}{m-2} \right) \\
&= \sum_{a_1 = k+\lambda_1 - 2}^{n} \sum_{t_1 = k - \lambda_1}^{a_1 - 2\lambda_1 + 2} \left(\frac{\binom{a_1 - \lceil p/2 \rceil}{\lambda_1 - 1}}{\binom{a_1 - 1}{\lambda_1 - 1}} \cdot \frac{1}{n^m} \binom{t_1 - k + \lambda_1 + m - 2}{m-2} \right) \\
&= \frac{1}{n^m} \sum_{a_1 = k+\lambda_1 - 2}^{n} \sum_{t_1 = k - \lambda_1}^{a_1 - 2\lambda_1 + 2} \left(\frac{\binom{\lfloor (a_1 - t_1)/2 \rfloor}{\lambda_1 - 1}}{\binom{a_1 - 1}{\lambda_1 - 1}} \cdot \binom{t_1 - k + \lambda_1 + m - 2}{m-2} \right)
\end{aligned}
$$

By the generality, we obtain

$$
P_n(\lambda) = \frac{\prod_{i=1}^{m} (n - \lambda_i + 1)}{n^m} - \sum_{i=1}^{m} \frac{\lambda_i}{n^m} \sum_{a_i = k+\lambda_i - 2}^{n} \sum_{t_i = k - \lambda_i}^{a_i - 2\lambda_i + 2} \frac{\binom{\lfloor (a_i - t_i)/2 \rfloor}{\lambda_i - 1}}{\binom{a_i - 1}{\lambda_i - 1}} \binom{t_i - k + \lambda_i + m - 2}{m-2}.
$$

□

Below are the two original problems. We apply our formula on them. As expected, the results are identical to the T. Kyle Petersen and Bridget Eileen Tenners'.

Corollary 1 (Pick-up bricks problem)**.** Let $n \geq 1, k \geq 3$ be positive integers. Select k sticks, each of which has length chosen from the uniform distribution on $\{1, 2, \ldots, n\}$. The probability that the resulting k sticks form a k-gon is

$$\begin{aligned}
P_n(\underbrace{1, 1, \ldots, 1}_{k}) &= 1 - \sum_{i=1}^{k} \frac{1}{n^k} \sum_{a=k-1}^{n} \sum_{t=k-1}^{a} \frac{\binom{\lfloor \frac{a-t}{2} \rfloor}{0}}{\binom{a-1}{0}} \binom{t-1}{k-2} \\
&= 1 - \sum_{i=1}^{k} \frac{1}{n^k} \sum_{a=k-1}^{n} \sum_{t=k-1}^{a} \binom{t-1}{k-2} \\
&= 1 - \sum_{i=1}^{k} \frac{1}{n^k} \sum_{a=k-1}^{n} \binom{a}{k-1} \\
&= 1 - \sum_{i=1}^{k} \frac{1}{n^k} \binom{n+1}{k} \\
&= 1 - \frac{k}{n^k} \binom{n+1}{k}.
\end{aligned}$$

Corollary 2 (Broken bricks problem)**.** Let $n \geq k \geq 3$ be positive integers, and consider a stick of length n. Pick $k{-}1$ distinct interior integer points on the stick, independently and at random, and cut the stick at these $k{-}1$ points. The probability that the resulting k pieces form a k-gon is

$$\begin{aligned}
1 - \frac{k}{1} \sum_{a=n}^{n} \sum_{t=0}^{0} \frac{\binom{\lfloor \frac{a-t}{2} \rfloor}{k-1}}{\binom{a-1}{k-1}} \binom{t-1}{-1} &= 1 - \frac{k}{1} \frac{\binom{\lfloor \frac{n}{2} \rfloor}{k-1}}{\binom{n-1}{k-1}} \binom{-1}{-1} \\
&= 1 - \frac{k \binom{\lfloor \frac{n}{2} \rfloor}{k-1}}{\binom{n-1}{k-1}}.
\end{aligned}$$

Theorem 2 (Premium chocolate bar)**.** Premium chocolate bar (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m premium chocolate bars chosen from a uniform distribution on $\{1, 2, \ldots, n\}$ of premium chocolate bar lengths. For a *Premium Chocolate Bar* of length l, $\{\frac{1}{2s}, \frac{2}{2s}, \ldots, l - \frac{1}{2s}\}$ are the only available cut points, i.e. there are $2sl - 1$ available cut points evenly distributed on it, where s is a sufficiently large integer. For each i, break the ith bar into λ_i pieces by choosing $\lambda_i - 1$ available cut points independently at random. Then the probability that the resulting k pieces form a k-gon is

$$P_{s,n}(\lambda) = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{n^m} \sum_{a_i=1}^{n} \sum_{t=1}^{a_i - \frac{\lambda_i - 1}{2s}} \frac{\binom{s(a_i - t)}{\lambda_i - 1}}{\binom{2sa_i - 1}{\lambda_i - 1}} \binom{t-1}{m-2}.$$

Proof. For $i = 1, 2, \ldots, m$, let X_i be the random variable representing the length of the ith chocolate bar that we pick. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \lambda_m$, let $X_{i,j}$ be the random variable representing the length of the jth pieces broken from the ith chocolate bar, so $X_i = X_{i,1} + X_{i,2} + \cdots + X_{i,\lambda_i}$.

$$P_{s,n}(\lambda) = 1 - \sum_{i=1}^{m} \sum_{j=1}^{\lambda_i} \mathbb{P}(X_{i,j} \geq \frac{1}{2} \sum_{l=1}^{n} X_l).$$

Given $a_1, a_2, \ldots, a_m$ and $p = a_1 + \cdots + a_m$. Let's take $\mathbb{P}(X_{1,1} \geq \frac{1}{2} \sum_{l=1}^{n} X_l$ and $\forall\, u = 1, 2, \ldots, m, X_u \geq \lambda_u)$ for example.

$$\mathbb{P}(X_{1,1} \geq \frac{1}{2} \sum_{l=1}^{n} X_l) = \sum_{1 \leq a_1, \ldots, a_m \leq n} \mathbb{P}(X_{1,1} \geq \frac{p}{2} \text{ and } \forall\, j = 1, 2, \ldots, m, X_j = a_j)$$

Since we're choosing bars from an uniform distribution on $\{1, 2, \ldots, n\}$, given $a_1 \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}(X_1 = a_1) = \frac{1}{n}.$$

We randomly choose $\lambda_1 - 1$ breakable point on X_1, given $a_{1,1} \in \{\frac{1}{2s}, \frac{2}{2s}, \ldots, a_1 - \frac{\lambda_1 - 1}{2s}\}$,

$$\mathbb{P}(X_{1,1} = a_{1,1} \mid X_1 = a_1) = \binom{2s(a_1 - a_{1,1}) - 1}{\lambda_1 - 2} \Big/ \binom{2sa_1 - 1}{\lambda_1 - 1}.$$

Subsequently, given $h \in \mathbb{N}$,

$$\mathbb{P}(X_{1,1} \geq h \mid X_1 = a_1) = \begin{cases} \binom{2s(a_1-h)}{\lambda_1 - 1} / \binom{2sa_1 - 1}{\lambda_1 - 1} & \text{if } h \leq a_1 - (\lambda_1 - 1)/(2s), \\ 0 & \text{if } h > a_1 - (\lambda_1 - 1)/(2s). \end{cases}$$

Let $t_1 = p - a_1$. Now we evaluate

$$\begin{aligned}
&\sum_{1 \leq a_1, \ldots, a_m \leq n} \mathbb{P}(X_{1,1} \geq \frac{1}{2} \sum_{l=1}^{n} X_l \text{ and } X_j = a_j \; \forall \; j = 1, 2, \ldots, m) \\
&= \sum_{1 \leq a_1 \leq n} \sum_{m-1 \leq t_1 \leq (m-1)n} \mathbb{P}\left(X_{1,1} \geq \frac{p}{2} \text{ and } (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1) \right) \\
&= \sum_{1 \leq a_1 \leq n} \sum_{m-1 \leq t_1 \leq (m-1)n} \left(\mathbb{P}(X_{1,1} \geq \frac{p}{2} \mid (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)) \cdot \mathbb{P}(X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1) \right) \\
&= \sum_{a_1=1}^{n} \sum_{t_1=m-1}^{a_1-(\lambda_1-1)/(2s)} \left(\mathbb{P}(X_{1,1} \geq \frac{p}{2} \mid (X_1 = a_1 \text{ and } \sum_{l=2}^{m} X_l = t_1)) \cdot \frac{1}{n^m} \binom{t_1 - 1}{m-2} \right) \\
&= \sum_{a_1=1}^{n} \sum_{t_1=m-1}^{a_1-(\lambda_1-1)/(2s)} \left(\frac{\binom{2s(a_1 - \lceil p/2 \rceil)}{\lambda_1 - 1}}{\binom{2sa_1 - 1}{\lambda_1 - 1}} \cdot \frac{1}{n^m} \binom{t_1 - 1}{m-2} \right) \\
&= \frac{1}{n^m} \sum_{a_1=1}^{n} \sum_{t_1=m-1}^{a_1-(\lambda_1-1)/(2s)} \left(\frac{\binom{s(a_1 - t_1)}{\lambda_1 - 1}}{\binom{2sa_1 - 1}{\lambda_1 - 1}} \cdot \binom{t_1 - 1}{m-2} \right).
\end{aligned}$$

By the generality, we obtain

$$P_{s,n}(\lambda) = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{n^m} \sum_{a_i=1}^{n} \sum_{t_i=m-1}^{a_i-(\lambda_i-1)/(2s)} \frac{\binom{s(a_i - t_i)}{\lambda_i - 1}}{\binom{2sa_i - 1}{\lambda_i - 1}} \binom{t_i - 1}{m-2}.$$

□

Note when $n \to \infty$ and $s \to \infty$, $P_{s,n}(\lambda)$ is exactly $P(\lambda)$.

Theorem 3.

$$\lim_{n\to\infty} P_n(\lambda) = 1 - \frac{1}{m}\sum_{i=1}^{m} \frac{\lambda_i!}{2^{\lambda_i-1}(m+\lambda_i-2)!}$$

Proof. Consider

$$\lim_{n\to\infty}\left(\frac{\prod_{i=1}^{m}(n-\lambda_i+1)}{n^m} - \sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=k+\lambda_i-2}^{n}\sum_{t=k-\lambda_i}^{a-2\lambda_i+2}\frac{\binom{\lfloor\frac{a-t}{2}\rfloor}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}}\binom{t-k+\lambda_i+m-2}{m-2}\right).$$

Claim. $\left\lfloor\dfrac{a-t}{2}\right\rfloor$ can be replaced by $\dfrac{a-t}{2}$ with same limits.

Proof. Firstly, we prove the following two lemmas.

Lemma. If two polynomial $f(x), g(x)$ with positive leading coefficients, and there exist a positive constant M such that $f(x) \geq g(x)\ \forall\ x > M$，then deg $f(x) \geq$ deg $g(x)$.

Proof. Let $h(x) = f(x) - g(x)$, and $h(x) > 0\ \forall\ x > M$. If deg $f(x) <$ deg $g(x)$, then the leading coefficient of $h(x)$ is negative. However, a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$ with negative leading coefficient is negative when

$$x > \frac{\sum_{i=0}^{n-1}|a_i|}{|a_n|},$$

contradiction. □

Lemma.

$$\sum_{t=k-\lambda_i}^{a-2\lambda_i+2}\frac{\binom{\lfloor\frac{a-t}{2}\rfloor}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}}\binom{t-k+\lambda_i+m-2}{m-2}$$

is a function of a single indeterminate a with degree $m-1$. And only the coefficient of the term a^{m-1} affects $\lim_{n\to\infty} P_n(\lambda)$.

Proof. Notice that after summation, a polynomial of a single indeterminate a with degree α becomes a polynomial of a single indeterminate n with degree $\alpha+1$. Therefore when $n \to \infty$, as the first term is the only one that matters, only the coefficient of the term a^{m-1} affects $\lim_{n\to\infty} P_n(\lambda)$. □

Back to the claim,

$$\sum_{t=k-\lambda_i}^{a-2\lambda_i+2} \frac{\binom{\lfloor\frac{a-t}{2}\rfloor}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2}$$

can be divided into

$$\sum_{t=k-\lambda_i}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t}{2}}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2} - \sum_{t=k-\lambda_i, 2\nmid a-t}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t}{2}}{\lambda_i-1} - \binom{\frac{a-t-1}{2}}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2} \quad (\star)$$

Notice that the highest order term of a, t is $\dfrac{a^\alpha t^{(\lambda_i-1-\alpha)} t^{m-2}}{a^{\lambda_i-1}}$, sum up by t will get the degree of a is $\alpha + \lambda_i - 1 - \alpha + m - 2 + 1 - (\lambda_i - 1) = m - 1$.The backward term can be regarded as

$$\begin{aligned}
&\sum_{t=k-\lambda_i, 2\nmid a-t}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t}{2}}{\lambda_i-1} - \binom{\frac{a-t-1}{2}}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2} \\
\le &\sum_{t=k-\lambda_i, 2\nmid a-t}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t}{2}}{\lambda_i-1} - \binom{\frac{a-t-2}{2}}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2} \\
= &\sum_{t=k-\lambda_i, 2\nmid a-t}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t-2}{2}}{\lambda_i-2}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2}
\end{aligned}$$

From the lemma, the degree of

$$\sum_{t=k-\lambda_i, 2\nmid a-t}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t-2}{2}}{\lambda_i-2}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2} \quad (\heartsuit)$$

must be not less than the second term of $(\star)$. However, the deg of $(\heartsuit)$ is $m-2$, since the deg of the second term of $(\star)$ is at most $m-2$. According to the lemma, we can omit this term. $\square$

Hence

$$\lim_{n\to\infty} P_n(\lambda) = \lim_{n\to\infty} \left(1 - \frac{1}{n^m} \sum_{i=1}^{m} \lambda_i \sum_{a=k+\lambda_i-2}^{n} \sum_{t=k-\lambda_i}^{a-2\lambda_i+2} \frac{\binom{\frac{a-t}{2}}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}} \binom{t-k+\lambda_i+m-2}{m-2}\right).$$

Claim. $\binom{\frac{a-t}{2}}{\lambda_i - 1}$ can be replaced by $\dfrac{\binom{a-t}{\lambda_i-1}}{2^{\lambda_i-1}}$ with same limits.

Proof.

$$\binom{\frac{a-t}{2}}{\lambda_i - 1} = \frac{1}{(\lambda_i - 1)!} \times \frac{a-t}{2} \times \frac{a-t-2}{2} \times \cdots \times \frac{a-t-2(\lambda_i - 2)}{2}$$
$$\frac{\binom{a-t}{\lambda_i - 1}}{2^{\lambda_i - 1}} = \frac{1}{(\lambda_i - 1)!} \times \frac{a-t}{2} \times \frac{a-t-1}{2} \times \cdots \times \frac{a-t-(\lambda_i - 2)}{2}$$

We view both equations as the polynomial of $(a-t)$, and get the coefficient of $(a-t)^{\lambda_i - 1}$ are the same，the following term don't have deg with $\lambda_i - 1$. By lemma, the term of deg which is less than $a^{\lambda_i - 2}$ won't affect the limits of $P_n(\lambda)$.

□

Back to the problem,

$$\lim_{n\to\infty} P_n(\lambda) = \lim_{n\to\infty} \left(1 - \frac{1}{n^m} \sum_{i=1}^{m} \frac{\lambda_i}{2^{\lambda_i - 1}} \sum_{a=k+\lambda_i-2}^{n} \frac{1}{\binom{a-1}{\lambda_i - 1}} \sum_{t=k-\lambda_i}^{a-2\lambda_i+2} \binom{a-t}{\lambda_i - 1} \binom{t-k+\lambda_i+m-2}{m-2} \right).$$

Using the combinatorial identity mentioned above,

$$\lim_{n\to\infty} P_n(\lambda) = \lim_{n\to\infty} \left(1 - \frac{1}{n^m} \sum_{i=1}^{m} \frac{\lambda_i}{2^{\lambda_i - 1}} \sum_{a=k+\lambda_i-2}^{n} \frac{1}{\binom{a-1}{\lambda_i - 1}} \binom{a-1+m+\lambda_i-k}{m+\lambda_i-2} \right).$$

and by proposition 5

$$\lim_{n\to\infty} P_n(\lambda) = \lim_{n\to\infty} \left(1 - \frac{1}{n^m} \sum_{i=1}^{m} \frac{\lambda_i}{2^{\lambda_i - 1}} \sum_{a=k+\lambda_i-2}^{n} \frac{(\lambda_i - 1)!}{a^{\lambda_i - 1}} \frac{a^{m+\lambda_i-2}}{(m+\lambda_i-2)!} \right).$$

Eventually

$$\lim_{n\to\infty} P_n(\lambda) = 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{\lambda_i!}{2^{\lambda_i - 1}(m+\lambda_i-2)!}.$$

□

Theorem 4 (The broken pick-up sticks problem)**.**

$$P(\lambda) = \lim_{n\to\infty}\lim_{s\to\infty} P_{s,n}(\lambda) = 1 - \frac{1}{m}\sum_{i=1}^{m}\frac{\lambda_i!}{2^{\lambda_i-1}(m+\lambda_i-2)!}.$$

Proof. Consider the result of premium chocolate bar when $n\to\infty$ and $s\to\infty$.

$$\begin{aligned}
P(\lambda) &= \lim_{n\to\infty}\lim_{s\to\infty} P_{s,n}(\lambda)\\
&= \lim_{n\to\infty}\lim_{s\to\infty}\left(1-\sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=1}^{n}\sum_{t=1}^{\lfloor a-\frac{\lambda_i-1}{2s}\rfloor}\frac{\binom{s(a-t)}{\lambda_i-1}}{\binom{2sa-1}{\lambda_i-1}}\binom{t-1}{m-2}\right)\\
&= \lim_{n\to\infty}\lim_{s\to\infty}\left(1-\sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=1}^{n}\sum_{t=1}^{\lfloor a-\frac{\lambda_i-1}{2s}\rfloor}\frac{s^{\lambda_i-1}(a-t)^{\lambda_i-1}}{2^{\lambda_i-1}s^{\lambda_i-1}a^{\lambda_i-1}}\binom{t-1}{m-2}\right)\\
&= \lim_{n\to\infty}\lim_{s\to\infty}\left(1-\sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=1}^{n}\sum_{t=1}^{\lfloor a-\frac{\lambda_i-1}{2s}\rfloor}\frac{\binom{a-t}{\lambda_i-1}}{\binom{2a}{\lambda_i-1}}\binom{t-1}{m-2}\right)\\
&= \lim_{n\to\infty}\lim_{s\to\infty}\left(1-\sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=1}^{n}\frac{1}{\binom{2a}{\lambda_i-1}}\binom{a}{m+\lambda_i-2}\right)\\
&= \lim_{n\to\infty}\lim_{s\to\infty}\left(1-\sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=1}^{n}\frac{(\lambda_i-1)!}{2^{\lambda_i-1}a^{\lambda_i-1}}\frac{a^{m+\lambda_i-2}}{(m+\lambda_i-2)!}\right)\\
&= 1-\frac{1}{m}\sum_{i=1}^{m}\frac{\lambda_i!}{2^{\lambda_i-1}(m+\lambda_i-2)!}
\end{aligned}$$

□

5 Comparing different $P(\lambda)$

Theorem 5. Consider all partitions while fixing m and k. $P(\lambda)$ has the largest value while $\forall\ 1 \le i \le j \le m, \lambda_i - \lambda_j = 0, 1$.

Proof. Suppose there's a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 \ge \lambda_2 \ge \ldots \lambda_m, \lambda_1 - 1 \le \lambda_2 + 1$. Let $\lambda' = (\lambda_1 - 1, \lambda_2 + 1, \ldots, \lambda_m)$. Let's prove $P(\lambda') - P(\lambda) > 0$.

$$
\begin{aligned}
& P(\lambda') - P(\lambda) > 0 \\
\Longleftrightarrow\ & \frac{(\lambda_1-1)!}{2^{\lambda_1-2}(m+\lambda_1-3)!} + \frac{(\lambda_2+1)!}{2^{\lambda_2}(m+\lambda_2-1)!} < \frac{\lambda_1!}{2^{\lambda_1-1}(m+\lambda_1-2)!} + \frac{\lambda_2!}{2^{\lambda_2-1}(m+\lambda_2-2)!} \\
\Longleftrightarrow\ & 2^{\lambda_1-\lambda_2-1}\frac{\lambda_2!(m+\lambda_1-3)!}{(\lambda_1-1)!(m+\lambda_2-2)!} > \frac{(m+\lambda_2-1)(2m+\lambda_1-4)}{(2m+\lambda_2-3)(m+\lambda_1-2)} \\
\Longleftrightarrow\ & 2^{\lambda_1-\lambda_2-1}\frac{\binom{m+\lambda_1+\lambda_2-3}{\lambda_1-1}}{\binom{m+\lambda_1+\lambda_2-3}{\lambda_2}} > \frac{(m+\lambda_2-1)(2m+\lambda_1-4)}{(2m+\lambda_2-3)(m+\lambda_1-2)} \\
\Longleftarrow\ & 2^{\lambda_1-\lambda_2-1}\frac{\binom{m+\lambda_1+\lambda_2-3}{\lambda_1-1}}{\binom{m+\lambda_1+\lambda_2-3}{\lambda_2}} > 1 \ge \frac{(m+\lambda_2-1)(2m+\lambda_1-4)}{(2m+\lambda_2-3)(m+\lambda_1-2)}
\end{aligned}
$$

For the inequality on the right side, it can be done by expanding. For the one on the left, we check the following two cases.

- $\dfrac{m+\lambda_1+\lambda_2-3}{2} > \lambda_1 - 1$
  Since $\lambda_1 - 1 \ge \lambda_2$, we have $\binom{m+\lambda_1+\lambda_2-3}{\lambda_1-1} > \binom{m+\lambda_1+\lambda_2-3}{\lambda_2}$.

- $\dfrac{m+\lambda_1+\lambda_2-3}{2} \le \lambda_1 - 1$
  Since $m+\lambda_2-2 > \lambda_2$, we have* $\binom{m+\lambda_1+\lambda_2-3}{\lambda_1-1} = \binom{m+\lambda_2-2}{\lambda_2} > \binom{m+\lambda_1+\lambda_2-3}{\lambda_2}$.

□

Theorem 6. Consider all partitions while fixing k, $P(1, 1, \ldots, 1)$ has the largest value.

Proof. Suppose there's a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Let $\lambda' = (\lambda_e + 1, \ldots \lambda_e + 1, \lambda_e \ldots, \lambda_e)$ where $\lambda_e + 1 + \cdots + \lambda_e + 1 + \lambda_e \cdots + \lambda_e = k$. From the previous theorem, $P(\lambda) \le P(\lambda')$. Suppose there are α $(\lambda_e + 1)$s and β (λ_e), where $\beta \geq 1$, then $\alpha + \beta = m$, and

$$P(1, 1, \ldots, 1) = 1 - \frac{1}{(\alpha(\lambda_e + 1) + \beta(\lambda_e) - 1)!}.$$

For $\alpha \geq 1$ or $\alpha = 0, \beta > 1, \lambda_e > 1$,

$$\begin{aligned} P(\lambda') &= 1 - \frac{1}{\alpha + \beta}\left(\frac{\alpha(\lambda_e + 1)!}{2^{\lambda_e}(\alpha + \beta + \lambda_e - 1)!} + \frac{\beta \lambda_e!}{2^{\lambda_e - 1}(\alpha + \beta + \lambda_e - 2)!}\right) \\ &= 1 - \frac{\lambda_e!(\alpha(\lambda_e + 1) + 2\beta(\alpha + \beta + \lambda_e - 1))}{2^{\lambda_e}(\alpha + \beta)(\alpha + \beta + \lambda_e - 1)!} \\ &\leq 1 - \frac{\lambda_e!(\alpha + \beta)(\lambda_e + 1)}{2^{\lambda_e}(\alpha + \beta)(\alpha + \beta + \lambda_e - 1)!} \\ &= 1 - \frac{(\lambda_e + 1)!}{2^{\lambda_e}(\alpha + \beta + \lambda_e - 1)!} \\ &\leq 1 - \frac{1}{(\alpha + \beta + \lambda_e - 1)!} \\ &\leq 1 - \frac{1}{(\alpha(\lambda_e + 1) + \beta(\lambda_e) - 1)!} = P(1, 1, \ldots, 1). \end{aligned}$$

For $\alpha = 0, \beta = 1$,

$$P(\lambda') = 1 - \frac{\lambda_e}{2^{\lambda_e - 1}} < 1 - \frac{1}{(\lambda_e - 1)!} = P(1, 1, \ldots, 1).$$

For $\alpha = 0, \lambda_e = 1$,

$$P(\lambda') = P(1, 1, \ldots, 1).$$

□

Theorem 7. Consider all partitions while fixing k, $P(k - 1, 1)$ has minimum. While $k = 3$, $P(2, 1) = P(3) = \frac{1}{4}$ get minimum.

Proof. We prove the following first.

Lemma. For $k, t \in \mathbb{N}$ where $k > t + 1$, $P(k - t, \underbrace{1, 1, \ldots, 1}_{t}) < P(k - t - 1, \underbrace{1, 1, \ldots, 1}_{t+1})$ ∘

Proof.

$$\begin{aligned}
&P(k-t-1,\underbrace{1,1,\ldots,1}_{t+1}) - P(k-t,\underbrace{1,1,\ldots,1}_{t}) > 0 \\
\iff &\frac{1}{t+1}\left(\frac{t}{t!} + \frac{(k-t)!}{2^{k-t-1}(k-1)!}\right) > \frac{1}{t+2}\left(\frac{t+1}{(t+1)!} + \frac{(k-t-1)!}{2^{k-t-2}(k-1)!}\right) \\
\iff &\frac{t^2+t-1}{(t+2)!} + \frac{(k-t-1)!\,((k-t)(t+2)-2(t+1))}{2^{k-t-1}(k-1)!(t+1)(t+2)} > 0
\end{aligned}$$

□

By the previous theorem, we have $P(k-m+1,\underbrace{1,1,\ldots,1}_{m-1})$ has minimum under fixed m and k. Then by the lemma

$$P(\underbrace{1,1,\ldots,1}_{k}) > P(2,\underbrace{1,1,\ldots,1}_{k-1}) > \cdots > P(k-1,1).$$

Lastly we compare $P(k-1,1)$ and $P(k)$.

$$P(k) - P(k-1,1) \geq 0 \iff \frac{k}{2^{k-1}} \leq \frac{1}{2}(\frac{1}{2^{k-1}} + \frac{1}{2}) \iff k \leq 1 + 2^{k-2},$$

which can be verified with induction on k. And equality holds when $k = 3$. □

6 Geometric properties of the triangles

Remark 6.1 (Notation)**.** Denote I, O, H as incenter, circumcenter, orthocenter of triangle.

Theorem 8. Pick 3 sticks chosen from the uniform distribution of length on $\{1, 2, \ldots, n\}$. When $n \to \infty$, the probability of these three sticks forming an obtuse triangle is $\frac{\pi}{4} - \frac{1}{2}$, while the probability of them forming an acute triangle is $1 - \frac{\pi}{4}$.

Proof. We prove several lemmas first.

Claim 6.1. Denote

$$\chi(x) = \begin{cases} 1 & \text{if } x \equiv 1 \pmod 4, \\ -1 & \text{if } x \equiv 3 \pmod 4, \\ 0 & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

Then the number of integer solutions to $x^2 + y^2 = n$ is

$$4 \sum_{d|n} \chi(d).$$

Proof. Since $\mathbb{Z}[i]$ is a unique factorization domain (UFD), we can get $(x + yi)(x - yi) = n$. Let the unique factorization in $\mathbb{Z}$ of

$$n = 2^v \prod_{s=1}^{m} p_s^{\alpha_s} \prod_{t=1}^{n} q_t^{\beta_t},$$

where p_s are primes $4k + 1$-like and q_t are primes $4k + 3$-like. By the unique factorization,

$$x + yi = u(1 + i)^v \prod_{s=1}^{m} (x_s + iy_s)^{\gamma_s} (x_s - iy_s)^{\alpha_s - \gamma_s} \prod_{t=1}^{n} q_t^{\beta_t/2}.$$

Notice that each β_t is even if only if $x^2 + y^2 = n$ has integer solutions, and if so, there are $4 \prod_{s=1}^{m} (\alpha_s + 1)$ pairs of (x, y). However, we find it equivalent to

$$(\chi(1) + \chi(2) + \chi(4) + \ldots + \chi(2^v)) \prod_{s=1}^{m} (1 + \chi(p_s) + \ldots + \chi(p_s^{\alpha_s})) \prod_{t=1}^{n} \left(1 + \chi(q_t) + \ldots + \chi(q_t^{\beta_t})\right).$$

According to the definition, we can easily get $\chi(x)$ is multiplicative, and we get our desired result. □

The number of solutions of $x^2 + y^2 \leq n$ is

$$\sum_{m=1}^{n} 4 \sum_{d|m} \chi(d) = 4 \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor \chi(i)$$

Note that if (x, y) is a solution, then $(\pm x, \pm y)$ are also solutions. Hence we focus on the solutions in quadrant I, and the number of solutions in quadrant I (excluding x and y-axis) is

$$\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor \chi(i) - \sqrt{n}.$$

For convenience, we introduce big O notation to describe the limiting behavior of a function.

Claim 6.2. The number of Pythagorean triples in $\{1, 2, 3, \ldots, n\}^3$ is at most $O(n^2)$.

Proof. We estimate the quantity of Pythagorean triples in $\{1, 2, 3, \ldots, n\}^3$. We have $(x, y, z) = (k(u^2 - v^2), 2kuv, k(u^2 + v^2))$. Since $u^2 + v^2 \leq n$, there are at most $\sqrt{n} \times \sqrt{n} = n$ pairs of (u, v). Meanwhile, k ranges from 1 to n. In addition, there are 3! arrangements of x, y, z, which implies that there are at most $6n^2$ solution. □

The way to pick up three pieces is n^3, hence the probability of forming a right triangle is 0 as $n \to \infty$.

Claim 6.3. $\lim_{n \to \infty} \mathbb{P}_n \left(x^2 + y^2 < z^2\right) = \frac{\pi}{12}$

The number of solutions to $x^2 + y^2 \leq z^2$ is

$$\sum_{i=1}^{z^2} \lfloor \frac{z^2}{i} \rfloor \chi(i) - z.$$

z ranges from $\{1, 2, 3, \ldots, n\}$, hence

$$\#\{(x, y, z) \mid x^2 + y^2 \leq z^2\} = \sum_{z=1}^{n} \left(\sum_{i=1}^{z^2} \lfloor \frac{z^2}{i} \rfloor \chi(i) - z \right).$$

Lemma. Denote $x - \lfloor x \rfloor$ by $\{x\}$. $\sum_{z=1}^{n} \left\{ \frac{z^2}{i} \right\} < n$

Proof. It's trivial that for all z, $\left\{ \frac{z^2}{i} \right\} \leq \frac{i-1}{i}$. Therefore $\sum_{z=1}^{n} \left\{ \frac{z^2}{i} \right\} \leq \frac{(i-1)}{i} n < n$. □

Lemma. We can replaced $\lfloor \frac{z^2}{i} \rfloor$ by $\frac{z^2}{i}$, and the difference between them will be at most $O(n^2)$.

Proof.

$$\begin{aligned}
&\sum_{z=1}^{n}\left(\sum_{i=1}^{z^2}\frac{z^2}{i}\chi(i)-z\right)-\sum_{z=1}^{n}\left(\sum_{i=1}^{z^2}\lfloor\frac{z^2}{i}\rfloor\chi(i)-z\right)\\
=&\sum_{z=1}^{n}\left(\sum_{i=1}^{z^2}\left\{\frac{z^2}{i}\right\}\chi(i)\right)\\
=&\sum_{i=1}^{n}\left(\sum_{z=1}^{n}\left\{\frac{z^2}{i}\right\}\chi(i)\right)\\
\leq&\sum_{i=1}^{n}\left(\sum_{z=1}^{n}\left\{\frac{z^2}{i}\right\}\right)\\
\leq&\sum_{i=1}^{n}n\\
=&n^2
\end{aligned}$$

□

Lemma. We can replace $\sum_{z=1}^{n}\left(\sum_{i=1}^{z^2}\frac{z^2}{i}\chi(i)-z\right)$ with $\sum_{z=1}^{n}\left(\sum_{i=1}^{n}\frac{z^2}{i}\chi(i)-z\right)$, and the difference between them will be at most $O(n^2)$.

Proof. Compare the two terms. We can see that the difference between them is

$$\sum_{i=3}^{n}\sum_{z=1}^{\lfloor\sqrt{i}\rfloor}\frac{z^2}{i}\chi(i)\leq\sum_{i=3}^{n}\frac{\sqrt{i}(\sqrt{i}+1)(2\sqrt{i}+1)}{6i}<\sum_{i=3}^{n}\frac{\sqrt{i}(2\sqrt{i})(3\sqrt{i})}{6i}=\sum_{i=3}^{n}\sqrt{i}<n\sqrt{n}$$

□

Lemma. $\lim_{n\to\infty}\sum_{z=1}^{n}\left(\sum_{i=1}^{n}\frac{z^2}{i}\chi(i)-z\right)=\frac{\pi}{12}n^3+O(n^2)$

Proof.

$$\lim_{n\to\infty}\sum_{z=1}^{n}\left(\sum_{i=1}^{n}\frac{z^2}{i}\chi(i)-z\right)=\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6}\cdot\sum_{i=1}^{n}\frac{\chi(i)}{i}-\frac{n(n-1)}{2}\right).$$

Notice that

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\chi(i)}{i}\right)=\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)=\frac{\pi}{4}.$$

The coefficient of the term n^3 is $\frac{\pi}{4} \cdot \frac{2}{6} = \frac{\pi}{12}$, hence the result. $\square$

To sum up,

$$\lim_{n\to\infty} \mathbb{P}_n\left(x^2+y^2<z^2\right) = \lim_{n\to\infty} \mathbb{P}_n\left(x^2+y^2\le z^2\right) = \lim_{n\to\infty} \frac{\#\{(x,y,z) \mid x^2+y^2\le z^2\}}{n^3}.$$

And

$$\lim_{n\to\infty} \frac{\#\{(x,y,z) \mid x^2+y^2\le z^2\}}{n^3} = \lim_{n\to\infty} \left(\left[\frac{n^3}{12}+O(n^2)\right]/n^3\right) = \frac{\pi}{12}.$$

Likewise,

$$\lim_{n\to\infty} \mathbb{P}_n\left(y^2+z^2<x^2\right) = \lim_{n\to\infty} \mathbb{P}_n\left(x^2+z^2<y^2\right) = \frac{\pi}{12}.$$

Also, we can easily find that if

$$x^2+y^2>z^2, y^2+z^2>x^2, z^2+x^2>y^2,$$

then x, y, z must form a triangle. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}_n(x^2+y^2>z^2 \ \wedge\ y^2+z^2>x^2 \ \wedge\ z^2+x^2>y^2) \\
=&1-\lim_{n\to\infty} \mathbb{P}_n\left(x^2+y^2<z^2\right)-\lim_{n\to\infty} \mathbb{P}_n\left(y^2+z^2<x^2\right)-\lim_{n\to\infty} \mathbb{P}_n\left(z^2+x^2<y^2\right) \\
=&1-\frac{\pi}{4}
\end{aligned}$$

Besides, we have $P_n(1,1,1) = \frac{1}{2}$, so the probability of forming obtuse triangle is $\frac{\pi}{4} - \frac{1}{2}$ $\square$

Corollary 3. Pick up 3 sticks chosen from a uniform distribution of stick lengths,. Then the probability for the three pieces forming the three medians of a triangle is equal to $P(1,1,1) = \frac{1}{2}$

Theorem 9. Pick up 2 sticks chosen from a uniform distribution of stick lengths, and break one of them into two pieces. Then the probability of existing a triangle $\triangle ABC$ with incenter I such that the lengths of $\overline{AI}, \overline{BI}, \overline{CI}$ are the chosen three pieces is equal to 1.

Proof. Here we provides another approaching method. Let DEF be I-cevian triangle.

Lemma 1. $\overline{AD}^2 = \overline{AB} \times \overline{AC} - \overline{BD} \times \overline{DC}$

Proof. We find a point X on $\overline{AC}$ such that $\triangle ABD \sim \triangle ADX$. Then

$$\overline{AB} : \overline{AD} = \overline{AD} : \overline{AX} \Longrightarrow \overline{AD}^2 = \overline{AB} \times \overline{AX}.$$

Also,

$$\measuredangle CDX = \measuredangle BDA + \measuredangle ADX = \measuredangle DXA + \measuredangle ADX = \measuredangle ADX,$$

so we have $\triangle CDX \sim \triangle CAD$, which implies

$$\overline{CX} : \overline{CD} = \overline{CD} : \overline{CA} = \overline{BD} : \overline{BA} \Longrightarrow \overline{CX} \times \overline{AB} = \overline{CD} \times \overline{BD}.$$

Sum up the two equations above to get the desired result. □

Back to the problem, let $\overline{BC} = a, \overline{CA} = b, \overline{AB} = c$. It's well known that $\overline{AI}^2 = bc\dfrac{b+c-a}{b+c+a}$. Let

$$\overline{AI}^2 = l, \overline{BI}^2 = m, \overline{CI}^2 = n, s = \frac{a+b+c}{2},$$

then $bc - l = ca - m = ab - n = \dfrac{abc}{s} = k$, implying that

$$a = \sqrt{\frac{(k+m)(k+n)}{(k+l)}}, b = \sqrt{\frac{(k+n)(k+l)}{(k+m)}}, c = \sqrt{\frac{(k+l)(k+m)}{(k+n)}}.$$

Plug the above into the equation $\dfrac{abc}{s} = k$, we have

$$f(k) = k^3 - (mn + nl + lm)k - 2lmn.$$

Notice that $f(0) < 0$. And when k is sufficiently large, $f(k) > 0$. So $f(k) = 0$ has positive root. Since the product of three roots is positive, and sum of them is negative, there can be only one positive root, which implies its uniqueness. □

Theorem 10. Pick up 2 sticks chosen from a uniform distribution of stick lengths, and break one of them into two pieces. Then the probability of existing a triangle $\triangle ABC$ with orthocenter H such that the lengths of $\overline{AH}, \overline{BH}, \overline{CH}$ are the chosen three pieces is equal to 1.

Proof. Note that if (l, m, n) describe the lengths we want, then (kl, km, kn) also satisfies the condition. Since if two triangle are similar, the length of corresponding side can be any multiple. By the previous theorem, for any (l, m, n), (l, m, n) can form $\overline{AI}, \overline{BI}, \overline{CI}$. Meanwhile, there must be a triangle such that $(\frac{1}{l}, \frac{1}{m}, \frac{1}{n})$ form $\overline{AI}, \overline{BI}, \overline{CI}$ as well. Then take an inversion with power 1 and center I. It's well-known that the incenter of triangle will become orthocenter, and so $\overline{A'H} = l, \overline{B'H} = m, \overline{C'H} = n$. Hence, for any l, m, n, there must be a triangle such that the distance between orthocenter and three vertices is l, m, n. ☐

Theorem 11. Pick up 2 sticks chosen from a uniform distribution of stick lengths, and break one of them into two pieces. Then the probability of existing a triangle $\triangle ABC$ and circumcenter O such that the distance of O to the sidelines are the chosen three pieces is equal to 1.

Proof. Since the distance of O to $\overline{BC}$ is half of $\overline{AH}$, so from the previous theorem and lemma we can get the results. ☐

Theorem 12. A stick is broken into three pieces with length α, β, γ at random. The probability of existing a triangle ABC such that $\overline{AH} = \alpha, \overline{BH} = \beta, \overline{CH} = \gamma$ and the angles of $\triangle ABC$ are all larger than 45° is

$$\frac{1}{4} - \frac{3}{4}[7 + 4\sqrt{2} - 8(10 + 7\sqrt{2}) \ln 2 + 4(10 + 7\sqrt{2}) \ln(2 + \sqrt{2})] \approx 0.21$$

Proof. To get the probability, we prove the condition is equivalent to

$$m^2 + n^2 + \sqrt{2}mn > l^2, n^2 + l^2 + \sqrt{2}nl > m^2, l^2 + m^2 + \sqrt{2}lm > n^2$$

Notice that all interior angles is less than 90°, hence $\overline{AH} = 2R \ \cos A < 2R \sin A = \overline{BC}$. Also, $\angle BHC > 135° \implies \overline{BH}^2 + \overline{CH^2} + \sqrt{2}\overline{BH} \times \overline{CH} > \overline{BH}^2 + \overline{CH^2} + 2\overline{BH} \times \overline{CH} \cos \angle BHC = \overline{BC}^2 > \overline{AH}^2$. Likewise, if $\angle A < 45°$, all the signs of inequalities will be inverted.

First of all, we consider the graph of these inequalities. We draw a equilateral triangle, and we consider P with its trilinear coordinate is $[m : n : l]$ (We normalize the coordinate such that $m + n + l = \sqrt{3}$). The inequalities are hyperbolas, and we are only concerned about the area between three functions.

We can use coordinate transformation to change into Cartesian coordinate (Proposition 6). We place the three vertices on $(0, \sqrt{3}), (-1, 0), (1, 0)$. We use the following transformation

$$m = y, n = \frac{\sqrt{3}(1+x) - y}{2}, l = \frac{\sqrt{3}(1-x) - y}{2}$$

Then $l^2 + m^2 + \sqrt{2}lm > n^2 \implies x < \frac{(2-\sqrt{2})y^2 + \sqrt{6}y}{6 + \sqrt{6}y - 2\sqrt{3}y}$, we can get the proportion of area between three hyperbolas is

$$\begin{aligned}
&\frac{1}{\sqrt{3}}[\frac{\sqrt{3}}{4} - 3\int_0^{\frac{\sqrt{3}}{2}} (\frac{y}{\sqrt{3}} - \frac{(2-\sqrt{2})y^2 + \sqrt{6}y}{6 + \sqrt{6}y - 2\sqrt{3}y})dy] \\
=&\frac{1}{4} - \frac{3}{4}[7 + 4\sqrt{2} - 8(10 + 7\sqrt{2})\ln 2 + 4(10 + 7\sqrt{2})\ln(2 + \sqrt{2})]
\end{aligned}$$

□

Theorem 13. A stick is broken into three pieces with length α, β, γ at random. The probability of existing a triangle ABC such that $\overline{AH} = \frac{1}{\alpha}, \overline{BH} = \frac{1}{\beta}, \overline{CH} = \frac{1}{\gamma}$ and $\triangle ABC$ is acute is

$$\frac{1}{4} - \frac{3}{4}[7 + 4\sqrt{2} - 8(10 + 7\sqrt{2})\ln 2 + 4(10 + 7\sqrt{2})\ln(2 + \sqrt{2})] \approx 0.21.$$

Proof. Consider a triangle $\triangle ABC$ with three interior angles exceed $45°$, let orthocenter be H. Perform an inversion centering H. Notice that inversion is conformal and H will become incenter$I$ of $\triangle A'B'C'$. Since $\angle B'IC' = \angle BHC < 135°$, $\angle B'A'C' = 2(\angle B'IC' - 90°) < 90°$, which implies this probability is equivalent to former one. □

7 Applications

A company is planning to share part of its profit with the employees at the end of the year. The committee held a meeting to discuss the matter. They would like to share the profit with a fancy entertainment show. Our work provides a solution. In the annual event where all k employees are present, there are m rooms, each with a random sum of money and a number λ representing the number of people it must hold. The employees are then randomly sent into the rooms so every room are filled with exactly as many as the room must hold. Lastly, the sum of money are randomly distributed to the employees in the room. Our work suggests that, for example, by adjusting the value of m or λ, we can tweak the probability that no one gets more than half of the profit. By randomly separating the employees into groups and randomly sharing the profit, the entertainment value is high, while a subtle balance is kept.

8 Conclusions

In this project, we solved the chocolate bars/bricks problem using random variables.

$$P_n(\lambda) = \frac{\prod_{i=1}^{m}(n-\lambda_i+1)}{n^m} - \sum_{i=1}^{m}\frac{\lambda_i}{n^m}\sum_{a=k+\lambda_i-2}^{n}\sum_{t=k-\lambda_i}^{a-2\lambda_i+2}\frac{\binom{\lfloor (a-t)/2 \rfloor}{\lambda_i-1}}{\binom{a-1}{\lambda_i-1}}\binom{t-k+\lambda_i+m-2}{m-2}.$$

And by considering premium chocolate bars and the limit behaviour, we obtained

$$P(\lambda) = 1 - \frac{1}{m}\sum_{i=1}^{m}\frac{\lambda_i!}{2^{\lambda_i-1}(m+\lambda_i-2)!}.$$

Through comparing different $P(\lambda)$, we get the maximum and minimum of $P(\lambda)$ under fixed k, and the maximum of $P(\lambda)$ under fixed m and k.

When forming a triangle with picked sticks, the probability of them sticks forming an obtuse triangle is $\frac{\pi}{4} - \frac{1}{2}$, while the probability of them forming an acute triangle is $1 - \frac{\pi}{4}$.

Several specific segments in a triangle are also studied.

Future research will look into several more variations of the spaghetti problem, as given below.

9 Open Problems

Problem 4. Stick (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m sticks chosen from a uniform distribution of stick lengths. For each i, break the ith stick into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. What is the probability that any k' pieces from the resulting k pieces form a k'-gon?

Problem 5. Stick (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = k$. Pick up m sticks chosen from a uniform distribution of stick lengths. For each i, break the ith stick into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. What is the probability that there exist k' pieces from the resulting k pieces such that they form a k-gon?

Problem 6. Stick (λ). Consider a partition $\lambda = (\lambda_1, ..., \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$, with $\sum \lambda_i = 6$. Pick up m sticks chosen from a uniform distribution of stick lengths. For each i, break the ith stick into λ_i pieces by choosing $\lambda_i - 1$ cut points independently at random. What is the probability that the resulting 6 pieces such that they form the six sides of a tetrahedron?

10 References

1. Ionascu, E. J., & Prajitura, G. (2010). Things to do with a broken stick. arXiv preprint arXiv:1009.0890. http://arxiv.org/abs/100

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Appendix

Proposition 1. $\sum_{i=0}^{k}\binom{n+i}{r}=\binom{n+k+1}{r+1}-\binom{n}{r+1}$ for $r\leq n$.

Proof. Pascal's Rule tells us

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k},$$

so

$$\sum_{i=0}^{n+k-r}\binom{r+i}{r}=\binom{r+1}{r+1}+\sum_{i=1}^{n+k-r}\binom{r+i}{r}=\binom{n+k+1}{r+1},$$

therefore

$$\sum_{i=0}^{k}\binom{n+i}{r}=\binom{n+k+1}{r+1}-\binom{n}{r+1}.$$

□

Proposition 2. $\sum_{i=a}^{n-b}\binom{i}{a}\binom{n-i}{b}=\binom{n+1}{a+b+1}$ for $a+b\leq n$.

Proof. In a class with $n+1$ students, in how many ways can a team of $a+b+1$ students be formed? The answer is

$$\binom{n+1}{a+b+1}.$$

We can give each student a number from $\{1,2,3,\ldots,n+1\}$, and any two of them have different number. The answer is also the sum over all possible values of i, of the number of team consisting of a students with number $< i+1$, b students with number $> i+1$ and the student with number $i+1$:

$$\sum_{i=a}^{n-b}\binom{i}{a}\binom{n-i}{b}.$$

We call the student with number $i+1$ is "special". If there exists two choices with different " special " students but the same student set. Let the number of "special" student of choice 1 is s; choice 2 is t. Since the definition of "special", s is the $a+1$-th less number of the student set of choice 1; t is the $a+1$-th less number of the student set of choice 2. If they are the same set, the arrangement should be the same, which implies $s=t$, contradiction. Hence any two of the choices are distinct. □

Proposition 3. Fix a positive integer $k\geq 3$ and a k-element multiset S of positive numbers.

There exists a (convex) polygon whose side lengths are the elements of S if and only if $x < \|S\| - x$ for each $x \in S$. ($\|S\|$ denotes the sum of the elements of S.)

Proof. First we prove that if $x < \|S\| - x$ for each $x \in S$, then there exists a (convex) polygon whose side lengths are the elements of S. We apply induction on k. The case where $k = 3$ is trivial. Suppose for all j-gon where $3 \leq j < k$, the statement holds. Let the elements of the k-element multiset S bijectively correspond to the side lengths of k-gon P. Any diagonal of P divides P into two smaller polygon whose numbers of sides are less than k. Collect the two polygons' side lengths and make them into sets, called S_1 and S_2, respectively. Let d be the length of the diagonal that cuts P in two. S_1 and S_2 both contain d and part of S. By our assumption, for any element $a \in S_1$,

$$a < \|S_1\| - a \quad \text{and} \quad d < \|S_2\| - d.$$

which leads to

$$a < \frac{\|S_1\|}{2} < \frac{\|S_1\|}{2} + (\frac{\|S_2\|}{2} - d) = \frac{\|S\|}{2}.$$

So it's proved.

Now for the other direction. $k = 3$ is trivial. Suppose for all j-element set where $3 \leq j < k$, the statement holds. Then let's take a look at the k-element set

$$\{a_1, a_2, \ldots, a_k\}.$$

By our assumption, once we find a segment of length d such that $\{a_1, a_2, d\}$ describe a triangle and $\{d, a_3, a_4, \ldots, a_k\}$ describe a $k-1$-gon, a k-gon can be described by the set S. WLOG let $a_1 \geq a_2$ and a_k be the largest element in $\{a_3, a_4, \ldots, a_k\}$. We need to solve the two following inequalities:

$$a_1 - a_2 < d < a_1 + a_2,$$

$$a_k - (a_3 + a_4 + \cdots + a_{k-1}) < d < a_k + (a_3 + a_4 + \cdots + a_{k-1}).$$

Except when

$$a_1 + a_2 \leq a_k - (a_3 + a_4 + \cdots + a_{k-1})$$

or

$$a_k + (a_3 + a_4 + \cdots + a_{k-1}) \leq a_1 - a_2,$$

d has a real solution. Yet the former inequality leads to

$$a_k \geq (a_1 + a_2 + \cdots + a_{k-1}),$$

the latter leads to

$$a_1 \geq (a_2 + a_3 + \cdots + a_k),$$

which both contradicts our assumption $a < \|S\| - a, \quad \forall\, a \in S$. So the result is proved. $\square$

Proposition 4. Given $n, k, p \in \mathbb{N}$, where $n \geq k$ and $n - (k-1) \geq p \geq n/2$, then there are $k\binom{n-p}{k-1}$ natural solution to

$$n = a_1 + a_2 + \cdots + a_k \quad \text{and } \exists\, i \text{ s.t. } a_i \geq p.$$

Proof. Since $p \geq \frac{n}{2}$, there exists at most one i such that $a_i \geq p$. WLOG let $a_1 \geq p$. Let $a_1' = a_1 - p + 1$, then

$$a_1' + a_2 + \cdots + a_k = n - p + 1$$

has $\binom{n-p}{k-1}$ natural solutions, each of which bijectively corresponds to the natural solutions to the original equation. $\square$

Proposition 5. For a super big m and a given j, $\binom{m}{j}$ can be approximated by $\frac{1}{j!}m^j$.

Proof.

$$\binom{m}{j} = \frac{1}{j!}m(m-1)\ldots(m-j+1) = \frac{1}{j!}m^j + \mathcal{O}(m^{j-1}),$$

$\square$

Proposition 6. On a plane, a point P with Cartesian coordinate (x, y) where $A = (0, \sqrt{3}), B =$

$(-1,0), C=(1,0)$ has trilinear coordinate $[p:q:r]$ with respect to the triangle ABC

$$\begin{cases} p = y, \\ q = \dfrac{\sqrt{3}(1+x)-y}{2}, \\ r = \dfrac{\sqrt{3}(1-x)-y}{2} \end{cases}$$

Proof. Denote$d(P,BC)$ represents the distance from P to BC. It's obvious that $p = d(P,BC) = y$. Let AP and BC intersect at D, then the coordinate of D is $(\dfrac{\sqrt{3}x}{\sqrt{3}-y}, 0)$.
Hence $CD : BD = \sqrt{3}(1-x)-y : \sqrt{3}(1-x)-y$. Because $AB = AC$, $CD : BD = S_{\triangle ACP} : S_{\triangle ABP} = \dfrac{AC \times q}{2} : \dfrac{AB \times r}{2} = q : r$. Due to $p+q+r = \sqrt{3}$, we can get the transformation. □

【評語】010026

本作品主要回答一個在 2020 年由 Petersen 及 Tenner 所提出的一個機率問題：將各種不同長度的巧克力棒再切成若干小段，則這些小段可以圍成一個多邊型的機率有多少？作者考慮此問題的離散型問題，用組合方法努力地計算出機率，並且嘗試用極限的技巧去推估原始的問題。整體而言，這是一件困難但是優秀的數學作品，需要許多數學的知識和分析的技巧，作者們的努力與成果值得嘉許。若是對於其中的一些數學估計能有更多的說明應該會讓作品更精進。