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得獎獎項　大會獎：二等獎

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關鍵字　區塊數、連續與不連續

作者簡介

我叫張家華，台中女中三年級，非常討厭寫自我介紹。

很榮幸再次有機會來到這個科學的殿堂與各地好手共襄盛舉。我在做科展的過程中對數學有了更深一層的理解，謝謝賴信志老師在過程中的指導。作品仍有許多的改進空間，希望能在這次活動中獲得經驗和成長。

我是林冠蓁，出生於台中市，目前就讀台中女中三年級。

遇上這個題目而燃起了興趣後，就一頭栽入了數學科。雖說對高中數學可謂得心應手，然而一踏入研究的領域，深度與課本內容實在全然不同。討論的過程中，我確切明白到數學此科的深廣及無窮。希望未來能更深入數學這個奇幻國度，探索它的不可思議。

摘要

平面上n條直線的分割區塊數R的最大值顯然為$\frac{n^2+n+2}{2}$，最小值顯然為$n+1$，在 Oleg A. Ivanov 教授的論文[3]裏探討了可能的分割區塊數R從某個定值開始到最大值$\frac{n^2+n+2}{2}$之間呈連續整數分布。我們進一步研究R可能的值從$n+1$到$\frac{n^2+n+2}{2}$之間在哪些區間產生不連續，我們稱這個不連續的區間為「跳躍」(jump)。以下是我們得到的結果：

當$n\geq 8$，平面上n條直線可能的分割區塊數R產生的「跳躍」區間為

$$\left[\frac{n^2+n+2}{2}-C_2^i+1,\frac{n^2+n+2}{2}-C_2^{i-1}-C_2^{n-i+1}-1\right]，$$

其中 $i=\left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor,\left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor+1,\cdots,n$ 。

Abstract

When there are n lines on a plane, it is obvious that the plane can be divided into R pieces, and $\frac{n^2+n+2}{2}$ is its maximum while $n+1$ is its minimum. In Professor Oleg A. Ivanov's essay, he mentions that every natural number in particular ranges actually occurs as the number R. We try to find out the ranges (which we name them "jump") where none of natural numbers in them occurs as the number R. Then we conclude:

When $n \geq 8$, the jump interval will appear as

$$\left[\frac{n^2+n+2}{2}-C_2^i+1, \frac{n^2+n+2}{2}-C_2^{i-1}-C_2^{n-i+1}-1\right],$$

in which $i=\left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor, \left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor+1, \cdots, n.$

壹、研究動機

在高一下學期數列與級數單元，我們學會利用遞迴數列求出平面上n條直線將平面分割出的最大區塊數，且進一步對這n條直線可能的分割區塊數感到好奇。在[3]中，Oleg A. Ivanov 教授發表的論文裏提及了關於n條直線切割平面產生的可能區塊數R必定從某個值開始到最大分割數間呈現連續整數分布。

例如：

n	R
4	5, 8, 9, 10, 11
5	6, 10, 12, 13, 14, 15, 16
6	7, 12, 15, 16, 17, 18, 19, 20, 21, 22

我們發現R值在產生連續整數分布之前就會有數段整數區間是不連續的，我們稱這些不連續的區間為「跳躍」(jump)，例如：$n=6$時，就會產生兩次「跳躍」，也就是整數區間$[8,11]$和$[13,14]$。我們對於「跳躍」是如何產生，又會如何分布感到很有興趣。

貳、研究目的

證明：

一、　平面上被n條直線分割出的區域數會產生連續分布的區間。

二、　平面上被n條直線分割出的區域數會產生「跳躍」的整數區間。

參、研究設備及器材

紙、筆

肆、研究過程

一、定義：

1. 一般位置：

在平面上，n條直線任二線不平行且任三線不共點，則我們稱這n條直線在一般位置。

2. R：平面上n條直線可分割出的區塊數。

例：$n=3, R=7$(如圖 1.1)。

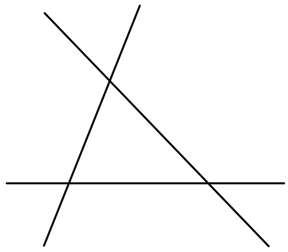


圖1.1

二、平面上n條直線的分割區塊數：

在[3]中我們得知了直線分割平面的區塊數R從一特定值起會呈連續分布，相對的，我們可以說區塊數R在一特定值之前可能會存在一些跳躍。藉由找出連續分布，我們可證明跳躍的不存在，或證明跳躍的存在來說明連續分布的不存在，進而了解區塊數R的整體分布情形。

由[2]，我們得知n條直線切割平面的區塊數R的公式可利用n條直線在一般位置時的分割區塊數$\frac{n^2+n+2}{2}=n+1+C_2^n$，再減掉因為平行或共點時所少掉的區塊數。因為當$\lambda_i$條直線從一般位置變為$\lambda_i$條直線共線時，其$R$值減少：

$$\frac{\lambda_i{}^2+\lambda_i+2}{2}-2\lambda_i=\frac{\lambda_i{}^2-3\lambda_i+2}{2}=\frac{(\lambda_i-1)(\lambda_i-2)}{2}=C_2^{\lambda_i-1}\text{ ，}$$

其中當$\lambda_i=2$時，$C_2^{\lambda_i-1}=0$。當μ_j條直線從一般位置變為μ_j條直線平行時，其R值減少：

$$\frac{\mu_j{}^2+\mu_j+2}{2}-(\mu_j+1)=\frac{\mu_j{}^2-\mu_j}{2}=\frac{\mu_j(\mu_j-1)}{2}=C_2^{\mu_j}\text{ 。}$$

所以我們可得到R的公式為：

$$R=n+1+C_2^n-\sum_{i=1}^{m}C_2^{\lambda_i-1}-\sum_{j=1}^{p}C_2^{\mu_j}\text{ ，}$$

其中λ_i為通過第i個交點的直線數，μ_j為第j個平行線集合中的直線數。其中，$n+1+C_2^n$為一般位置時的區塊數。我們設$Q=\sum_{i=1}^{m}C_2^{\lambda_{i-1}}+\sum_{j=1}^{p}C_2^{\mu_j}$，顯然$Q=n+1+C_2^n-R$，以下我們藉由討論$Q$的範圍來了解$R$值的分佈情況。

【定理 2.1】

平面上有n條線，若此n條直線中最多有μ條直線構成同一組平行線，最多有λ條直線交於一點，則：

(1) 當$\mu\geq\frac{n}{2}-1$時，則$C_2^{\mu}\leq Q\leq C_2^{\mu}+C_2^{n-\mu}$，且在此範圍中的所有$Q$值皆可對應到平面上一種$n$條直線的放法。

(2) 當$\lambda \ge \frac{n}{2}$時，則$C_2^{\lambda-1} \le Q \le C_2^{\lambda-1} + C_2^{n-\lambda+1}$，且在此範圍中的所有$Q$值皆可對應到平面上一種$n$條直線的放法。

<pf>

(1) 先固定平面上的μ條平行線，顯然每增加一條直線就會增加$a+1$個區塊數，其中a為增加的直線與本來存在於平面直線的交點數。當加上第一條直線，因為此線不能再與原μ條平行線平行，所以與原本μ條平行線的交點數$a_1 = \mu$；接著加入第二條直線，交點數a_2滿足：$\mu \le a_2 \le \mu+1$；加入第三條直線，交點數a_3滿足：$\mu \le a_3 \le \mu+2$；依此類推，加入第i條直線，交點數a_i滿足：$\mu \le a_i \le \mu+(i-1)$，$i = 1,2,\cdots,n-\mu$。所以平面的分割區塊數$R$滿足：

$$\mu+1+(n-\mu)(\mu+1) \le R \le \mu+1+\frac{(n+\mu+1)(n-\mu)}{2}$$ 。

又$Q = \frac{n^2+n+2}{2} - R$，則：

$$\frac{n^2+n+2}{2} - \mu - 1 - \frac{(n+\mu+1)(n-\mu)}{2} \le Q \le \frac{n^2+n+2}{2} - \mu - 1 - (n-\mu)(\mu+1)$$

$$\frac{\mu(\mu-1)}{2} \le Q \le \frac{(n^2-2n\mu+\mu^2)-(n-\mu)+(\mu^2-\mu)}{2} \Rightarrow C_2^{\mu} \le Q \le C_2^{\mu} + C_2^{n-\mu}$$ 。

接著我們證明在這個範圍中的每個Q值皆可對應到一種n條直線的放法。設平面上有兩組直線數分別為μ和$n-\mu$且等距分佈的平行線（如圖 2.1），此時$Q = C_2^{\mu} + C_2^{n-\mu}$。我們將第一組平行線由上至下編號為$l_1, l_2, ..., l_\mu$，第二組平行線由右至左逐次編號$L_1, L_2, ..., L_{n-\mu}$，並從$L_1$開始移動，因為$\mu \ge \frac{n}{2} - 1$，所以$\mu \ge n-\mu-2$，我們必能使$L_1$通過$l_1$與$L_{n-\mu-1}$的交點、$l_2$與$L_{n-\mu-2}$的交點、…、$l_{n-\mu-2}$與$L_2$的交點(如圖 2.2)。

此時Q值減少一，意即$Q = C_2^{\mu} + C_2^{n-\mu} - 1$。接著我們將$L_{n-\mu-1}$往左作些許平移，使之不通過$L_1$與$l_1$的交點，則此時$Q$值再減少一，意即$Q = C_2^{\mu} + C_2^{n-\mu} - 2$。緊接著我們再將$L_{n-\mu-2}$往左作等量的些許平移，使之不通過$L_1$與$l_2$的交點，依此類推，…，將$L_2$往左作等量的些許平移，使之不通過$L_1$與$l_{n-\mu-2}$(如圖 2.3)，這時$Q$值已比原來兩組平行線時的情況減少了$n-\mu-1$，此時$Q = C_2^{\mu} + C_2^{n-\mu} - (n-\mu-1)$。

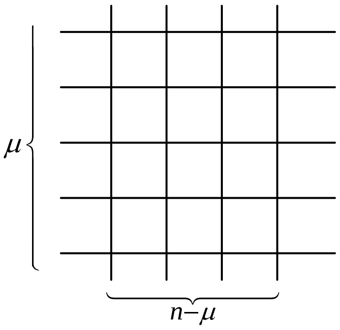


圖2.1

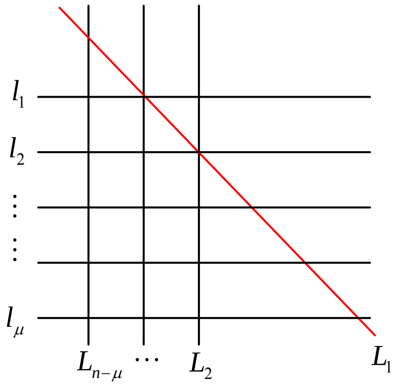


圖2.2

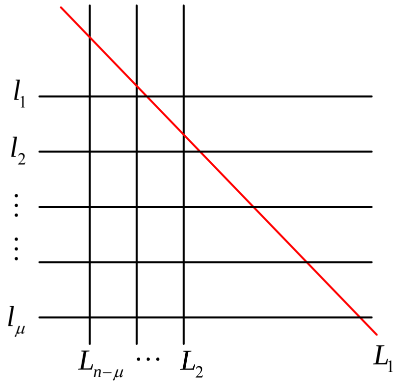


圖2.3

接下來我們比照剛才的方式，移動 L_2 使之通過 l_1 與 $L_{n-\mu-1}$ 的交點、l_2 與 $L_{n-\mu-2}$ 的交點、⋯、$l_{n-\mu-3}$ 與 L_3 的交點，這時 Q 值再減少一。接著再將 $L_{n-\mu-1}$ 往左作些許平移，逐次移動使之不通過上述的交點，則每次移動 Q 值即減少一，此時 $Q=C_2^{\mu}+C_2^{n-\mu}-(n-\mu-1)-(n-\mu-2)$。以下再以同樣的方式，依序移動 $L_{3,}L_{4,},\ldots,L_{n-\mu-2}$，$Q$ 值分別依序減少 $(n-\mu-3),\cdots,2$，最後再將 $L_{n-\mu-1}$ 作些許微調，使之與 $L_{n-\mu}$ 相交，Q 值再減少一，因此此時

$$Q=C_2^{\mu}+C_2^{n-\mu}-\left[(n-\mu-1)+(n-\mu-2)+\cdots+1\right]=C_2^{\mu}+C_2^{n-\mu}-\frac{(n-\mu)(n-\mu-1)}{2}=C_2^{\mu}$$ 。

又因為每次移動皆使 Q 值減一，所以我們可得知當 $\mu\geq\frac{n}{2}-1$ 時，則 $C_2^{\mu}\leq Q\leq C_2^{\mu}+C_2^{n-\mu}$，且在此範圍中的所有 Q 值皆可對應到一種平面上 n 條直線的放法。

(2) 先固定平面上的 λ 條共點直線，當加入第一條直線，此線不能再通過原本 λ 條直線所共的點，但可與其中一條線平行或皆不平行，故與原本 λ 條直線的交點數 a_1，滿足：$\lambda-1\leq a_1\leq\lambda$；接著加入第二條直線，交點數 a_2 滿足：$\lambda-1\leq a_2\leq\lambda+1$；加入第三條直線，交點數 a_3 滿足：$\lambda-1\leq a_3\leq\lambda+2$；依此類推，加入第 i 條直線，交點數 a_i 滿足：

$\lambda-1\leq a_i\leq\lambda+(i-1)$，$i=1,2,\cdots,n-\lambda$。所以平面的分割區塊數 R 滿足：

$$2\lambda+(n-\lambda)\lambda\leq R\leq 2\lambda+\frac{(n+\lambda+1)(n-\lambda)}{2}$$ 。

又 $Q=\frac{n^2+n+2}{2}-R$，則：

$$\frac{n^2+n+2}{2}-2\lambda-\frac{(n+\lambda+1)(n-\lambda)}{2}\leq Q\leq\frac{n^2+n+2}{2}-2\lambda-(n-\lambda)\lambda$$

$$\frac{(\lambda-1)(\lambda-2)}{2}\leq Q\leq\frac{(\lambda^2-3\lambda+2)+(n-\lambda)+(n^2-2n\mu+\lambda^2)}{2}\Rightarrow C_2^{\lambda-1}\leq Q\leq C_2^{\lambda-1}+C_2^{n-\lambda+1}$$ 。

接著我們證明在這個範圍中的每個Q值皆可對應到一種n條直線的放法。設平面上只有一組平行線，及一組通過同一共點的線(平行線組中有一條線通過此共點)，且$\lambda_1=\lambda,\ \mu_1=n-\lambda+1$(如圖 2.4)，此時$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}$。我們將共點之線由左至右編號為$l_1,l_2,...,l_\lambda$，平行線由下至上逐次編號$L_1,L_2,...,L_{n-\lambda+1}$，並從$L_1$開始移動，因為$\lambda\geq\dfrac{n}{2}$，所以$\lambda-1\geq n-\lambda-1$，我們必能使$L_1$通過$l_1$與$L_{n-\lambda}$的交點、$l_2$與$L_{n-\lambda-1}$的交點、…、$l_{n-\lambda-1}$與$L_2$的交點(如圖 2.5)。此時$Q$值減少一，意即$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}-1$。接著我們再$L_{n-\lambda}$往上些許平移，使之不通過$l_1$與$L_1$的交點。此時$Q$值再減少一，意即$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}-2$。緊接著我們再將$L_{n-\lambda-1}$往上些許平移，使之不通過$l_2$與$L_1$的交點，依此類推，…，將$L_2$往上些許平移，使之不通過$l_{n-\lambda-1}$與$L_1$的交點(如圖 2.6)，這時$Q$值已比原來一組平行線及一組共點線的情況減少了$n-\lambda$，此時$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}-(n-\lambda)$。

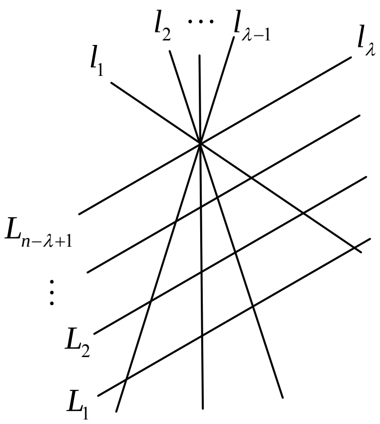


圖2.4

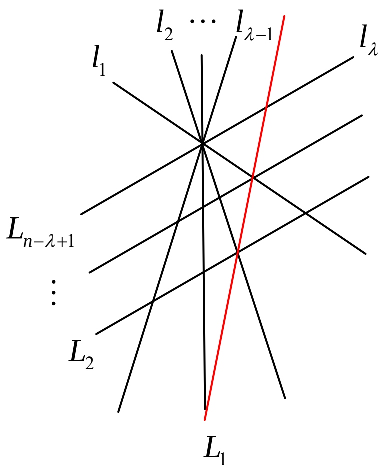


圖2.5

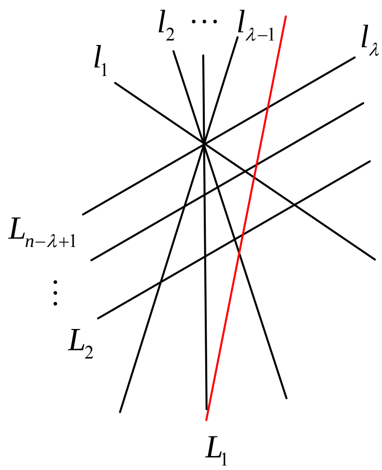


圖2.6

接下來我們比照剛才的方式，移動L_2使之通過l_1與$L_{n-\lambda}$的交點、l_2與$L_{n-\lambda-1}$的交點、…、$l_{n-\lambda-2}$與L_3的交點，這時Q值再減少一。接著再將$L_{n-\lambda}$往上些許平移，逐次移動使之不通過上述的交點，則每次移動Q值即減少一，此時
$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}-(n-\lambda)-(n-\lambda-1)$。

以下再以同樣的方式，依序移動$L_3,L_4,...,L_{n-\lambda-1}$後，$Q$值分別依序減少$n-\lambda-2,\cdots,2$，最後再將$L_{n-\lambda}$作些許微調，使之與$L_{n-\lambda+1}$相交，$Q$值再減一，因此這時

$$Q=C_2^{\lambda-1}+C_2^{n-\lambda+1}-\left[\left(n-\lambda\right)+\left(n-\lambda-1\right)+\cdots+1\right]=C_2^{\lambda-1}+C_2^{n-\lambda+1}-\frac{(n-\lambda+1)(n-\lambda)}{2}=C_2^{\lambda-1}\text{。}$$

又因為每次移動皆使Q值減一，所以我們可得知當$\lambda\geq\dfrac{n}{2}+1$時，則
$C_2^{\lambda-1}\leq Q\leq C_2^{\lambda-1}+C_2^{n-\lambda+1}$，且在此範圍中的所有$Q$值皆可對應到平面上一種$n$條直線的放法。

⊠

【定理 2.2】

平面上，對每一個整數 R 滿足 $R_n \le R \le \frac{n^2+n+2}{2}$，皆存在 n 條直線的圖形恰將平面分割成 R 塊區域，其中

$$R_n = \begin{cases} \frac{(n+1)(n+3)}{4}, \text{當}n\text{為奇數，} \\ \frac{(n+2)^2}{4}, \text{當}n\text{為偶數。} \end{cases}$$

<pf>

我們利用數學歸納法加以證明。當 $n=1$、$n=2$ 時，R_n 的值恰為 $\frac{n^2+n+2}{2}$，顯然成立，當 $n=3$ 時，$R_n=6$，顯然將三線交於一點時 $R=6$，又三線在一般位置時 $R=\frac{n^2+n+2}{2}=7$，故定理成立。接著我們分別依 n 的奇、偶分成兩個部分討論。

(1) 設當 $n=k\ge 3$，k 為奇數時定理成立，亦即滿足 $\frac{(k+1)(k+3)}{4} \le R \le \frac{k^2+k+2}{2}$ 的整數 R，皆存在分割區塊數恰為 R 的 n 條直線圖形。當 $n=k+1$ 時，此時再加上一條直線，並使此直線不與前 k 條直線中的任一條平行，也不通過任何已存在的交點。因此 $\frac{(k+1)(k+3)}{4}+k+1$ 到 $\frac{k^2+k+2}{2}+k+1$ 的所有整數皆為 n 條直線分割區塊數的可能值。

由【定理 2.1】得知當 $\lambda \ge \frac{n}{2}$ 時，$C_2^{\lambda-1} \le Q \le C_2^{\lambda-1}+C_2^{n-\lambda+1}$，且在此範圍中的所有 Q 值皆可對應到平面上一種 n 條直線的放法，所以 $\frac{n^2+n+2}{2}-C_2^{\lambda-1}-C_2^{n-\lambda+1}$ 到 $\frac{n^2+n+2}{2}-C_2^{\lambda-1}$ 的所有整數皆為 n 條直線分割區塊數的可能值。使 $n=k+1$，取 $\lambda=\frac{k+3}{2}$，則得：

$\frac{(k+3)^2}{4}$ 到 $\frac{(k+3)^2}{4}+\frac{k^2-1}{8}$ 的所有整數皆為 n 條直線分割區塊數的可能值。又

$$\frac{(k+1)(k+3)}{4}+k+1 \le \frac{(k+3)^2}{4}+\frac{k^2-1}{8} \Leftrightarrow 2\left(k^2+8k+7\right) \le 2\left(k^2+6k+9\right)+k^2-1$$

$$\Leftrightarrow (k-3)(k-1) \ge 0 \Leftrightarrow k \ge 3 \text{ or } k \le 1,$$

因此 $R_n=\frac{(k+3)^2}{4}$ 到 $\frac{k^2+k+2}{2}+k+1$ 的所有整數皆為 n 條直線分割區塊數的可能值，定理成立。

(2)　　設當 $n=k\geq 2$，k 為偶數時定理成立，亦即滿足 $\frac{(k+2)^2}{4}\leq R\leq \frac{k^2+k+2}{2}$ 的整數 R，

皆存在分割區塊數恰為 R 的 n 條直線圖形。此時再加上一條直線，並使此直線不與前 k 條直線中的任一條平行，也不通過任何已存在的交點。此時 $\frac{(k+2)^2}{4}+k+1$ 到 $\frac{k^2+k+2}{2}+k+1$ 的所有整數皆為 n 條直線分割區塊數的可能值。再次由【定理 2.1】，使 $n=k+1$，取 $\lambda=\frac{k+2}{2}$，則得：$\frac{(k+2)(k+4)}{4}$ 到 $\frac{(k+2)(k+4)}{4}+\frac{(k+2)k}{8}$ 的所有整數皆為 n 條直線分割區塊數的可能值，又

$$\frac{(k+2)^2}{4}+k+1\leq \frac{(k+2)(k+4)}{4}+\frac{(k+2)k}{8}\Leftrightarrow 2\left(k^2+4k+4\right)+8k+8\leq 2\left(k^2+6k+8\right)+k^2+2k$$

$\Leftrightarrow k(k-2)\geq 0\Leftrightarrow k\geq 2 \text{ or } k\leq 0$，

因此 $R_n=\frac{(k+2)(k+4)}{4}$ 到 $\frac{k^2+k+2}{2}+k+1$ 的所有整數皆為 n 條直線分割區塊數的可能值，定理成立。 ⊠

⊠

【定理 2.3】

當 $n\geq 8$，平面上 n 條直線可能的分割區塊數 R 產生的「跳躍」區間為

$$\left[\frac{n^2+n+2}{2}-C_2^i+1,\frac{n^2+n+2}{2}-C_2^{i-1}-C_2^{n-i+1}-1\right],$$

其中 $i=\left\lfloor \frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor,\left\lfloor \frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor+1,\cdots,n$。

<pf>

由【定理 2.1】得知：當 $\mu\geq\frac{n}{2}$ 時，$C_2^{\mu}\leq Q\leq C_2^{\mu}+C_2^{n-\mu}$；當 $\lambda\geq\frac{n}{2}+1$ 時，

$C_2^{\lambda-1}\leq Q\leq C_2^{\lambda-1}+C_2^{n-\lambda+1}$。所以當 $\mu\geq\frac{n}{2}$ 或 $\lambda\geq\frac{n}{2}+1$ 時，所有 Q 值皆滿足：

$C_2^i\leq Q\leq C_2^i+C_2^{n-i}$，其中 $\frac{n}{2}\leq i\leq n$。又

$$C_2^{i-1}+C_2^{n-i+1}+1\leq C_2^i-1\Leftrightarrow i^2-(2n+3)i+\left(n^2+n+6\right)\leq 0$$

$$\Leftrightarrow \frac{2n+3-\sqrt{8n-15}}{2}\leq i\leq \frac{2n+3+\sqrt{8n-15}}{2},$$

所以顯然當 $i\ge\frac{2n+3-\sqrt{8n-15}}{2}$ 時，在 $C_2^{i-1}+C_2^{n-i+1}+1$ 到 $C_2^{i}-1$ 之間必不存在任何其 $\mu\ge\frac{n}{2}$ 或 $\lambda\ge\frac{n}{2}+1$ 的 Q 值。

接著我們討論 $\mu<\frac{n}{2}$ 且 $\lambda<\frac{n}{2}+1$ 的狀況。因為

$$C_2^{i-1}+C_2^{n-i+1}=i^2-(2+n)i+\frac{n^2+n+2}{2}=\left(i-\frac{n+2}{2}\right)^2+\frac{n^2-2n}{4}\text{，}$$

所以當 $i\ge\frac{n+2}{2}$ 時，$C_2^{i-1}+C_2^{n-i+1}$ 為嚴格遞增函數，又

$$\frac{n+2}{2}\le\frac{2n+3-\sqrt{8n-15}}{2}\Leftrightarrow n+1\ge\sqrt{8n-15}\Leftrightarrow n^2-6n+16\ge 0\text{，}$$

此不等式顯然恆成立。故在 $\mu<\frac{n}{2}$ 且 $\lambda<\frac{n}{2}+1$ 的狀況中，我們只需要證明：Q 的最大值小於等於 $C_2^{i-1}+C_2^{n-i+1}$（其中 $i=\left|\frac{2n+3-\sqrt{8n-15}}{2}\right|$）。即可得證：對於任意的正整數 i 滿足：

$\frac{2n+3-\sqrt{8n-15}}{2}\le i\le n$，在 $C_2^{i-1}+C_2^{n-i+1}+1$ 到 $C_2^{i}-1$ 之間必不存在任何 Q 值。

考慮當 $\mu<\frac{n}{2}$ 且 $\lambda<\frac{n}{2}+1$ 時 Q 值可能的最大值。顯然當 μ 或 λ 取到最大時 Q 值會得到最大值，即 $\mu=\left\lfloor\frac{n-1}{2}\right\rfloor$、$\lambda=\left\lfloor\frac{n+1}{2}\right\rfloor$，故我們就 n 的奇偶性討論所有可能的組合狀況。首先，當 n 為奇數時，則 $\mu=\frac{n-1}{2}$、$\lambda=\frac{n+1}{2}$，可能的組合為：

(1) 二組 $\frac{n-1}{2}$ 條平行線和一條通過最多交點的直線(如圖 2.7)，則 $Q=2\times C_2^{\frac{n-1}{2}}+\frac{n-1}{2}$。

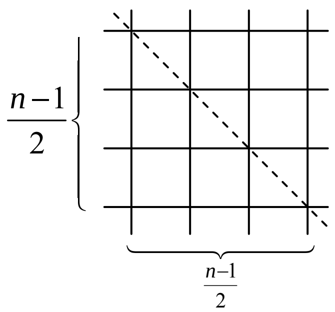


圖2.7

(2) 一組 $\frac{n-1}{2}$ 條平行線、一組 $\frac{n+1}{2}$ 條共點直線和一條通過最多交點且恰成一組兩線平行的直線(如圖 2.8)，則 $Q=C_2^{\frac{n-1}{2}}+C_2^{\frac{n+1}{2}-1}+\left(\frac{n-1}{2}-1\right)\times C_2^{3-1}+C_2^{2}=2\times C_2^{\frac{n-1}{2}}+\frac{n-1}{2}$。

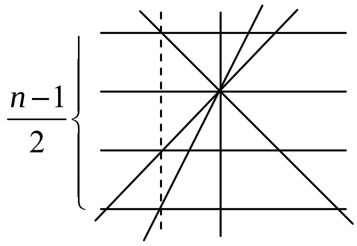


圖2.8

(3) 二組$\frac{n+1}{2}$共點直線，且讓這兩組共點直線除了共用的直線之外其餘直線皆構成兩兩平行（如圖 2.9），則$Q=2\times C_2^{\frac{n+1}{2}-1}+\left(\frac{n+1}{2}-1\right)\times C_2^2=2\times C_2^{\frac{n-1}{2}}+\frac{n-1}{2}$。

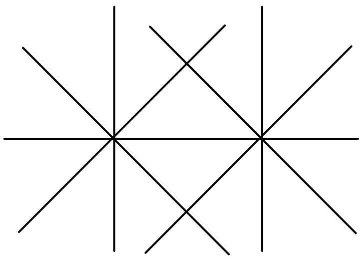


圖2.9

當n為偶數時，則$\mu_1=\frac{n}{2}-1$、$\lambda_1=\frac{n}{2}$，可能的組合為：

(1) 二組$\frac{n}{2}-1$條平行線和二條通過最多交點的直線(如圖 2.10)，則

$$Q=\begin{cases}2\times C_2^{\frac{n}{2}-1}+2\left(\frac{n}{2}-2\right)\times C_2^{3-1}+C_2^{4-1}=2\times C_2^{\frac{n}{2}-1}+n-1\text{ , }n\equiv 0(\bmod 4)\\2\times C_2^{\frac{n}{2}-1}+2\left(\frac{n}{2}-1\right)\times C_2^{3-1}=2\times C_2^{\frac{n}{2}-1}+n-2\text{ , }n\equiv 2(\bmod 4)\end{cases}$$ 。

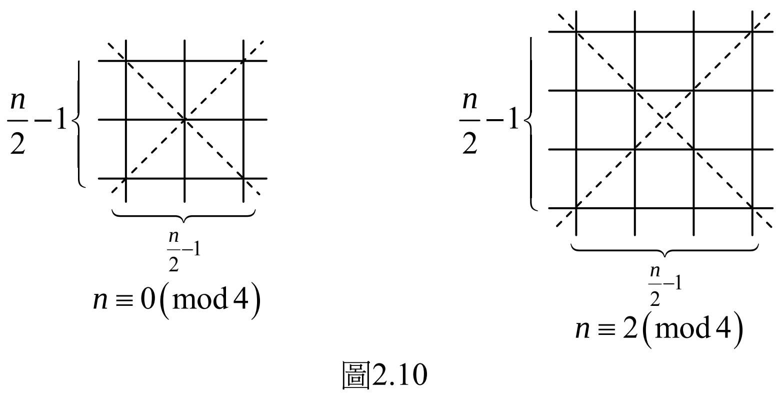
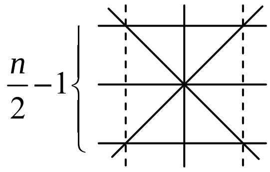


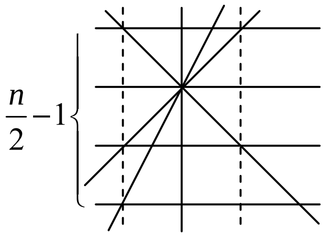
圖2.10

(2) 一組$\frac{n}{2}-1$條平行線、一組$\frac{n}{2}$條共點直線和二條平行同一條直線且通過最多交點的直線(如圖 2.11)，則

$$Q=\begin{cases} C_2^{\frac{n}{2}-1}+C_2^{\frac{n}{2}-1}+C_2^3+2\times\left(\frac{n}{2}-2\right)\times C_2^{3-1}=2\times C_2^{\frac{n}{2}-1}+n-1\ ,\ n\equiv 0(\bmod 4)\\ C_2^{\frac{n}{2}-1}+C_2^{\frac{n}{2}-1}+C_2^3+\left(\frac{n}{2}-2\right)\times C_2^{3-1}+\left(\frac{n}{2}-3\right)\times C_2^{3-1}=2\times C_2^{\frac{n}{2}-1}+n-2\ ,\ n\equiv 2(\bmod 4)\end{cases}$$ 。



$n\equiv 0(\bmod 4)$

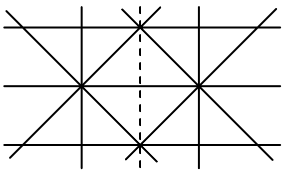


$n\equiv 2(\bmod 4)$

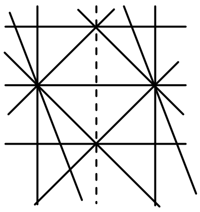
圖2.11

(3) 二組$\frac{n}{2}$條共點直線(共用一組直線且除了共用的直線之外其餘直線皆構成兩兩平行)，和一條通過最多交點且平行原本一組二平行線的直線(如圖 2.12)，則

$$Q=\begin{cases} 2\times C_2^{\frac{n}{2}-1}+\left(\frac{n}{2}-2\right)\times C_2^2+\left(\frac{n}{2}-2\right)\times C_2^{3-1}+C_2^3=2\times C_2^{\frac{n}{2}-1}+n-1\ ,\ n\equiv 0(\bmod 4)\\ 2\times C_2^{\frac{n}{2}-1}+\left(\frac{n}{2}-2\right)\times C_2^2+\left(\frac{n}{2}-3\right)\times C_2^{3-1}+C_2^3=2\times C_2^{\frac{n}{2}-1}+n-2\ ,\ n\equiv 2(\bmod 4)\end{cases}$$ 。



$n\equiv 0(\bmod 4)$



$n\equiv 2(\bmod 4)$

圖2.12

由以上討論可知，我們只需要證明：

$2\times C_2^{\frac{n-1}{2}}+\frac{n-1}{2}\le C_2^{i-1}+C_2^{n-i+1}$，及 $2\times C_2^{\frac{n}{2}-1}+n-1\le C_2^{i-1}+C_2^{n-i+1}$，其中 $i=\left|\frac{2n+3-\sqrt{8n-15}}{2}\right|$，將兩個不等式的右式減掉左式得：

$$C_2^{i-1}+C_2^{n-i+1}-2\times C_2^{\frac{n-1}{2}}-\frac{n-1}{2}=\frac{(i-1)(i-2)}{2}+\frac{(n-i+1)(n-i)}{2}-\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right)-\frac{n-1}{2}$$

$$=\frac{n^2+4n+3+4i^2-8i-4ni}{4}=\frac{1}{4}\left[(n-2i)^2+4(n-2i)+3\right]=\frac{1}{4}(n-2i+3)(n-2i+1)\ ;$$

$$C_2^{i-1}+C_2^{n-i+1}-2\times C_2^{\frac{n}{2}-1}-n+1=\frac{(i-1)(i-2)}{2}+\frac{(n-i+1)(n-i)}{2}-\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)-n+1$$

$$=\frac{n^2+4n+4i^2-8i-4ni}{4}=\frac{1}{4}\left[(n-2i)^2+4(n-2i)\right]=\frac{1}{4}(n-2i)(n-2i+4)$$ 。

因為

$$\frac{2n+3-\sqrt{8n-15}}{2}\le i<\frac{2n+3-\sqrt{8n-15}}{2}+1\Rightarrow -2n-3+\sqrt{8n-15}\ge -2i>-2n-5+\sqrt{8n-15}$$

$$\Rightarrow -n+1+\sqrt{8n-15}\ge n-2i+4>-n-1+\sqrt{8n-15}$$ ，且

$$-n+1+\sqrt{8n-15}\le 0\Leftrightarrow (n-1)^2\ge 8n-15\Leftrightarrow (n-8)(n-2)\ge 0\Leftrightarrow n\ge 8 \text{ or } n\le 2$$ ，

故當 $n\ge 8$ 時，$n-2i<n-2i+1<n-2i+3<n-2i+4\le 0$ ，所以

$2\times C_2^{\frac{n-1}{2}}+\frac{n-1}{2}\le C_2^{i-1}+C_2^{n-i+1}$ 且 $2\times C_2^{\frac{n}{2}-1}+n-1\le C_2^{i-1}+C_2^{n-i+1}$ 。

綜合以上所述，可得知當 $n\ge 8$ 時，對於任意的正整數 i 滿足：$i\ge\frac{2n+3-\sqrt{8n-15}}{2}$ ，$C_2^{i-1}+C_2^{n-i+1}+1$ 到 C_2^i-1 範圍內的 Q 值必不存在。意即 n 條直線的分割區塊數 R 必不可能存在於在區間 $\left[\frac{n^2+n+2}{2}-C_2^i+1,\frac{n^2+n+2}{2}-C_2^{i-1}-C_2^{n-i+1}-1\right]$ 之中。

由證明一開始的討論可知，當 $\frac{n}{2}\le i\le\left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor-1$ 時，區間 $\left[C_2^i,C_2^i+C_2^{n-i}\right]$ 在相鄰的兩個區間必產生重疊，且由【定理 2.1】得知在這些區間中必存在可對應 n 條直線擺放的圖形，故區間

$$\bigcup_{i=\left\lceil\frac{n}{2}\right\rceil}^{\left\lfloor\frac{2n+3-\sqrt{8n-15}}{2}\right\rfloor-1}\left[\frac{n^2+n+2}{2}-C_2^i-C_2^{n-i},\frac{n^2+n+2}{2}-C_2^i\right]$$

必是一個上界為 $\frac{n^2+n+2}{2}-C_2^{\left\lfloor\frac{n}{2}\right\rfloor}$ 的區間，且區間中所有整數必為 n 條直線分割區塊數的可能值，又由【定理 2.1】得知：區間 $\left[R_n,\frac{n^2+n+2}{2}\right]$ 中所有整數必為 n 條直線分割區塊數的可能值，所以我們只需要證明：$R_n\le\frac{n^2+n+2}{2}-C_2^{\left\lfloor\frac{n}{2}\right\rfloor}$ 即可得證

$\left[\frac{n^2+n+2}{2}-C_2^i+1,\frac{n^2+n+2}{2}-C_2^{i-1}-C_2^{n-i+1}-1\right]$ ，其中 $i=\left\lceil\frac{2n+3-\sqrt{8n-15}}{2}\right\rceil,\cdots,n$ ，

是平面上 n 條直線可能分割區塊數的所有「跳躍」區間。接著我們依 n 的奇偶性來證明這個不等式必成立：

(1) n為奇數時，

$$R_n = \frac{(n+1)(n+3)}{4} \le \frac{n^2+n+2}{2} - C_2^{\frac{n+1}{2}} \Leftrightarrow 2\left(n^2+4n+3\right) \le 4\left(n^2+n+2\right) - (n+1)(n-1)$$

$\Leftrightarrow (n-3)(n-1) \ge 0 \Leftrightarrow n \ge 3 \text{ or } n \le 1$，

因為$n \ge 8$，故不等式成立。

(2) 當n為偶數

$$R_n = \frac{(n+2)^2}{4} \le \frac{n^2+n+2}{2} - C_2^{\frac{n}{2}} \Leftrightarrow 2\left(n^2+4n+4\right) \le 4\left(n^2+n+2\right) - n(n-2)$$

$\Leftrightarrow n(n-2) \ge 0 \Leftrightarrow n \ge 2 \text{ or } n \le 0$，

☒

因為$n \ge 8$，故不等式成立。

伍、研究結果

當$n \ge 8$，平面上n條直線可能的分割區域數R產生的「跳躍」區間為

$$\left[\frac{n^2+n+2}{2} - C_2^i + 1, \frac{n^2+n+2}{2} - C_2^{i-1} - C_2^{n-i+1} - 1\right],$$

其中$i = \left\lfloor \frac{2n+3-\sqrt{8n-15}}{2} \right\rfloor, \left\lfloor \frac{2n+3-\sqrt{8n-15}}{2} \right\rfloor + 1, \cdots, n$。

陸、結論

平面上，被n條直線所切割出的區塊數可能值，除特定區間外分布於所有介於最大值到最小值的所有整數。

柒、參考資料

1. John E. Wetzel, 1978 , On the Division of the Plane by Lines, The American Mathematical Monthly, Vol. 85, No. 8 , pp. 647-656.
2. Oleg A. Ivanov, 2010 , On the Number of Regions into Which n Straight Lines Divide the Plane, The Mathematical Association of America, 117, p.881~888.

Introduction

Given n lines on the plane, Roberts gave a formula to count the number of regions r formed by any arbitrary arrangement [1]. While the minimum number $r_{\min}$ and the maximum number $r_{\max}$ of Roberts' formula can be easily determined and they form the two endpoints of an interval of number of regions J_n, not every natural number between the two endpoints can be included in the interval J_n. In other words, one will notice that there are "gaps" in the interval J_n, and they form the gap integral interval G_n. The purpose of this study is to identify every number can or cannot be realized as the number of regions.

Definition

1. Define Ω as a set of every arbitrary arrangement construct by n lines, then define ω as an arbitrary arrangement, and $\omega \in \Omega$.
2. Define r as the number of regions which the plane is divided into by n lines, and as $r_{\min}$ is the least possible number of regions which n lines may divide the plane into, $r_{\max}$ is the greatest possible number of regions which n lines may divide the plane into.
3. In Roberts' formula we count the number of regions

$$r = R(\omega) = n + 1 + C_2^n - \sum_{i=1}^{m} C_2^{\lambda_i - 1} - \sum_{j=1}^{p} C_2^{\mu_j} ,$$

in which λ_i denote the number of lines which pass through the point i (and there are m points in the arrangement ω), μ_j denote the number of lines in the parallel family j (and there are p parallel families in the arrangement ω).
4. $J_n = \{ r = R(\omega) | \omega \in \Omega \}$, $G_n = \{ g | r_{\min} \le g \le r_{\max} \wedge g \notin J_n \}$. While J_n is an interval of every possible number of regions which n lines can divide the plane into, G_n is an interval of numbers between $r_{\min}$ and $r_{\max}$ that n lines cannot divide the plane into.
5. $\lambda = \Lambda(\omega) = \max\{\lambda_1 \lambda_2, ..., \lambda_m\}$, $\mu = \mathrm{M}(\omega) = \max\{\mu_1, \mu_2, ..., \mu_p\}$. So λ is the greatest number of lines passing through a point, and μ is the greatest number of lines that parallel each other.
6. $J_{n,\mu} = \{ r = R(\omega) | \omega \in \Omega \wedge \mathrm{M}(\omega) = \mu \}$, $J_{n,\lambda} = \{ r = R(\omega) | \omega \in \Omega \wedge \Lambda(\omega) = \lambda \}$. $J_{n,\mu}$ and $J_{n,\lambda}$ are subintervals of every possible number of regions which the plane is divided into by n lines under the condition where there are at most μ lines parallel each other or at most λ lines intersect at a point, respectively.

Proof

Theorem 1.

Suppose there are n lines on the plane, when

(1) given the first group of parallel lines μ_1, $(\mu_1+1)(n-\mu_1+1) \le r \le \frac{n^2+n-\mu_1^2+\mu_1+2}{2}$.

(2) given the first group of lines that intersect a point λ_1, $\lambda_1(n-\lambda_1+2) \le r \le \frac{n^2+n-\lambda_1^2+3\lambda_1}{2}$.

<proof>

(1) Suppose there are μ_1 lines, L_1, $L_2,\ldots,$ L_{μ_1}, paralleling each other on the plane. It is obvious that when one line is added on the plane and the lines intersect other lines at a points, the number of regions will increase by $a+1$. When line L_{μ_1+1} is added on the plane, because line L_{μ_1+1} can parallel with none of the μ_1 lines, so line L_{μ_1+1} will pass through at least and at most μ_1 lines at different points($\mu_1 \le a_1 \le \mu_1$). When line L_{μ_1+2} is added on the plane, we will find that the line also can pass through μ_1 points at least and pass through the μ_1+1 at most ($\mu_1 \le a_2 \le \mu_1+1$). When line L_{μ_1+3} is added on the plane, $\mu_1 \le a_3 \le \mu_1+2$. In conclusion, when line L_i is added on the plane, $\mu_1 \le a_i \le \mu_1+i-1$, and the range of number of regions can be identified

$$\mu_1+1+(n-\mu_1)(\mu_1+1) \le r \le \mu_1+1+\frac{(n-\mu_1)(n+\mu_1+1)}{2}$$

(2) Suppose there are λ_1 lines, L_1, $L_2,\ldots,$ L_{λ_1} intersecting at one point on the plane. When line L_{λ_1+1} is added on the plane, as line L_{λ_1+1} parallel or not parallel with one of the lines in the λ_1 lines , line L_{λ_1+1} will pass through at least λ_1-1 and at most λ_1 lines at different points($\lambda_1-1 \le a_1 \le \lambda_1$). When line L_{λ_1+2} is added on the plane, we will find that the line can also pass through λ_1-1 points at least and can pass through the λ_1+1 ($\lambda_1-1 \le a_2 \le \lambda_1+1$) points at most. When line L_{λ_1+3} is added on the plane, $\lambda_1-1 \le a_3 \le \lambda_1+2$. In conclusion, when line L_i is added on the plane, $\lambda_1-1 \le a_i \le \lambda_1+i-1$.

In addition, because when $\mu=\mu_1$, there are at least a group of μ_1 parallel lines, or when $\lambda=\lambda_1$, there are at least λ_1 lines passing through the same point, we can say that

$$J_{n,\mu=\mu_1} \subseteq \left\{ r \,\middle|\, (\mu_1+1)(n-\mu_1+1) \le r \le \frac{n^2+n-\mu_1^2+\mu_1+2}{2} \right\} \text{ or,}$$

$$J_{n,\lambda=\lambda_1} \subseteq \left\{ r \,\middle|\, \lambda_1(n-\lambda_1+2) \le r \le \frac{n^2+n-\lambda_1^2+3\lambda_1}{2} \right\}.$$

Theorem 2.

Suppose there are n lines on the plane,

(1) when $\mu \ge \frac{n}{2}$, $J_{n,\mu} = \left\{ r \,\middle|\, (\mu+1)(n-\mu+1) \le r \le \frac{n^2+n-\mu^2+\mu+2}{2} \right\}$.

(2) when $\lambda \ge \frac{n}{2}$, $J_{n,\lambda} = \left\{ r \,\middle|\, \lambda(n-\lambda+2) \le r \le \frac{n^2+n-\lambda^2+3\lambda}{2} \right\}$.

<proof>

(1) We construct every integral in the subinterval $J_{n,\mu}$ to prove that every number in the subinterval can be arranged as the number of regions. First we arrange the n lines into μ parallel lines and $n-\mu$ parallel lines, and the plane is divided into the least possible number of regions $r=(\mu+1)(n-\mu+1)$ (figure 2.1). Then we move line $L_{\mu+1}$ and make it pass through L_1 and L_{n-1} at one point, L_2 and L_{n-2} at one point,…, L_i and L_{n-i} at a point ($n-\mu-2 \ge i \ge 1$) (figure 2.2) and make the number of regions r increase by one. Next we move L_{n-1} to make it not intersect L_1 and $L_{\mu+1}$ at a point, move L_{n-2} to make it not intersect L_2 and $L_{\mu+1}$ at a point, ... , move $L_{\mu+i}$ to make it not intersect $L_{n-\mu-i}$ and $L_{\mu+1}$ at a point ($n-\mu-1 \ge i \ge 2$), and the number of regions r will increase by one at a time, $r=(\mu+1)(n-\mu+1)+(n-\mu-1)$ (figure 2.3).

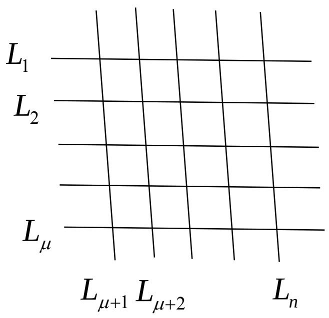


figure 2.1

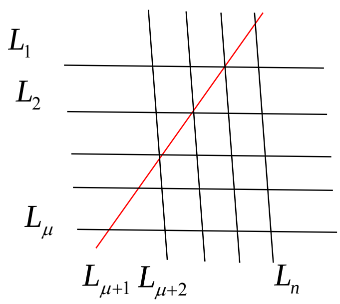


figure 2.2

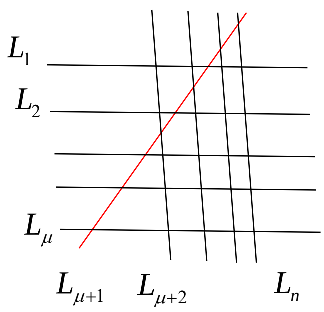


figure 2.3

Following the moving mode above, we move line $L_{\mu+2}$ and make it pass through L_1 and L_{n-1} at one point, L_2 and L_{n-2} at one point,…, L_i and L_{n-i} ($n-\mu-3 \ge i \ge 1$) at a point and make the number of regions r increase by one. Next we move line L_{n-1}, L_{n-2}, …, $L_{\mu+3}$ to make it not intersect L_1, L_2, …, $L_{n-\mu-3}$ and $L_{\mu+2}$ at a point, and the number of regions r will increase by one at a time, $r=(\mu+1)(n-\mu+1)+(n-\mu-1)+(n-\mu-2)$.

In addition, line $L_{\mu+3}$, $L_{\mu+4}$, …, L_{n-2} can also follow the same pattern as line $L_{\mu+1}$ and $L_{\mu+2}$, and the number of regions r will increase by $n-\mu-3$ to 2,

$$r=(\mu+1)(n-\mu+1)+\left[(n-\mu-1)+(n-\mu-2)+\cdots+1\right]=\frac{n^2+n-\mu^2+\mu+2}{2}.$$

When we move the lines, the number of regions r will increase by one, so every number between $(\mu+1)(n-\mu+1)$ and $\frac{n^2+n-\mu^2+\mu+2}{2}$ can be arranged as the number of regions.

(2) We construct every integral in the subinterval $J_{n,\lambda}$ to prove that every number in the subinterval can be arranged as the number of regions. First we arrange the n lines into λ lines that pass through a point and $n-\lambda$ lines that parallel to one of the λ lines, and the plane is divided into the least possible number of regions $r=\lambda(n-\lambda+2)$ (figure 2.4). Then we move line $L_{\lambda+1}$ and make it pass through $L_{\lambda-1}$ and L_n at one point, $L_{\lambda-2}$ and L_{n-1} at one point,…, $L_{\lambda-1}$ and L_{n-i+1} at a point ($n-\lambda-1\geq i\geq 1$) (figure 2.5) and make the number of regions r increase by one. Next we move L_n to make it not intersect $L_{\lambda-1}$ and $L_{\lambda+1}$ at a point, move L_{n-1} to make it not intersect $L_{\lambda-2}$ and $L_{\lambda+1}$ at a point, ... , move L_{n-i+1} to make it not intersect $L_{\lambda-1}$ and $L_{\lambda+1}$ at a point ($n-\lambda-1\geq i\geq 1$), and the number of regions r will increase by one at a time, $r=\lambda(n-\lambda+2)+(n-\lambda)$ (figure 2.6).

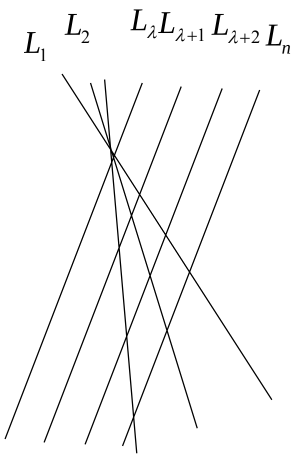


figure 2.4

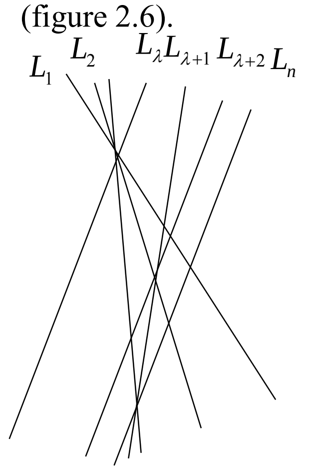


figure 2.5

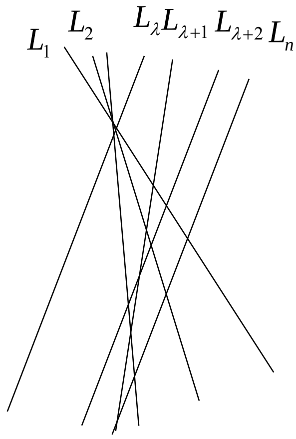


figure 2.6

Following the moving mode above, we move line $L_{\lambda+2}$ and make it pass through $L_{\lambda-1}$ and L_n at one point, $L_{\lambda-2}$ and L_{n-1} at one point,…, $L_{\lambda-i}$ and L_{n-i+1} ($n-\lambda-2\geq i\geq 1$) at a point and make the number of regions r increase by one. Next we move line L_n, L_{n-1}, …, $L_{\lambda+3}$ to make it not intersect $L_{\lambda-1}$, $L_{\lambda-1}$, …, $L_{2\lambda-n+2}$ and $L_{\lambda+2}$ at a point, and the number of regions r will increase by one at a time, $r=\lambda(n-\lambda+2)+(n-\lambda)+(n-\lambda-1)$.

In addition, line $L_{\lambda+3}$, $L_{\lambda+4}$, …, L_{n-2} can also follow the same pattern as $L_{\lambda+1}$ and $L_{\lambda+2}$, and increase the number of regions r by $n-\lambda-2$ to 3,

$$r=\lambda(n-\lambda+2)+\left[(n-\lambda)+(n-\lambda-1)+\cdots+1\right]=\frac{n^2+n-\lambda^2+3\lambda}{2}.$$

When we move the lines, the number of regions r will increase by one, so every number between $\lambda(n-\lambda+2)$ and $\frac{n^2+n-\lambda^2+3\lambda}{2}$ can be arranged as the number of regions.

Theorem 3

Suppose $k=\max\{\lambda,\mu\}$,

$$\frac{n^2}{k}+1\le r.$$

<proof>

From [1] we know that

$$R(\omega)=r=\frac{n^2+n+2}{2}+\sum_{i=1}^{m}\frac{(\lambda_i-1)(\lambda_i-2)}{2}+\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2},$$

in which λ_i denote the number of lines which passes through the point i (and there are m points in the arrangement ω), μ_j denote the number of lines in the parallel family j (and there are p parallel families in the arrangement ω).

The formula can be rewritten as

$$\frac{n^2+n+2}{2}=r+\sum_{i=1}^{m}\frac{(\lambda_i-1)(\lambda_i-2)}{2}+\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2},$$

and then we list an inequality

$$\frac{n^2+n+2}{2}=r+\sum_{i=1}^{m}\frac{(\lambda_i-1)(\lambda_i-2)}{2}+\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}\le r+\frac{(k-2)}{2}\sum_{i=1}^{m}(\lambda_i-1)+\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}.$$

From [2] we are given another formula that counts the number of regions r,

$$r=n+1+\sum_{i=1}^{m}(\lambda_i-1),$$

so the inequality can be rewritten as

$$\frac{n^2+n+2}{2}\le r+\frac{(k-2)}{2}(r-n-1)+\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}.$$

Next, for the right formula, we minus $\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}$, and for the left formula, we minus the maximum possible number of $\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}$.

Suppose there are x lines and y lines on the plane, when $x+y\le k$, $\frac{x(x-1)}{2}+\frac{y(y-1)}{2}\le\frac{(x+y)(x+y-1)}{2}$, and when $x+y>k$, $\frac{x(x-1)}{2}+\frac{y(y-1)}{2}\le\frac{k(k-1)}{2}+\frac{(x+y-k)(x+y-k-1)}{2}$. In addition, every lines can only be included in one parallel family, so the best strategy to arrange the n lines to make $\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}$ the maximum possible number is to make $\left\lfloor \frac{n}{k} \right\rfloor$ group k parallel lines and make the rest $n-k\left\lfloor \frac{n}{k} \right\rfloor$ lines parallel to each other, $\sum_{j=1}^{p}\frac{\mu_j(\mu_j-1)}{2}=\left\lfloor \frac{n}{k} \right\rfloor\frac{k^2-k}{2}+\frac{(n-k\left\lfloor \frac{n}{k} \right\rfloor)(n-\left\lfloor \frac{n}{k} \right\rfloor-1)}{2}$.

Because $\left\lfloor \frac{n}{k} \right\rfloor\frac{k^2-k}{2}+\frac{(n-k\left\lfloor \frac{n}{k} \right\rfloor)(n-\left\lfloor \frac{n}{k} \right\rfloor-1)}{2}\le\frac{nk(k-1)}{2k}$, for the left formula of the inequality, we minus $\frac{nk(k-1)}{2k}$, and the new inequality $\frac{n^2+n+2}{2}-\frac{n(k-1)}{2}\le r+\frac{(k-2)(r-n-1)}{2}$.

The inequality is equivalent to $\frac{n^2}{k}+1\le r$.

Theorem 4

(1) Every gap subinterval must be included in the subintervals,

$$G_n \subseteq \left\{ g \,\middle|\, (j+1)n - \frac{j^2+j+2}{2} + 1 \le g \le (j+2)n - (j+1)^2 \wedge 0 \le j \le \frac{-3+\sqrt{8n-15}}{2} \right\}.$$

(2) Every number in the subintervals cannot be arranged as the number of regions.

$$\left\{ g \,\middle|\, (j+1)n - \frac{j^2+j+2}{2} + 1 \le g \le (j+2)n - (j+1)^2 \wedge 0 \le j \le \frac{n-2-\sqrt{n(n-2\sqrt{2n}+4)}}{2} \right\} \subseteq G_n$$

<proof>

(1) From theorem 1, we know that when $\mu \ge \frac{n}{2}$,

$J_{n,\mu} = \left\{ (\mu+1)(n-\mu+1), \ldots, \frac{n^2+n-\mu^2+\mu+2}{2} \right\}$, or when $\lambda \ge \frac{n}{2}$,

$J_{n,\lambda} = \left\{ \lambda(n-\lambda+2), \ldots, \frac{n^2+n-\lambda^2+3\lambda}{n} \right\}$

and every number in the subinterval can realized as the number of region into which n lines divide the plane. Thus when $\mu \ge \frac{n}{2}$ or $\lambda \ge \frac{n}{2}$ the number of regions must be included in the subintervals $\left\{ (i+1)(n-i+1) \le r \le \frac{n^2+n-i^2+i+2}{n} \right\}$ as $i = \mu$ or $\lambda - 1$ and $\frac{n}{2} \le i \le n$. The subinterval can also be rewritten as $\left\{ (j+1)n - j^2 + 1 \le r \le (j+1)n - \frac{j^2+j+2}{2} \right\}$ while $j = n - i$ $(0 \le j \le \frac{n}{2})$.

When $\mu \ge \frac{n}{2}$ or $\lambda \ge \frac{n}{2}$, the lower bound and the upper bound of the subintervals decrease as μ or λ increase so one subinterval can overlap at most two subintervals. Since when $j \le \frac{\sqrt{8n-15}-3}{2}$, $(j+1)n - \frac{j^2+j+2}{2} + 1 \le (j+2)n - (j+1)^2 + 1 - 1$, one notice that when $\frac{n}{2} \ge j > \frac{\sqrt{8n-15}-3}{2}$ every two subintervals overlap and form a larger subinterval in which every number can be realized as the number of regions. Moreover, there may be possible gaps between each two subintervals when $j \le \frac{\sqrt{8n-15}-3}{2}$.

From [2] we know that every number from $\frac{(n+2)^2}{4}$ (when n is even) or $\frac{n(n+3)}{4}$ (when n is odd) to $\frac{n^2+n+2}{2}$ can be arranged as the number of regions r, and one can prove every number which is not proved to be the possible gaps or number of regions must be the number of regions.

(2) For the lower bound of subintervals $J_{n,\mu=i}$ and $J_{n,\mu=n-i}$, $J_{n,\lambda=i}$ and $J_{n,\lambda=n-i}$ is the same, no number can be realized as the number of regions when $\mu \geq \frac{\sqrt{8n-15}-3}{2}$ or $\lambda \geq \frac{\sqrt{8n-15}-1}{2}$.

Based on theorem 3 (i.e. $\frac{n^2}{k}+1 \leq r$), when $\mu < \frac{\sqrt{8n-15}-3}{2}$ and $\lambda < \frac{\sqrt{8n-15}-1}{2}$, $r \geq n^2 \div \frac{\sqrt{8n-15}-3}{2}+1 \geq n^2 \div \frac{\sqrt{8n}}{2}+1 = 2n\sqrt{2n}+1$. Also, when $j \leq \frac{n-2-\sqrt{n^2-2n\sqrt{2n}+4n}}{2}$, $2n\sqrt{2n}+1 \geq (j+2)n-(j+1)^2+1$, and every number in subintervals $\left\{(j+1)n-\frac{j^2+j}{2},\ldots,(j+2)n-(j+1)^2\right\}$ ($j \leq \frac{n-2-\sqrt{n^2-2n\sqrt{2n}+4n}}{2} \wedge j \leq \frac{\sqrt{8n-15}-3}{2}$) cannot be realized as the number of regions.

Discussions and Conclusions

1. The plane can be divided into every number between $r_{\min}$ and $r_{\max}$ except the possible gap subintervals,
$$\left\{ r \middle| r_{\min} \le r \le r_{\min} \wedge r \notin \left\{ (j+1)n - \frac{j^2+j}{2}, \ldots, (j+2)n - (j+1)^2 \right\} \wedge 0 \le j \le \frac{\sqrt{8n-15}-3}{2} \right\} \subseteq J_n .$$

2. Every number in the gap subintervals cannot be realized as the number of regions,
$$\left\{ g \middle| (j+1)n - \frac{j^2+j+2}{2} + 1 \le g \le (j+2)n - (j+1)^2 \wedge 0 \le j \le \frac{n-2-\sqrt{n(n-2\sqrt{2n}+4)}}{2} \right\} \subseteq G_n .$$

3. Except the numbers in the subintervals,
$$\left\{ (j+1)n - \frac{j^2+j}{2}, \ldots, (j+2)n - (j+1)^2 \right\} \quad \left(\frac{n-2-\sqrt{n(n-2\sqrt{2n}+4)}}{2} < j < \frac{\sqrt{8n-15}-3}{2} \right),$$
we can tell any number can or cannot the plane be divided into by n lines.

4. Compared to Arnold's result, this study proved more gap subintervals.

5. Compared to Ivanov's result, this study discusses the question on a normal plane instead of a projective plane. It can be demonstrated that the gaps identified in this study still constitute gaps on the projective plane.

References

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【評語】010050

本作品是第二次參賽，在第一次參賽評審給予的建議是處理不連續跳躍區間。然而，作者真的在這個困難的問題上做出努力，且獲得某種程度的進展。但是這部連續跳躍區間真的是一個困難的問題，要寄望高中生能完全解決應該是不太可能。本作品有不錯的想法，且作品具延伸性，第二次比起第一次進展相當大，是一件不錯的科展作品。