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參展科別 **數學科**

作品名稱 **整係數多項式裡有乾坤-平衡多項式**

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關 鍵 字　平衡多項式、分圓多項式

作者簡介

我們是朱翊豪(左上圖)、黃哲鈺(右上圖)，國立鳳山高中二年級。

來自不同的國中，個性也不相同，但我們的身上都有一個共通點，就是我們熱愛數學，數學對我們來說都有一種不可抵抗的魔力，雖然研究途中遇到了許多困難，但我們一起克服了。科展提升了我們的報告與邏輯思考能力，讓我們的數學不再局限於寫制式的考卷，而是能從生活中發現數學問題，並且解決，科展對高中的我們來說是一份獨特的禮物，也是挑戰。

我們很普通，但因為我們的努力，我們的作品不平凡。

摘要

本文主要研究以數學方法探究物理上的平衡問題，將物理與數學的代數、數論、組合及幾何做了緊密的結合，並導出幾個有趣的結果。

文中得出 n 的因數結構與平衡多項式的關係，以及平衡多項式與平衡問題一般解的關係，並進一步探討其平衡的排序策略及個數。再則將平衡多項式引入複數平面，可以看到許多有趣的幾何性質，並進而回歸至物理的應用。

Abstract

This study concerns the mathematical models to solve the physic balance problem. We combine several mathematical tools including algebra, number theory, combinatorics and geometry to derive some interesting properties.

In our argument, the relationship between the factors structure and the balance polynomials, and the relationship between the balance polynomials and general solutions to the physic balance problem are explored. Furthermore, we explore which permutation of weights will get balance and the number of the correct permutations. Finally we conduct the complex number theory to discuss the balance polynomials. In this way we could investigate many interesting phenomena consisting to the physical application.

壹、研究動機

有一個遊戲是:「將 1g,2g,3g,4g,5g,6g 等砝碼分置於一個圓盤上使之平衡。」(如圖 1)。玩了許久僅發現只有兩組解及其排序是 1,4,5,2,3,6 以及 1,5,3,4,2,6 的兩組(如圖 2、3),心想是否還有其他的排序?而將此兩組係數建構一對應的多項式分別可得$1+4x+5x^2+2x^3+3x^4+6x^5$以及$1+5x+3x^2+4x^3+2x^4+6x^5$而此兩多項式均有一共同因式x^2-x+1,於是我們好奇為何有此兩平衡多項式這一個因式,且$x^2-x+1=0$之根又均在$x^6=1$的分圓上,因此展開我們一連串的探索!

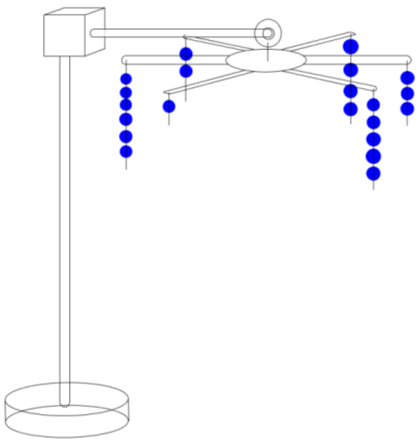
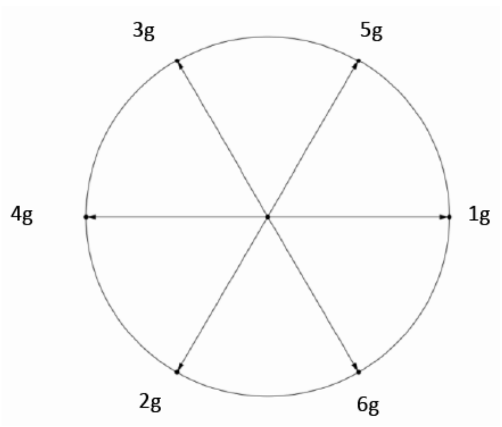


圖 1



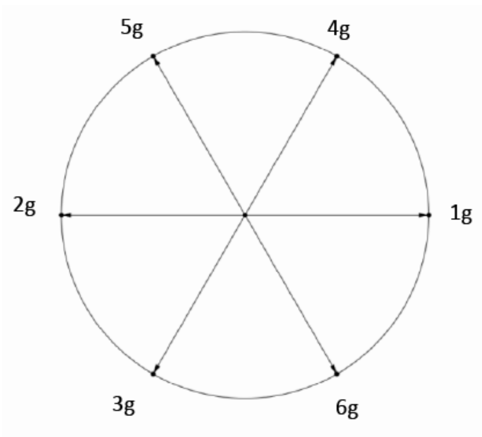


圖 2 圖 3

貳、名詞定義

1. 一階平衡多項式：$f_n^1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$，其中 $a_0, a_1, \cdots, a_{n-1}$ 為 $1,2,\cdots,n$ 的一種排列，且 $a_0, a_1, \cdots, a_{n-1}$ 可圍成等角 n 邊形。

2. t 階平衡多項式：$f_n^t = a_0^t + a_1^t x + a_2^t x^2 + a_3^t x^3 + \cdots + a_{n-1}^t x^{n-1}$，其中 $a_0^t, a_1^t, \cdots, a_{n-1}^t$ 為 $1^t, 2^t, \cdots, n^t$ 的一種排列，且 $a_0^t, a_1^t, \cdots, a_{n-1}^t$ 可圍成等角 n 邊形。

3. 分圓多項式：$\Phi_n(x) = \prod_{\substack{1 \le k < n \\ \gcd(k,n)=1}} (x - \omega^k)$，其中 ω 為 $x^n - 1 = 0$ 的根。

參、研究目的

1. 探討一階平衡多項式 $f_n^1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$ 的性質。

2. 探討一階平衡多項式 f_n^1 與 $x^n = 1$ 的根的關係及 n 的結構與性質。

3. 探討 $a_0 + a_1 \omega + a_2 \omega^2 + a_3 \omega^3 + \cdots + a_{n-1} \omega^{n-1} = 0$ 在複數平面上的幾何意義。

4. 推廣一階平衡多項式中的係數 $a_0, a_1, \cdots, a_{n-1}$ 至 $1^2, 2^2, \cdots, n^2$，甚 $1^t, 2^t, \cdots, n^t$。

5. 推廣一階平衡多項式中的係數 $a_0, a_1, \cdots, a_{n-1}$ 至 $1^2, 2^2, \cdots, n^2$，甚

$1^t, 2^t, \cdots, n^t$ 。

6. 探討分圓多項式與平衡多項式的關係。

肆、研究方法及步驟

一、當 $n=6$ 時：

因為1,2,3,4,5,6自由排列後會有 $6!$ 種，但我們發現只有下述兩種有

$x^2-x+1=0$ 的根，即 $\begin{array}{l} x^2-x+1 \mid 1+4x+5x^2+2x^3+3x^4+6x^5 \\ x^2-x+1 \mid 1+5x+3x^2+4x^3+2x^4+6x^5 \end{array}$

又 $x^2-x+1=0$ 的根為 ω, ω^5 且 $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5=0$，所以想從

$x^6-1=0$ 的六個根 $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ 來尋找 f_6。

$$\because \begin{cases} \omega^0+\omega^3=0 \\ \omega^1+\omega^4=0 \\ \omega^2+\omega^5=0 \end{cases}，且 \begin{cases} \omega^0+\omega^2+\omega^4=0 \\ \omega^1+\omega^3+\omega^5=0 \end{cases}$$

$$\therefore \begin{cases} A_1(\omega^0+\omega^3)=0 \\ A_2(\omega^1+\omega^4)=0 \\ A_3(\omega^2+\omega^5)=0 \end{cases}，且 \begin{cases} B_1(\omega^0+\omega^2+\omega^4)=0 \\ B_2(\omega^1+\omega^3+\omega^5)=0 \end{cases}$$

計算可得 $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5$，但因計算過於複雜，於是引入中國剩餘定理

(1) 令 $A_1=1$，$A_2=2$，$A_2=3$。$B_1=0$，$B_2=3$。

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5
A	A_1	A_2	A_3	A_1	A_2	A_3
	1	2	3	1	2	3
B	B_1	B_2	B_1	B_2	B_1	B_2
	0	3	0	3	0	3
係數	1	5	3	4	2	6

可得 $1+5\omega+3\omega^2+4\omega^3+2\omega^4+6\omega^5=0$

故 $f_6=1+5x+3x^2+4x^3+2x^4+6x^5$

將 A_1,B_1 固定，A_2,A_3 互換可得 $f_6=1+6x+2x^2+4x^3+3x^4+5x^5$

此和 $f_6=1+5x+3x^2+4x^3+2x^4+6x^5$ 所圍出的六邊形為左右對稱的等角六邊形，我們將其視為同一個。(如下頁圖 4 及圖 5)

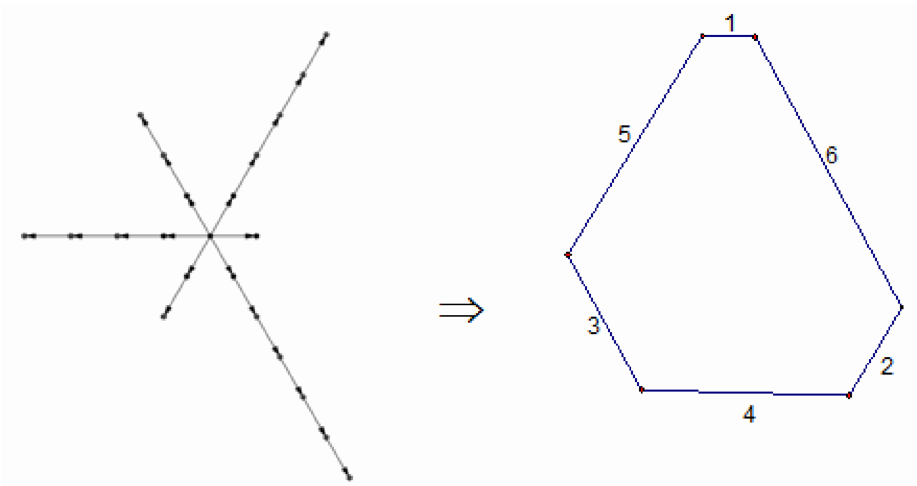


圖 4

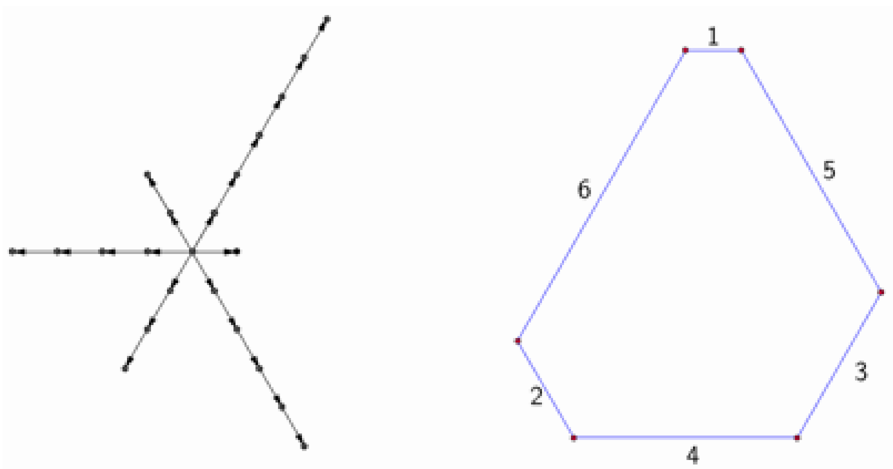


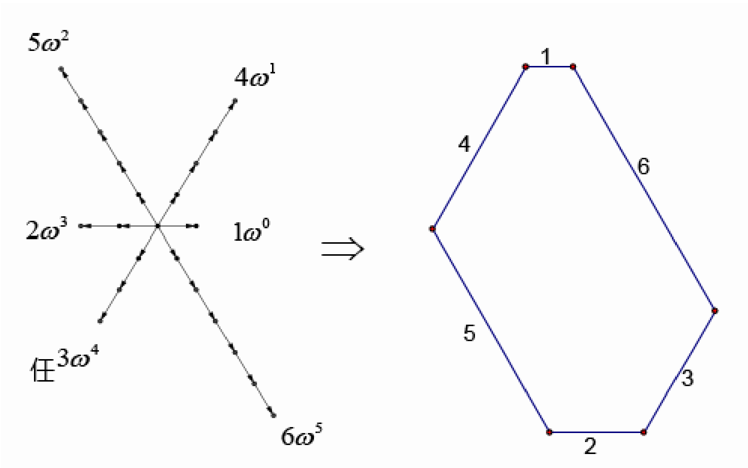
圖 5

令 $A_1=0$，$A_2=2$，$A_2=4$。$B_1=1$，$B_2=2$。

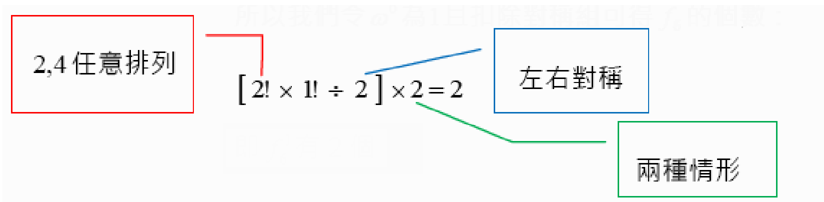
	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5
A	A_1	A_2	A_3	A_1	A_2	A_3
	0	2	4	0	2	4
B	B_1	B_2	B_1	B_2	B_1	B_2
	1	2	1	2	1	2
係數	1	4	5	2	3	6

可得$1+4\omega+5\omega^2+2\omega^3+3\omega^4+6\omega^5=0$

故 $f_6=1+4x+5x^2+2x^3+3x^4+6x^5$



所以我們令 ω^0 為1且扣除對稱組可得 f_6 的個數：



即 f_6^1 有 2 個

又$\because a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5=0$

將 ω 代入，可得

$\Rightarrow a_0+a_1\omega+a_2\omega^2+a_3\omega^3+a_4\omega^4+a_5\omega^5=0$

令 $\omega^3=-1$ 代入得

$$\Rightarrow a_0+a_1\omega+a_2\omega^2-a_3-a_4\omega-a_5\omega^2=0$$

$$\Rightarrow (a_0-a_3)-(a_4-a_1)\omega+(a_2-a_5)\omega^2=0$$

$\therefore a_0-a_3=a_4-a_1=a_2-a_5=1,3$，那公差為 1 或 3

$\therefore f_6$ 的係數所圍六邊形具備對邊等差的結構

且利用 $x^2-x+1 | f_6$ 的性質，我們利用電腦 C 語言驗證 f_6 有 2 種

二、當 $n=10$ 時：

我們延續 $n=6$ 的方法來尋找 f_{10}

因為 $\omega^0+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8+\omega^9=0$

又 $\begin{cases}\omega^0+\omega^5=0\\ \omega^1+\omega^6=0\\ \omega^2+\omega^7=0\\ \omega^3+\omega^8=0\\ \omega^4+\omega^9=0\end{cases}$，且 $\begin{cases}\omega^0+\omega^2+\omega^4+\omega^6+\omega^8=0\\ \omega^1+\omega^3+\omega^5+\omega^7+\omega^9=0\end{cases}$

$\therefore \begin{cases}A_1(\omega^0+\omega^5)=0\\ A_2(\omega^1+\omega^6)=0\\ A_3(\omega^2+\omega^7)=0\\ A_4(\omega^3+\omega^8)=0\\ A_5(\omega^4+\omega^9)=0\end{cases}$，且 $\begin{cases}B_1(\omega^0+\omega^2+\omega^4+\omega^6+\omega^8)=0\\ B_2(\omega^1+\omega^3+\omega^5+\omega^7+\omega^9)=0\end{cases}$

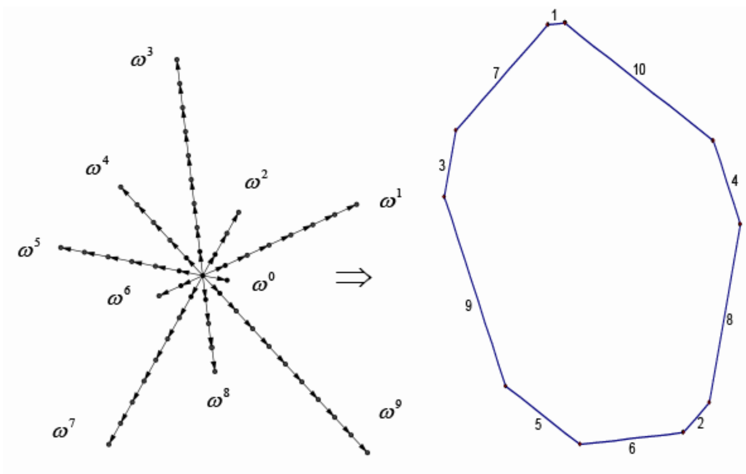
(1) 令 $A_1=1$，$A_2=2$，$A_3=3$，$A_4=4$，$A_5=5$

$B_1=0$，$B_2=5$。

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9
A	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	1	2	3	4	5	1	2	3	4	5
B	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2
	0	5	0	5	0	5	0	5	0	5
係數	1	7	3	9	5	6	2	8	4	10

可得$1+7\omega+3\omega^2+9\omega^3+5\omega^4+6\omega^5+2\omega^6+8\omega^7+4\omega^8+10\omega^9=0$

故 $f_{10}=1+7x+3x^2+9x^3+5x^4+6x^5+2x^6+8x^7+4x^8+10x^9$



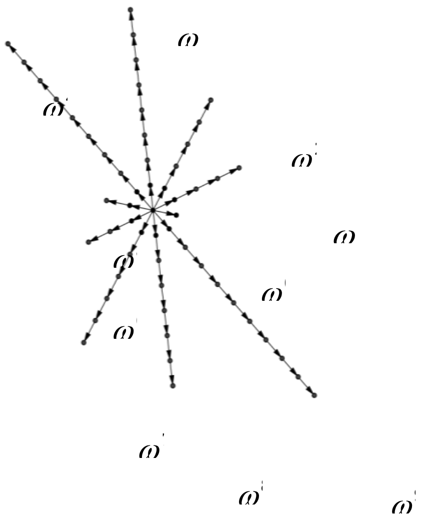
(2) 令 $A_1=0$，$A_2=2$，$A_3=4$，$A_4=6$，$A_5=8$

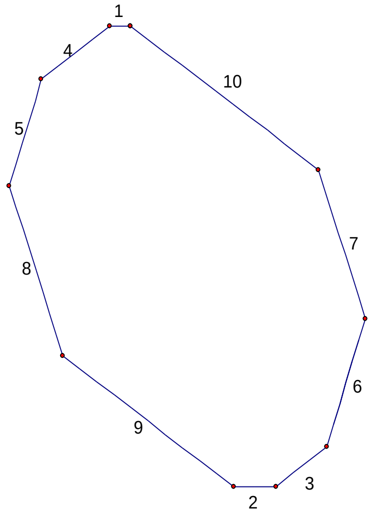
$B_1=1$，$B_2=2$。

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9
A	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	2	4	6	8	0	2	4	6	8
B	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2
	1	2	1	2	1	2	1	2	1	2
係數	1	4	5	8	9	2	3	6	7	10

可得$1+4\omega+5\omega^2+8\omega^3+9\omega^4+2\omega^5+3\omega^6+6\omega^7+7\omega^8+10\omega^9=0$

故 $f_{10}=1+4x+5x^2+8x^3+9x^4+2x^5+3x^6+6x^7+7x^8+10x^9$





個數： $[\,4!\times 1!\div 2\,]\times 2=24$

我們同時也發現 $\begin{matrix} f_{10}=1+7x+3x^2+9x^3+5x^4+6x^5+2x^6+8x^7+4x^8+10x^9 \\ f_{10}=1+4x+5x^2+8x^3+9x^4+2x^5+3x^6+6x^7+7x^8+10x^9 \end{matrix}$

均有 $\omega,\omega^3,\omega^7,\omega^9$ **的根，即** $x^4-x^3+x^2-x+1\mid f_{10}$

三、當 $n=14$ 時：

$$\omega^0+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8+\omega^9+\omega^{10}+\omega^{11}+\omega^{12}+\omega^{13}=0$$

又 $\begin{cases} \omega^0+\omega^7=0 \\ \omega^1+\omega^8=0 \\ \omega^2+\omega^9=0 \\ \omega^3+\omega^{10}=0 \\ \omega^4+\omega^{11}=0 \\ \omega^5+\omega^{12}=0 \\ \omega^6+\omega^{13}=0 \end{cases}$，且 $\begin{cases} \omega^0+\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}+\omega^{12}=0 \\ \omega^1+\omega^3+\omega^5+\omega^7+\omega^9+\omega^{11}+\omega^{13}=0 \end{cases}$

$$\therefore \begin{cases} A_1(\omega^0+\omega^7)=0 \\ A_2(\omega^1+\omega^8)=0 \\ A_3(\omega^2+\omega^9)=0 \\ A_4(\omega^3+\omega^{10})=0 \\ A_5(\omega^4+\omega^{11})=0 \\ A_6(\omega^5+\omega^{12})=0 \\ A_7(\omega^6+\omega^{13})=0 \end{cases}，且 \begin{cases} B_1(\omega^0+\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}+\omega^{12})=0 \\ B_2(\omega^1+\omega^3+\omega^5+\omega^7+\omega^9+\omega^{11}+\omega^{13})=0 \end{cases}$$

(1) $A_1=1$，$A_2=2$，$A_3=3$，$A_4=4$，$A_5=5$，$A_6=6$，$A_7=7$

$B_1=0$，$B_2=7$。

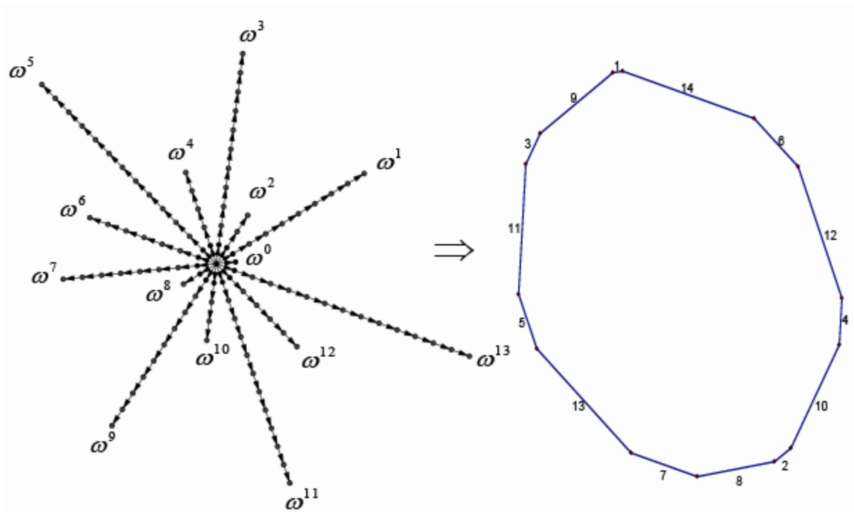
	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}
A	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_2	A_3	A_1	A_2	A_3	A_2	A_3
	1	2	3	4	5	6	7	1	2	3	4	5	6	7
B	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2
	0	7	0	7	0	7	0	7	0	7	0	7	0	7
係數	1	9	3	11	5	13	7	8	2	10	4	12	6	14

可得

$$1+9\omega+3\omega^2+11\omega^3+5\omega^4+13\omega^5+7\omega^6+8\omega^7+2\omega^8+10\omega^9+4\omega^{10}+12\omega^{11}+6\omega^{12}+14\omega^{13}=0$$

故

$$f_{14}=1+9x+3x^2+11x^3+5x^4+13x^5+7x^6+8x^7+2x^8+10x^9+4x^{10}+12x^{11}+6x^{12}+14x^{13}$$



(2) 令 $A_1=0$，$A_2=2$，$A_3=4$，$A_4=6$，$A_5=8$，$A_6=10$，$A_7=12$

$B_1=1$，$B_2=2$。

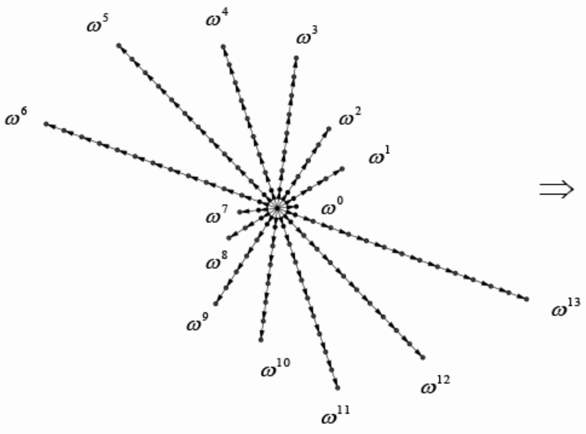
	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}
A	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_1	A_2	A_3	A_4	A_5	A_6	A_7
	0	2	4	6	8	10	12	0	2	4	6	8	10	12
B	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2	B_1	B_2
	1	2	1	2	1	2	1	2	1	2	1	2	1	2
係數	1	4	5	8	9	12	13	2	3	6	7	10	11	14

可得

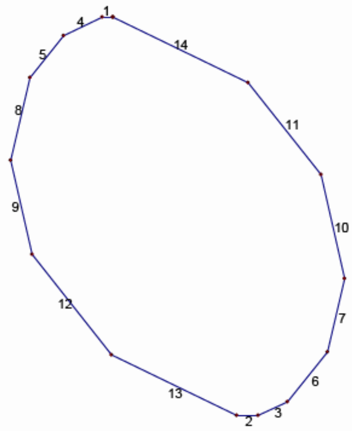
$$1+4\omega+5\omega^2+8\omega^3+9\omega^4+12\omega^5+13\omega^6+2\omega^7+3\omega^8+6\omega^9+7\omega^{10}+10\omega^{11}+11\omega^{12}+14\omega^{13}=0$$

故

$$f_{14}=1+4x+5x^2+8x^3+9x^4+12x^5+13x^6+2x^7+3x^8+6x^9+7x^{10}+10x^{11}+11x^{12}+14x^{13}$$



$\Rightarrow$



個數：$\left[\,6!\times 1!\div 2\,\right]\times 2=720$

我們同時也發現

$f_{14}=1+9x+3x^{2}+11x^{3}+5x^{4}+13x^{5}+7x^{6}+8x^{7}+2x^{8}+10x^{9}+4x^{10}+12x^{11}+6x^{12}+14x^{13}$
$f_{14}=1+4x+5x^{2}+8x^{3}+9x^{4}+12x^{5}+13x^{6}+2x^{7}+3x^{8}+6x^{9}+7x^{10}+10x^{11}+11x^{12}+14x^{13}$ **均有**

$\omega,\omega^{3},\omega^{5},\omega^{9},\omega^{11},\omega^{13}$**的根，即** $x^{6}-x^{5}+x^{4}-x^{3}+x^{2}-x+1\mid f_{14}$

四、證明存在平衡多項式 f_{2p}^{1}，其中 p 為奇質數

〈證明〉

$$\because\ \omega^{0}+\omega^{1}+\omega^{2}+\cdots+\omega^{2p-2}+\omega^{2p-1}=0$$

$$\text{又}\begin{cases}\omega^{0}+\omega^{p}=0\\ \omega^{1}+\omega^{p+1}=0\\ \omega^{2}+\omega^{p+2}=0\\ \quad\vdots\\ \quad\vdots\\ \omega^{p-1}+\omega^{2p-1}=0\end{cases}\text{，且}\begin{cases}\omega^{0}+\omega^{2}+\omega^{4}+\cdots+\omega^{2p-2}=0\\ \omega^{1}+\omega^{3}+\omega^{5}+\cdots+\omega^{2p-1}=0\end{cases}$$

$$\therefore \begin{cases} A_1(\omega^0+\omega^p)=0 \\ A_2(\omega^1+\omega^{p+1})=0 \\ A_3(\omega^2+\omega^{p+2})=0 \\ \quad\vdots \\ \quad\vdots \\ A_p(\omega^{p-1}+\omega^{2p-1})=0 \end{cases} \text{，且} \begin{cases} B_1(\omega^0+\omega^2+\omega^4+\cdots+\omega^{2p-2})=0 \\ B_2(\omega^1+\omega^3+\omega^5+\cdots+\omega^{2p-1})=0 \end{cases}$$

令 $A_1=1$，$A_2=2$，$A_3=3\cdots\cdots A_p=p$。$B_1=0$，$B_2=p$。

	ω^0	ω^1	ω^2		ω^{p-1}	ω^p		ω^{2p-1}
A	A_1	A_2	A_3		A_P	A_{P+1}		A_{2P}
	1	2	3		p	1		p
B	B_1	B_2	B_1	……	B_1	B_2	……	B_2
	0	p	0		0	p		p
係數	1	$2+p$	3		p	$1+p$		$2p$

故可得 $f_{2p}=1+(p+2)x+3x^2+\cdots+2px^{2p-1}$

(2) 令 $A_1=0$，$A_2=2$，$A_3=4\cdots\cdots A_p=2p-2$。$B_1=1$，$B_2=2$。

	ω^0	ω^1	ω^2		ω^{p-1}	ω^p		ω^{2p-1}
A	A_1	A_2	A_3		A_P	A_{P+1}		A_{2P}
	0	2	4		$2p-2$	0		$2p-2$
B	B_1	B_2	B_1	……	B_1	B_2	……	B_2
	1	2	1		1	2		2
係數	1	4	5		$2p-1$	2		$2p$

可得 $f_{2p}=1+4x+5x^2+\cdots+2px^{2p-1}$

且 $x^{p-1}-x^{p-2}+x^{p-3}+\cdots+1 \mid f_{2p}$

又 f_{2p} 的個數為：$\left[(p-1)!\times 1!\div 2\right]\times 2=(p-1)!$

故 f_{2p} 的個數為 $(p-1)!$ 個

五、證明 $n=p^r$ 時不存在平衡多項式，其中 $r\in Z$ 且 $p\in$ 質數

〈證明〉

令當 $n=p^{k+m}$ 時存在平衡多項式，其中 $k,m\in Z$ 且 $k\geq m$

又將 ω 代入 f_{2p} 可得 $a_0+a_1\omega+a_2\omega^2+\cdots+a_{2p-1}\omega^{2p-1}$

再將 $\omega^p=-1$ 代入上式中，可得

$(a_0-a_p)+(a_1-a_{p+1})\omega+(a_2-a_{p+2})+\cdots+(a_{p-1}-a_{2p-1})\omega^{n-1}=0$

$\therefore a_0-a_p=a_1-a_{p+1}=a_2-a_{p+2}=\cdots=a_{p-1}-a_{2p-1}=1 or\ p$

$\therefore f_{2p}$ 的係數所圍成的 $2p$ 邊形具有對邊等差的結構

$$
\text{則}\begin{cases}
a_0=1+0=1\\
a_1=2+p^k=2+p^k\\
a_2=3+2p^k=3+2p^k\\
\quad\vdots\\
\quad\vdots\\
a_{p^m-1}=p^m+(p^m-1)p^k=p^{m+k}+p^m-p^k-1\\
a_{p^m}=p^m+1+(0)p^k=p^m+1\\
\quad\vdots\\
\quad\vdots\\
\boxed{a_{p^k-1}=p^k+(p^m-1)p^k=p^{mk}}\\
a_{p^k}=1+(0)p^k=1\\
\quad\vdots\\
\quad\vdots\\
\boxed{a_{p^{k+m}}=p^k+(p^m-1)p^k=p^{mk}}
\end{cases}
$$

由 $a_{p^k-1}=a_{p^{m+k}}$ 可知與平衡多項式的定義矛盾

$\therefore n=p^{k+m}$ 時不存在平衡多項式

六、接下來我們探討$n \neq 2p$(p為奇質數)的其他種情形

(一)當$n=12$時：

我們延續$n=6$的方法來尋找 f_{12}^{1}

$$\because \omega^0+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8+\omega^9+\omega^{10}+\omega^{11}=0$$

又$\begin{cases}\omega^0+\omega^4+\omega^8=0\\ \omega^1+\omega^5+\omega^9=0\\ \omega^2+\omega^6+\omega^{10}=0\\ \omega^3+\omega^7+\omega^{11}=0\end{cases}$，且$\begin{cases}\omega^0+\omega^3+\omega^6+\omega^9=0\\ \omega^1+\omega^4+\omega^7+\omega^{10}=0\\ \omega^2+\omega^5+\omega^8+\omega^{11}=0\end{cases}$

$\therefore\begin{cases}A_1(\omega^0+\omega^4+\omega^8)=0\\ A_2(\omega^1+\omega^5+\omega^9)=0\\ A_3(\omega^2+\omega^6+\omega^{10})=0\\ A_4(\omega^3+\omega^7+\omega^{11})=0\end{cases}$，且$\begin{cases}B_1(\omega^0+\omega^3+\omega^6+\omega^9)=0\\ B_2(\omega^1+\omega^4+\omega^7+\omega^{10})=0\\ B_3(\omega^2+\omega^5+\omega^8+\omega^{11})=0\end{cases}$

(1) 令$A_1=1$，$A_1=2$，$A_1=3$，$A_1=4$。$B_1=0$，$B_1=4$，$B_1=8$。

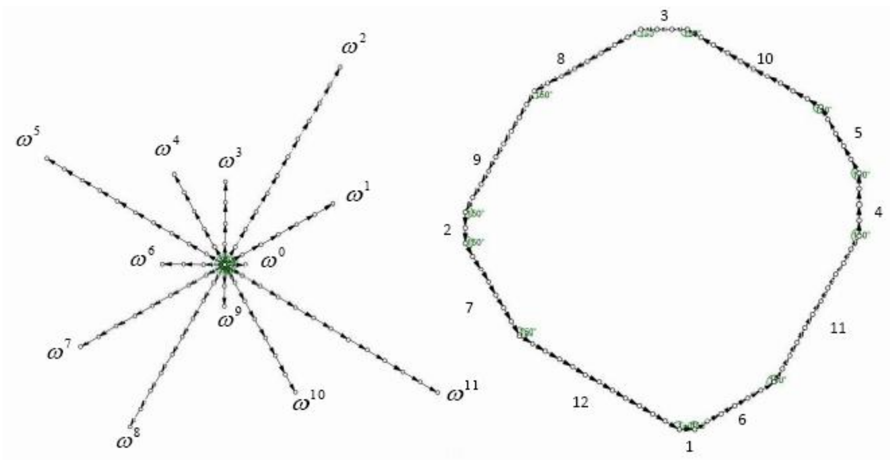
		ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}
A		A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
		1	2	3	4	1	2	3	4	1	2	3	4
B		B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
		0	4	8	0	4	8	0	4	8	0	4	8
係數		1	6	11	4	5	10	3	8	9	2	7	12

可得

$$1+6\omega+11\omega^2+4\omega^3+5\omega^4+10\omega^5+3\omega^6+8\omega^7+9\omega^8+2\omega^9+7\omega^{10}+12\omega^{11}=0$$

故

$$f_{12}=1+6x+11x^2+4x^3+5x^4+10x^5+3x^6+8x^7+9x^8+2x^9+7x^{10}+12x^{11}$$



(2) 令 $A_1=1$，$A_1=3$，$A_1=6$，$A_1=9$。$B_1=1$，$B_1=2$，$B_1=3$。

		ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}
A		A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
		0	3	6	9	0	3	6	9	0	3	6	9
B		B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
		1	2	3	1	2	3	1	2	3	1	2	3
係數		1	5	9	10	2	6	7	11	3	4	8	12

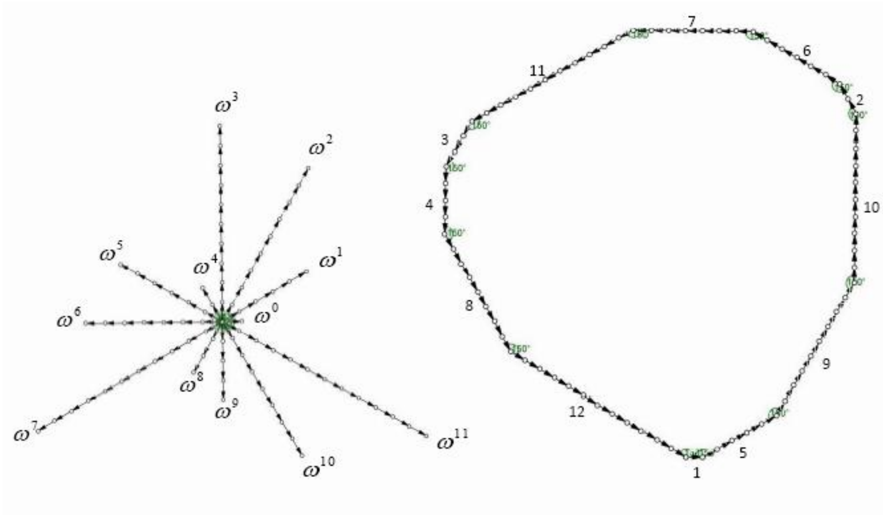
可

得

$$1+5\omega+9\omega^2+10\omega^3+2\omega^4+6\omega^5+7\omega^6+11\omega^7+3\omega^8+4\omega^9+8\omega^{10}+12\omega^{11}=0$$

故

$$f_{12}=1+5x+9x^2+10x^3+2x^4+6x^5+7x^6+11x^7+3x^8+4x^9+8x^{10}+12x^{11}$$



我們經由上述 f_{12}^1 的中國剩餘定理，可以求得 f_{12}^1 的個數為 $[3!\times 2!\div 2\times 2]=12$ 。

但利用分圓多項式必整除平衡多項式的性質，藉由 C 語言程式算出 f_{12}^1 的個數，出乎我們的意料之外，個數竟然有 732 個，這引起了我們的好奇，所以經過了長時間的探討與驗證，發現分圓多項式所找出的 732 組皆符合和力為零且皆可圍成等角 12 邊形，故我們將 $n=12$ 做以下的分類：

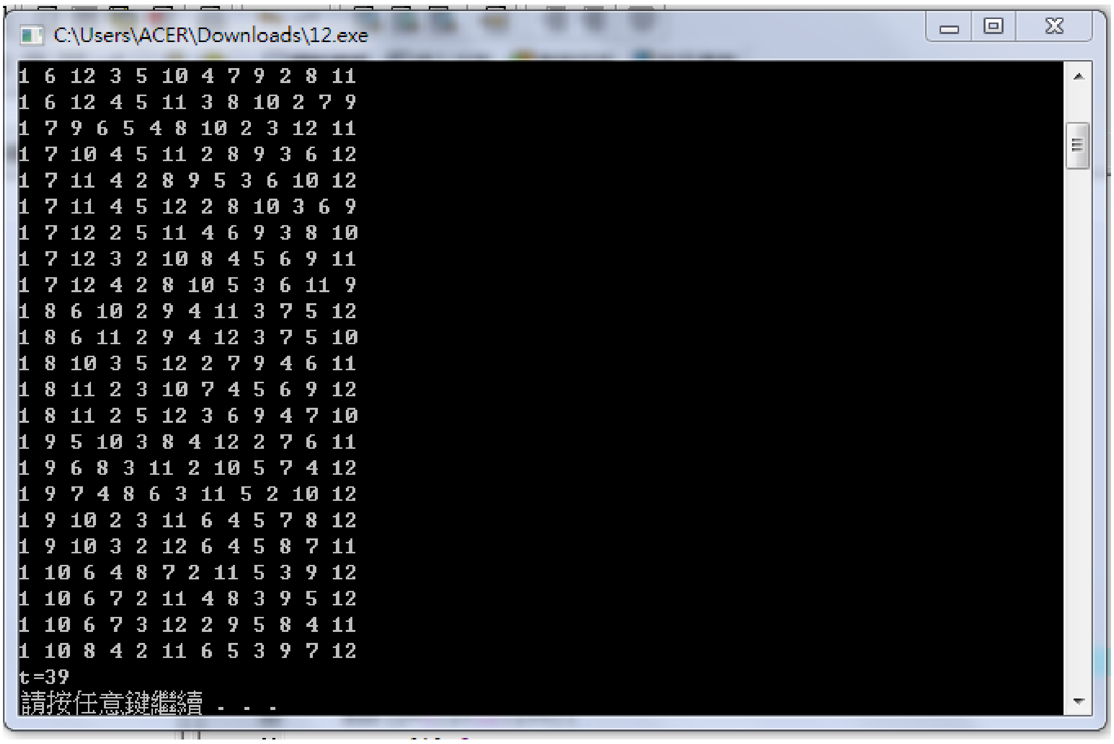
1. 可被 $\Phi_2\times\Phi_{12}$ 整除的個數有：48 個

2. 可被 $\Phi_3\times\Phi_{12}$ 整除的個數有：32 個

3. 可被 $\Phi_6\times\Phi_{12}$ 整除的個數有：12 個

4. 剩餘可被 Φ_{12} 整除的個數則有：628 個

以下為部分的程式執行結果



（二）當$n=15$時：

$$\omega^0+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8+\omega^9+\omega^{10}+\omega^{11}+\omega^{12}+\omega^{13}+\omega^{14}=0$$

又 $\begin{cases}\omega^0+\omega^5+\omega^{10}=0\\ \omega^1+\omega^6+\omega^{11}=0\\ \omega^2+\omega^7+\omega^{12}=0\\ \omega^3+\omega^8+\omega^{13}=0\\ \omega^4+\omega^9+\omega^{14}=0\end{cases}$，且 $\begin{cases}\omega^0+\omega^3+\omega^6+\omega^9+\omega^{12}=0\\ \omega^1+\omega^4+\omega^7+\omega^{10}+\omega^{13}=0\\ \omega^2+\omega^5+\omega^8+\omega^{11}+\omega^{14}=0\end{cases}$

$\therefore \begin{cases}A_1(\omega^0+\omega^5+\omega^{10})=0\\ A_2(\omega^1+\omega^6+\omega^{11})=0\\ A_3(\omega^2+\omega^7+\omega^{12})=0\\ A_4(\omega^3+\omega^8+\omega^{13})=0\\ A_5(\omega^4+\omega^9+\omega^{14})=0\end{cases}$，且 $\begin{cases}B_1(\omega^0+\omega^3+\omega^6+\omega^9+\omega^{12})=0\\ B_2(\omega^1+\omega^4+\omega^7+\omega^{10}+\omega^{13})=0\\ B_3(\omega^2+\omega^5+\omega^8+\omega^{11}+\omega^{14})=0\end{cases}$

(1) 令 $A_1=0$，$A_1=3$，$A_1=6$，$A_1=9$，$A_1=12$。$B_1=1$，$B_1=2$，$B_1=3$。

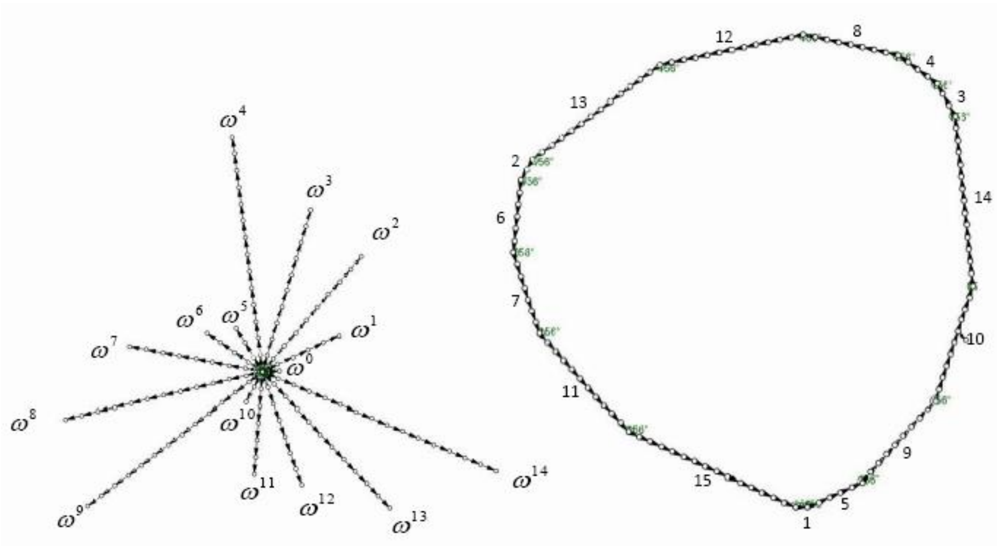
	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}
A	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
B	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
係數	1	5	9	10	14	3	4	8	12	13	2	6	7	11	15

可得

$$1+5\omega+9\omega^2+10\omega^3+14\omega^4+3\omega^5+4\omega^6+8\omega^7+12\omega^8+13\omega^9+2\omega^{10}+6\omega^{11}+7\omega^{12}+11\omega^{13}+15\omega^{14}=0$$

故

$$f_{15}=1+5x+9x^2+10x^3+14x^4+3x^5+4x^6+8x^7+12x^8+13x^9+2x^{10}+6x^{11}+7x^{12}+11x^{13}+15x^{14}$$



(2) 令 $A_1=1$，$A_1=2$，$A_1=3$，$A_1=4$，$A_1=5$。$B_1=0$，$B_1=5$，$B_1=10$。

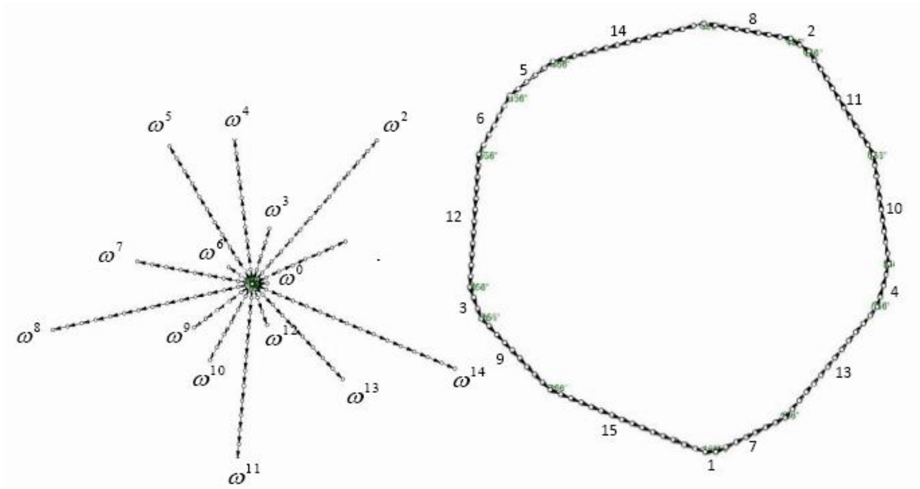
	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}
A	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
B	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10
係數	1	7	13	4	10	11	2	8	14	5	6	12	3	9	15

可得

$$1+7\omega+13\omega^{2}+4\omega^{3}+10\omega^{4}+11\omega^{5}+2\omega^{6}+8\omega^{7}+14\omega^{8}+5\omega^{9}+6\omega^{10}+12\omega^{11}+3\omega^{12}+9\omega^{13}+15\omega^{14}=0$$

故

$$f_{15}=1+7x+13x^{2}+4x^{3}+10x^{4}+11x^{5}+2x^{6}+8x^{7}+14x^{8}+5x^{9}+6x^{10}+12x^{11}+3x^{12}+9x^{13}+15x^{14}$$



又 f_{15} 的個數為：$\left[\,4!\times 2!\div 2\,\right]\times 2=48$

推廣至 $n=pq$，其中 $(p,q)=1$、p,q 皆為質數

則 f_{pq} 的個數為 $(p-1)!(q-1)!$

七、證明當 $n=rs$，$(r,s)=1$ 時，必存在平衡多項式

〈證明〉

$$\because\ \omega^{0}+\omega^{1}+\omega^{2}+\cdots+\omega^{rs-2}+\omega^{rs-1}=0$$

又 $\begin{cases}\omega^{0}+\cdots+\omega^{rs-s}=0\\ \omega^{1}+\cdots+\omega^{rs-s+1}=0\\ \omega^{2}+\cdots+\omega^{rs-s+2}=0\\ \quad\vdots\\ \omega^{s-1}+\cdots+\omega^{rs-1}=0\end{cases}$，且 $\begin{cases}\omega^{0}+\cdots+\omega^{rs-r}=0\\ \omega^{1}+\cdots+\omega^{rs-r+1}=0\\ \omega^{2}+\cdots+\omega^{rs-r+2}=0\\ \\ \omega^{r-1}+\cdots+\omega^{rs-1}=0\end{cases}$

$$\therefore \begin{cases} A_1(\omega^0+\cdots+\omega^{rs-s})=0 \\ A_2(\omega^1+\cdots+\omega^{rs-s+1})=0 \\ A_3(\omega^2+\cdots+\omega^{rs-s+2})=0 \\ \vdots \\ A_s(\omega^{s-1}+\cdots+\omega^{rs-1})=0 \end{cases}，且 \begin{cases} B_1(\omega^0+\cdots+\omega^{rs-r})=0 \\ B_2(\omega^1+\cdots+\omega^{rs-r+1})=0 \\ B_3(\omega^2+\cdots+\omega^{rs-r+2})=0 \\ \vdots \\ B_r(\omega^{r-1}+\cdots+\omega^{rs-1})=0 \end{cases}$$

(1) 令 $A_1=1$，$A_2=2$，$A_3=3\cdots A_s=s$。$B_1=0$，$B_2=s$，$B_3=2s\cdots B_r=(r-1)s$。

	ω^0	ω^1	ω^2		ω^{rs-2}	ω^{rs-1}
A	A_1	A_2	A_3		A_{rs-1}	A_{rs}
	1	2	3		$s-1$	s
B	B_1	B_2	B_3	…….	B_{rs-1}	B_{rs}
	0	s	$2s$		$(r-2)s$	$(r-1)s$
係數	1	$2+s$	$3+2s$		$rs-s-1$	rs

故可得 $f_{rs}=1+(2+s)x+(3+2s)x^2+\cdots+(rs-s-1)x^{rs-2}+rsx^{rs-1}$

(2) 令 $A_1=1$，$A_2=2$，$A_3=3\cdots A_s=r$。$B_1=0$，$B_2=r$，$B_3=2r\cdots B_r=(s-1)r$。

	ω^0	ω^1	ω^2		ω^{rs-2}	ω^{rs-1}
A	A_1	A_2	A_3		A_{rs-1}	A_{rs}
	1	2	3		$r-1$	r
B	B_1	B_2	B_3	…….	B_{rs-1}	B_{rs}
	0	r	$2r$		$(s-2)r$	$(s-1)r$
係數	1	$2+r$	$3+2r$		$rs-r-1$	rs

故可得 $f_{rs}=1+(2+r)x+(3+2r)x^2+\cdots+(rs-r-1)x^{rs-2}+rsx^{rs-1}$

且 $\prod_{i=1}^{m}(x-\omega^i) \mid f_{rs}$ **，其中** $(i,n)=1$ **，** $m \le n$

八、分圓多項式 $\Phi_n(x)$ 與 f_n 的關係

令 ω 為 $x^n=1$ 的原根，我們發現

(1) ω,ω^5 均為 $f_6=0$ 的根，且

$(x-\omega)(x-\omega^5)=x^2-x+1$，即為 $\Phi_6(x)$

(2) $\omega,\omega^3,\omega^7,\omega^9$ 均為 $f_{10}=0$ 的根，且

$(x-\omega)(x-\omega^3)(x-\omega^7)(x-\omega^9)=x^4-x^3+x^2-x+1$，即為 $\Phi_{10}(x)$

(3) $\omega,\omega^5,\omega^7,\omega^{11}$ 均為 $f_{12}=0$ 的根，且

$(x-\omega)(x-\omega^5)(x-\omega^7)(x-\omega^{11})=x^4-x^2+1$，即為 $\Phi_{12}(x)$

如此計算過於複雜，為了算出分圓多項式 $\Phi_n(x)$，我們利用分圓多項式的性質

1. $\Phi_{pm}(x)=\Phi_m(x^p)\ \ when\ \ p\ \ divides\ \ m$

2. $\Phi_{pm}(x)=\dfrac{\Phi_m(x^p)}{\Phi_p(x)}\ \ when\ \ p\ \ does\ \ not\ \ divides\ \ m$

可推出

$$\Phi_6(x)=\frac{\Phi_3(x^2)}{\Phi_3(x)}=\frac{x^4+x^2+1}{x^2+x+1}=x^2-x+1$$

$$\Phi_{10}(x)=\frac{\Phi_5(x^2)}{\Phi_5(x)}=\frac{x^8+x^6+x^4+x^2+1}{x^4+x^3+x^2+x+1}=x^4-x^3+x^2-x+1$$

$$\Phi_{12}(x)=\frac{\Phi_4(x^3)}{\Phi_4(x)}=\frac{x^6+1}{x^2+1}=x^4-x^2+1$$

$$\Phi_{15}(x)=\frac{\Phi_5(x^3)}{\Phi_5(x)}=\frac{x^{12}+x^9+x^6+x^3+1}{x^4+x^3+x^2+x+1}=x^8-x^7+x^5-x^4+x^3-x^2+x-1$$

$$\Phi_{30}(x)=\frac{\Phi_6(x^5)}{\Phi_6(x)}=\frac{x^{10}-x^5+1}{x^2-x+1}=x^8+x^7-x^6-x^5-x^4-x^3+x+1$$

⋮

⋮

並利用這樣的想法，我們利用 C 語言，找出了 f_n 的個數

九、證明分圓多項式 $\Phi_n(x) \mid f_n$

〈證明〉

$\because f_n=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{n-1}x^{n-1}$

其中 $a_0,a_1,a_2,\cdots,a_{n-1}$ 為 $1,2,\cdots,n$ 的一種排列

且 $a_0,a_1,a_2,\cdots,a_{n-1}$ 為等角 n 邊形的邊長

$\therefore f_n(\omega)=a_0+a_1\omega+a_2\omega^2+a_3\omega^3+\cdots+a_{n-1}\omega^{n-1}$

$\because a_0+a_1\omega+a_2\omega^2+a_3\omega^3+\cdots+a_{n-1}\omega^{n-1}=0$

必由

$$\begin{cases}\omega^i+\omega^{i+k}+\omega^{i+2k}+\cdots+\omega^{i+(r-1)k}=0\ ,\ rk=n, i=0,1,2,\cdots,(k-1)\\ \omega^j+\omega^{j+t}+\omega^{j+2t}+\cdots+\omega^{j+(s-1)t}=0\ ,\ st=n, j=0,1,2,\cdots,(t-1)\end{cases}$$ 所組成

且 $\because (m,n)=1$ 、 $(m,k)=1$ ，且 $(m,t)=1$

$\therefore (\omega^m)^i, (\omega^m)^{i+k}, (\omega^m)^{i+2k}, \cdots, (\omega^m)^{i+(r-1)k}$ ， $i = 0,1,2,\cdots,(k-1)$

必可產生 $\omega^i, \omega^{i+k}, \omega^{i+2k}, \cdots, \omega^{i+(r-1)k}$

$\therefore (\omega^m)^j, (\omega^m)^{j+t}, (\omega^m)^{j+2t}, \cdots, (\omega^m)^{j+(s-1)t}$ ， $j = 0,1,2,\cdots,(t-1)$

必可產生 $\omega^j, \omega^{j+t}, \omega^{j+2t}, \cdots, \omega^{j+(s-1)t}$

$\therefore f_n(\omega^m) = 0$ ，其中 $(m,n) = 1$, $1 \le m < n$

$\therefore f_n(\omega) = 0$ ，其中 $(m,n) = 1$, $1 \le m < n$

$$\therefore \prod_{\substack{1 \le k < n \\ (m,n)=1}} (x - \omega^m) \Big| f_n$$

即 $\Phi_n(x) \big| f_n$

十、探討二階平衡多項式 $f_n^2 = a_0^2 + a_1^2 x + a_2^2 x^2 + a_3^2 x^3 + \cdots + a_{n-1}^2 x^{n-1}$，其中 $a_0^2, a_1^2, a_2^2, \cdots, a_{n-1}^2$ 為 $1^2, 2^2, \cdots, n^2$ 的一種排列

(一)當 $n = 30$ 時，$\because n = 5 \times 6$，且當 $n = 6$ 時存在最小的 f_n，則利用 $f_n(x)$ 的係數搭配中國剩餘定理可得：

$$\begin{cases} 1(\omega^0 + \omega^6 + \omega^{12} + \omega^{18} + \omega^{24}) = 0 \\ 4(\omega^1 + \omega^7 + \omega^{13} + \omega^{19} + \omega^{25}) = 0 \\ 5(\omega^2 + \omega^8 + \omega^{14} + \omega^{20} + \omega^{26}) = 0 \\ 2(\omega^3 + \omega^9 + \omega^{15} + \omega^{21} + \omega^{27}) = 0 \\ 3(\omega^4 + \omega^{10} + \omega^{16} + \omega^{22} + \omega^{28}) = 0 \\ 6(\omega^5 + \omega^{11} + \omega^{17} + \omega^{23} + \omega^{29}) = 0 \end{cases} \text{------}(1)$$

$$\begin{cases}(6\times 0)(\omega^0+\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25})=0\\(6\times 1)(\omega^1+\omega^6+\omega^{11}+\omega^{16}+\omega^{21}+\omega^{26})=0\\(6\times 2)(\omega^2+\omega^7+\omega^{12}+\omega^{17}+\omega^{22}+\omega^{27})=0------(2)\\(6\times 3)(\omega^3+\omega^8+\omega^{13}+\omega^{18}+\omega^{23}+\omega^{28})=0\\(6\times 4)(\omega^4+\omega^9+\omega^{14}+\omega^{19}+\omega^{24}+\omega^{29})=0\end{cases}$$

即

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}	ω^{15}
A	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3	A_4
	1	4	5	2	3	6	1	4	5	2	3	6	1	4	5	2
B	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5	B_1
	0	6	12	18	24	0	6	12	18	24	0	6	12	18	24	0
係數	1	10	17	20	27	6	7	16	23	26	3	12	13	22	29	2

	ω^{16}	ω^{17}	ω^{18}	ω^{19}	ω^{20}	ω^{21}	ω^{22}	ω^{23}	ω^{24}	ω^{25}	ω^{26}	ω^{27}	ω^{28}	ω^{29}
A	A_5	A_6	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3	A_4	A_5	A_6
	3	6	1	4	5	2	3	6	1	4	5	2	3	6
B	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
	6	12	18	24	0	6	12	18	24	0	6	12	18	24
係數	9	18	19	28	5	8	15	24	25	4	11	14	21	30

故可得

$$\begin{aligned}f_{30}=&1+10x+17x^2+20x^3+27x^4+6x^5+7x^6+16x^7+23x^8+26x^9+3x^{10}+12x^{11}+13x^{12}+\\&22x^{13}+29x^{14}+2x^{15}+9x^{16}+18x^{17}+19x^{18}+28x^{19}+5x^{20}+8x^{21}+15x^{22}+24x^{23}+25x^{24}+\\&4x^{25}+11x^{26}+14x^{27}+21x^{28}+30x^{29}\end{aligned}$$

而在尋找 f_{30} 的同時，我們發現只要適當的調整係數，將(1)(2)的係數代入並延伸中國剩餘定理就可得：

$$\begin{cases}1^2(\omega^0+\omega^6+\omega^{12}+\omega^{18}+\omega^{24})=0\\4^2(\omega^1+\omega^7+\omega^{13}+\omega^{19}+\omega^{25})=0\\5^2(\omega^2+\omega^8+\omega^{14}+\omega^{20}+\omega^{26})=0\\2^2(\omega^3+\omega^9+\omega^{15}+\omega^{21}+\omega^{27})=0\\3^2(\omega^4+\omega^{10}+\omega^{16}+\omega^{22}+\omega^{28})=0\\6^2(\omega^5+\omega^{11}+\omega^{17}+\omega^{23}+\omega^{29})=0\end{cases}$$

$$\begin{cases}(6\times0)^2(\omega^0+\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25})=0\\(6\times1)^2(\omega^1+\omega^6+\omega^{11}+\omega^{16}+\omega^{21}+\omega^{26})=0\\(6\times2)^2(\omega^2+\omega^7+\omega^{12}+\omega^{17}+\omega^{22}+\omega^{27})=0\\(6\times3)^2(\omega^3+\omega^8+\omega^{13}+\omega^{18}+\omega^{23}+\omega^{28})=0\\(6\times4)^2(\omega^4+\omega^9+\omega^{14}+\omega^{19}+\omega^{24}+\omega^{29})=0\end{cases}$$

$$\begin{cases}2(6\times0)(1\omega^0+6\omega^5+3\omega^{10}+2\omega^{15}+5\omega^{20}+4\omega^{25})=0\\2(6\times1)(4\omega^1+1\omega^6+6\omega^{11}+3\omega^{16}+2\omega^{21}+5\omega^{26})=0\\2(6\times2)(5\omega^2+4\omega^7+1\omega^{12}+6\omega^{17}+3\omega^{22}+2\omega^{27})=0\\2(6\times3)(4\omega^3+5\omega^8+4\omega^{13}+1\omega^{18}+6\omega^{23}+3\omega^{28})=0\\2(6\times4)(3\omega^4+2\omega^9+5\omega^{14}+4\omega^{19}+1\omega^{24}+6\omega^{29})=0\end{cases}$$

下表為 $n=5\times6=30$ 的二階係數順序

ω^0	$1^2+(6\times 0)^2+2(6\times 0)\times 1=1^2$	ω^1	$4^2+(6\times 1)^2+2(6\times 1)\times 4=10^2$
ω^2	$5^2+(6\times 2)^2+2(6\times 2)\times 5=17^2$	ω^3	$2^2+(6\times 3)^2+2(6\times 3)\times 2=20^2$
ω^4	$3^2+(6\times 4)^2+2(6\times 4)\times 3=27^2$	ω^5	$6^2+(6\times 0)^2+2(6\times 0)\times 6=6^2$
ω^6	$1^2+(6\times 1)^2+2(6\times 1)\times 1=7^2$	ω^7	$4^2+(6\times 2)^2+2(6\times 2)\times 4=16^2$
ω^8	$5^2+(6\times 3)^2+2(6\times 3)\times 5=23^2$	ω^9	$2^2+(6\times 4)^2+2(6\times 4)\times 2=26^2$
ω^{10}	$3^2+(6\times 0)^2+2(6\times 0)\times 3=3^2$	ω^{11}	$6^2+(6\times 1)^2+2(6\times 1)\times 6=12^2$
ω^{12}	$1^2+(6\times 2)^2+2(6\times 2)\times 1=13^2$	ω^{13}	$4^2+(6\times 3)^2+2(6\times 3)\times 4=22^2$
ω^{14}	$5^2+(6\times 4)^2+2(6\times 4)\times 5=29^2$	ω^{15}	$2^2+(6\times 0)^2+2(6\times 0)\times 2=2^2$
ω^{16}	$3^2+(6\times 1)^2+2(6\times 1)\times 3=9^2$	ω^{17}	$6^2+(6\times 2)^2+2(6\times 2)\times 6=18^2$
ω^{18}	$1^2+(6\times 3)^2+2(6\times 3)\times 1=19^2$	ω^{19}	$4^2+(6\times 4)^2+2(6\times 4)\times 4=28^2$
ω^{20}	$5^2+(6\times 0)^2+2(6\times 0)\times 5=5^2$	ω^{21}	$2^2+(6\times 1)^2+2(6\times 1)\times 2=8^2$
ω^{22}	$3^2+(6\times 2)^2+2(6\times 2)\times 3=15^2$	ω^{23}	$6^2+(6\times 3)^2+2(6\times 3)\times 6=24^2$
ω^{24}	$1^2+(6\times 4)^2+2(6\times 4)\times 1=25^2$	ω^{25}	$4^2+(6\times 0)^2+2(6\times 0)\times 4=4^2$
ω^{26}	$5^2+(6\times 1)^2+2(6\times 1)\times 5=11^2$	ω^{27}	$2^2+(6\times 2)^2+2(6\times 2)\times 2=14^2$
ω^{28}	$3^2+(6\times 3)^2+2(6\times 3)\times 3=21^2$	ω^{29}	$6^2+(6\times 4)^2+2(6\times 4)\times 6=30^2$

$$f_{30}^2=1^2+10^2x+17x^2+20^2x^3+27^2x^4+6^2x^5+7^2x^6+16^2x^7+23^2x^8+26^2x^9+3^2x^{10}+12^2x^{11}+ 13^2x^{12}+22^2x^{13}+29^2x^{14}+2^2x^{15}+9^2x^{16}+18^2x^{17}+19^2x^{18}+28^2x^{19}+5^2x^{20}+8^2x^{21}+15^2x^{22}+ 24^2x^{23}+25^2x^{24}+4^2x^{25}+11^2x^{26}+14^2x^{27}+21^2x^{28}+30^2x^{29}$$

其係數為 f_{30}^1 的平方

（二）又 $30=10\times3$ 或 15×2 ，當 $n=30=10\times3$ 時，同 $n=6\times5$ 時的想法，利用 f_{10} 的係數，搭配中國剩餘定理可得二階係數：

$$\begin{cases}1^2(\omega^0+\omega^{10}+\omega^{20})=0\\7^2(\omega^1+\omega^{11}+\omega^{21})=0\\3^2(\omega^2+\omega^{12}+\omega^{22})=0\\9^2(\omega^3+\omega^{13}+\omega^{23})=0\\5^2(\omega^4+\omega^{14}+\omega^{24})=0\\6^2(\omega^5+\omega^{15}+\omega^{25})=0\\2^2(\omega^6+\omega^{16}+\omega^{26})=0\\8^2(\omega^7+\omega^{17}+\omega^{27})=0\\4^2(\omega^8+\omega^{18}+\omega^{28})=0\\10^2(\omega^9+\omega^{19}+\omega^{29})=0\end{cases}$$

且 $$\begin{cases}(10\times0)^2(\omega^0+\omega^3+\omega^6+\omega^9+\omega^{12}+\omega^{15}+\omega^{18}+\omega^{21}+\omega^{24}+\omega^{27})=0\\(10\times1)^2(\omega^1+\omega^4+\omega^7+\omega^{10}+\omega^{13}+\omega^{16}+\omega^{19}+\omega^{22}+\omega^{25}+\omega^{28})=0\\(10\times2)^2(\omega^2+\omega^5+\omega^8+\omega^{11}+\omega^{14}+\omega^{17}+\omega^{20}+\omega^{23}+\omega^{26}+\omega^{29})=0\end{cases}$$

又

$$\begin{cases}2(10\times0)(1\omega^0+9\omega^3+2\omega^6+10\omega^9+3\omega^{12}+6\omega^{15}+4\omega^{18}+7\omega^{21}+5\omega^{24}+8\omega^{27})=0\\2(10\times1)(7\omega^1+5\omega^4+8\omega^7+1\omega^{10}+9\omega^{13}+2\omega^{16}+10\omega^{19}+3\omega^{22}+6\omega^{25}+4\omega^{28})=0\\2(10\times2)(3\omega^2+6\omega^5+4\omega^8+7\omega^{11}+5\omega^{14}+8\omega^{17}+1\omega^{20}+9\omega^{23}+2\omega^{26}+10\omega^{29})=0\end{cases}$$

下表為 $n=3\times10=30$ 的二階係數順序

ω^0	$1^2+(10\times0)^2+2(10\times0)\times1=1^2$	ω^1	$7^2+(10\times1)^2+2(10\times1)\times7=17^2$
ω^2	$3^2+(10\times2)^2+2(10\times2)\times3=23^2$	ω^3	$9^2+(10\times0)^2+2(10\times0)\times9=9^2$
ω^4	$5^2+(10\times1)^2+2(10\times1)\times5=15^2$	ω^5	$6^2+(10\times2)^2+2(10\times2)\times6=26^2$
ω^6	$2^2+(10\times0)^2+2(10\times0)\times2=2^2$	ω^7	$8^2+(10\times1)^2+2(10\times1)\times8=18^2$
ω^8	$4^2+(10\times2)^2+2(10\times2)\times4=24^2$	ω^9	$10^2+(10\times0)^2+2(10\times0)\times10=10^2$
ω^{10}	$1^2+(10\times1)^2+2(10\times1)\times1=11^2$	ω^{11}	$7^2+(10\times2)^2+2(10\times2)\times7=27^2$
ω^{12}	$3^2+(10\times0)^2+2(10\times0)\times3=3^2$	ω^{13}	$9^2+(10\times1)^2+2(10\times1)\times9=19^2$
ω^{14}	$5^2+(10\times2)^2+2(10\times2)\times5=25^2$	ω^{15}	$6^2+(10\times0)^2+2(10\times0)\times6=6^2$
ω^{16}	$2^2+(10\times1)^2+2(10\times1)\times2=12^2$	ω^{17}	$8^2+(10\times2)^2+2(10\times2)\times8=28^2$
ω^{18}	$4^2+(10\times0)^2+2(10\times0)\times4=4^2$	ω^{19}	$10^2+(10\times1)^2+2(10\times1)\times10=20^2$
ω^{20}	$1^2+(10\times2)^2+2(10\times2)\times1=21^2$	ω^{21}	$7^2+(10\times0)^2+2(10\times0)\times7=7^2$
ω^{22}	$3^2+(10\times1)^2+2(10\times1)\times3=13^2$	ω^{23}	$9^2+(10\times2)^2+2(10\times2)\times9=29^2$
ω^{24}	$5^2+(10\times0)^2+2(10\times0)\times5=5^2$	ω^{25}	$6^2+(10\times1)^2+2(10\times1)\times6=16^2$
ω^{26}	$2^2+(10\times2)^2+2(10\times2)\times2=22^2$	ω^{27}	$8^2+(10\times0)^2+2(10\times0)\times8=8^2$
ω^{28}	$4^2+(10\times1)^2+2(10\times1)\times4=14^2$	ω^{29}	$10^2+(10\times2)^2+2(10\times2)\times10=30^2$

$$f_{30}^2=1^2+17^2x+23x^2+9^2x^3+15^2x^4+26^2x^5+2^2x^6+18^2x^7+24^2x^8+10^2x^9+11^2x^{10}+27^2x^{11}+$$
$$3^2x^{12}+19^2x^{13}+25^2x^{14}+6^2x^{15}+12^2x^{16}+28^2x^{17}+4^2x^{18}+20^2x^{19}+21^2x^{20}+7^2x^{21}+13^2x^{22}+$$
$$29^2x^{23}+5^2x^{24}+16^2x^{25}+22^2x^{26}+8^2x^{27}+14^2x^{28}+30^2x^{29}$$

其係數為 f_{30}^1 的平方

（三）當 $n=30=2\times15$ 時，利用 f_{15} 的係數，再搭配中國剩餘定理的想法可得：

$$\begin{cases} 1^2(\omega^0+\omega^{15})=0 \\ 5^2(\omega^1+\omega^{16})=0 \\ 9^2(\omega^2+\omega^{17})=0 \\ 10^2(\omega^3+\omega^{18})=0 \\ 14^2(\omega^4+\omega^{19})=0 \\ 3^2(\omega^5+\omega^{20})=0 \\ 4^2(\omega^6+\omega^{21})=0 \\ 8^2(\omega^7+\omega^{22})=0 \\ 12^2(\omega^8+\omega^{23})=0 \\ 13^2(\omega^9+\omega^{24})=0 \\ 2^2(\omega^{10}+\omega^{25})=0 \\ 6^2(\omega^{11}+\omega^{26})=0 \\ 7^2(\omega^{12}+\omega^{27})=0 \\ 11^2(\omega^{13}+\omega^{28})=0 \\ 15^2(\omega^{14}+\omega^{29})=0 \end{cases}$$

且

$$\begin{cases} (15\times0)^2(\omega^0+\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}+\omega^{12}+\omega^{14}+\omega^{16}+\omega^{18}+\omega^{20}+\omega^{22}+\omega^{24}+\omega^{26}+\omega^{28})=0 \\ (15\times1)^2(\omega^1+\omega^3+\omega^5+\omega^7+\omega^9+\omega^{11}+\omega^{13}+\omega^{15}+\omega^{17}+\omega^{19}+\omega^{21}+\omega^{23}+\omega^{25}+\omega^{27}+\omega^{29})=0 \end{cases}$$

又

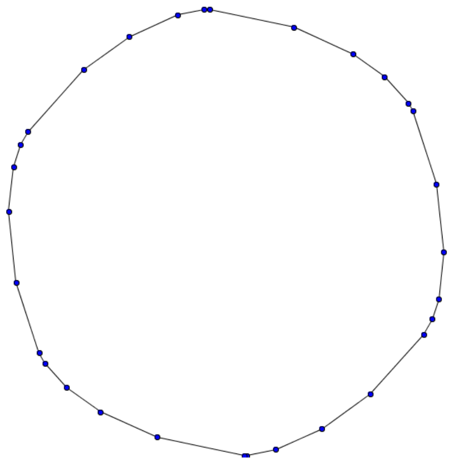
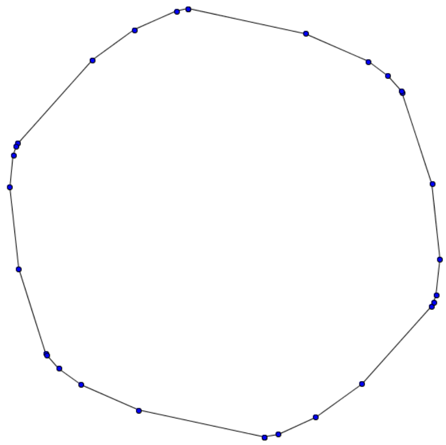
$$\begin{cases} 2(15\times0)(1\omega^0+9\omega^2+14\omega^4+4\omega^6+12\omega^8+2\omega^{10}+7\omega^{12}+15\omega^{14}+5\omega^{16}+10\omega^{18}+3\omega^{20} \\ \qquad +8\omega^{22}+13\omega^{24}+6\omega^{26}+11\omega^{28})=0 \\ 2(15\times1)(5\omega^1+10\omega^3+3\omega^5+8\omega^7+13\omega^9+6\omega^{11}+11\omega^{13}+1\omega^{15}+9\omega^{17}+14\omega^{19}+4\omega^{21} \\ \qquad +12\omega^{23}+2\omega^{25}+7\omega^{27}+15\omega^{29})=0 \end{cases}$$

下表為 $n=2\times15=30$ 的二階係數順

ω^0	$1^2+(15\times0)^2+2(15\times0)\times1=1^2$	ω^1	$5^2+(15\times1)^2+2(15\times1)\times5=20^2$
ω^2	$9^2+(15\times0)^2+2(15\times0)\times9=9^2$	ω^3	$10^2+(15\times1)^2+2(15\times1)\times10=25^2$
ω^4	$14^2+(15\times0)^2+2(15\times0)\times14=14^2$	ω^5	$3^2+(15\times1)^2+2(15\times1)\times3=18^2$
ω^6	$4^2+(15\times0)^2+2(15\times0)\times4=4^2$	ω^7	$8^2+(15\times1)^2+2(15\times1)\times8=23^2$
ω^8	$12^2+(15\times0)^2+2(15\times0)\times12=12^2$	ω^9	$13^2+(15\times1)^2+2(15\times1)\times13=28^2$
ω^{10}	$2^2+(15\times0)^2+2(15\times0)\times2=2^2$	ω^{11}	$6^2+(15\times1)^2+2(15\times1)\times6=21^2$
ω^{12}	$7^2+(15\times0)^2+2(15\times0)\times7=7^2$	ω^{13}	$11^2+(15\times1)^2+2(15\times1)\times11=26^2$
ω^{14}	$15^2+(15\times0)^2+2(15\times0)\times15=15^2$	ω^{15}	$1^2+(15\times1)^2+2(15\times1)\times1=16^2$
ω^{16}	$5^2+(15\times0)^2+2(15\times0)\times5=5^2$	ω^{17}	$9^2+(15\times1)^2+2(15\times1)\times9=24^2$
ω^{18}	$10^2+(15\times0)^2+2(15\times0)\times10=10^2$	ω^{19}	$14^2+(15\times1)^2+2(15\times1)\times14=29^2$
ω^{20}	$3^2+(15\times0)^2+2(15\times0)\times3=3^2$	ω^{21}	$4^2+(15\times1)^2+2(15\times1)\times4=19^2$
ω^{22}	$8^2+(15\times0)^2+2(15\times0)\times8=8^2$	ω^{23}	$12^2+(15\times1)^2+2(15\times1)\times12=27^2$
ω^{24}	$13^2+(15\times0)^2+2(15\times0)\times13=13^2$	ω^{25}	$2^2+(15\times1)^2+2(15\times1)\times2=17^2$
ω^{26}	$6^2+(15\times0)^2+2(15\times0)\times6=6^2$	ω^{27}	$7^2+(15\times1)^2+2(15\times1)\times7=22^2$
ω^{28}	$11^2+(15\times0)^2+2(15\times0)\times11=11^2$	ω^{29}	$15^2+(15\times1)^2+2(15\times1)\times15=30^2$

$$f_{30}^2=1^2+20^2x+9x^2+25^2x^3+14^2x^4+18^2x^5+4^2x^6+23^2x^7+12^2x^8+28^2x^9+2^2x^{10}+21^2x^{11}+7^2x^{12}+26^2x^{13}+15^2x^{14}+16^2x^{15}+5^2x^{16}+24^2x^{17}+10^2x^{18}+29^2x^{19}+3^2x^{20}+19^2x^{21}+8^2x^{22}+27^2x^{23}+13^2x^{24}+17^2x^{25}+6^2x^{26}+22^2x^{27}+11^2x^{28}+30^2x^{29}$$

其係數為 f_{30}^1 的平方

f_{30}^{1}	f_{30}^{2}
	

$\therefore$ 由(一)(二)(三)可得

$\because 30=5\times6=10\times3=15\times2$ 且 $f_{6}^{1}, f_{10}^{1}, f_{15}^{1}$ 均存在

$\therefore f_{30}^{2}$ 有三種基本型，其個數分別為：

①當 $n=5\times6$ 時， f_{30}^{2} 有 $2\times4!=48$ 個

②當 $n=10\times3$ 時， f_{30}^{2} 有 $2!\times4!=48$ 個

③當 $n=15\times2$ 時， f_{30}^{2} 有 $2\times4!\times1=48$ 個

十一、證明 n 含有 3 個以上的質因子時，必存在係數為 $1^2,2^2,3^2,\cdots,n^2$ 的 t 階平衡多項式

〈證明〉

令 $\omega^{k}=\cos\dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{n}$ ，$k=0,1,2,\cdots,n-1$

設 $n=p\times q$ ，$p,q>1$，q 存在邊長為 $1,2,\cdots,q$ 的等角序列 q 邊形，即

$f_q=a_0+a_1x+a_2x^2+\cdots+a_{q-1}x^{q-1}$

$\therefore a_0 + a_1\omega^p + a_2\omega^{2p} + \cdots + a_{q-1}\omega^{(q-1)p} = 0$

即 $\sum_{j=0}^{q-1} a_j \omega^{jp} = 0 ------(1)$

又 $\omega^0 + \omega^q + \omega^{2q} + \cdots + \omega^{(p-1)q} = 0$ ，即 $\sum_{j=0}^{p-1} \omega^{jq} = 0 ------(2)$

且 $\omega^0 + \omega^p + \omega^{2p} + \cdots + \omega^{(q-1)p} = 0$ ，即 $\sum_{j=0}^{q-1} \omega^{jp} = 0 ------(3)$

對 $(1)\times 2(s-1)q\omega^{(s-1)q}$ ，$1 \le s \le p ------(4)$

$(2)\times a_{t-1}^2 \omega^{(t-1)p}$ ，$1 \le t \le q ------(5)$

$(3)\times (s-1)^2 q^2 \omega^{(s-1)q}$ ，$1 \le s \le p ------(6)$

$(4)+(5)+(6)$ 得

$$\sum_{s=1}^{p} 2(s-1)q\omega^{(s-1)q} \sum_{j=0}^{q-1} a_j \omega^{jp} + \sum_{t=1}^{q} a_{t-1}^2 \omega^{(t-1)p} \sum_{j=0}^{p-1} \omega^{jp} + \sum_{s=1}^{p} [(s-1)q]^2 \omega^{(s-1)q} \sum_{j=0}^{q-1} \omega^{jp} = 0$$

$\Rightarrow \sum_{k=0}^{n-1} b_k \omega^k = 0$ ，其中 $b_k = \left[a_{t-1}^2 + 2a_{t-1}(s-1)q + [(s-1)q]^2\right] = [a_{t-1} + (s-1)q]^2$

$\because (p,q) = 1$ ，且 $t \in \{1,2,\cdots,q\}$, $s \in \{1,2,\cdots,p\}$

又 $k \equiv (t-1)p + (s-1)q \pmod{n}$

$\therefore k \in \{0,1,2,\cdots,n-1\}$

故 $\{b_0, b_1, \cdots, b_n\}$ 可為 $1^2, 2^2, 3^2, \cdots, n^2$ 的一個排列

即存在 $f_n^2 = a_0{}^2 + a_1{}^2 x + a_2{}^2 x^2 + a_3{}^2 x^3 + \cdots + a_{n-1}{}^2 x^{n-1}$

其中 $\{a_0{}^2, a_1{}^2, a_2{}^2, \cdots, a_{n-1}{}^2\} = \{1^2, 2^2, 3^2, \cdots, n^2\}$

十二、探討 3 階 $f_n^3 = a_0^3 + a_1^3 x + a_2^3 x^2 + a_3^3 x^3 + \cdots + a_{n-1}^3 x^{n-1}$，其中 $a_0^3, a_1^3, a_2^3, \cdots, a_{n-1}^3$ 為 $1^3, 2^3, \cdots, n^3$ 的一種排列

當 $n = 210$ 時，$\because 210 = 2 \times 3 \times 5 \times 7$ 為最小含有 4 個互質因數的自然數

我們發現：

(一) 當 $n = 210$ 時具有 1 階平衡多項式

	ω^0	ω^1	ω^2		ω^7		ω^{209}
A	A_1	A_2	A_3		A_8		A_{30}
	1	10	17	$\cdots$	16	$\cdots$	30
B	B_1	B_2	B_3	$\cdots$	B_1	$\cdots$	B_7
	0	30	60		0		180
係數	1	40	77		16		210

$\therefore f_{210} = 1 + 40x + 77x^2 + \cdots + 16x^7 + \cdots + 210x^{209}$

(二) 當 $n = 210$ 時具有 2 階平衡多項式

	ω^0	ω^1	ω^2		ω^7		ω^{209}
a^2	1^2	10^2	17^2		16^2		30^2
b^2	$(30\times 0)^2$	$(30\times 1)^2$	$(30\times 2)^2$	$\cdots$	$(30\times 0)^2$	$\cdots$	$(30\times 6)^2$
$2ab$	$2(1)(30\times 0)$	$2(10)(30\times 1)$	$2(17)(30\times 2)$		$2(16)(30\times 0)$		$2(30)(30\times 6)$
$(a+b)^2$	1^2	40^2	77^2		16^2		210^2

$\therefore f_{210}^2 = 1^2 + 40^2 x + 77^2 x^2 + \cdots + 16^2 x^7 + \cdots + 210^2 x^{209}$

(三) 當 $n = 210 = 30 \times 7$ 時更具有 3 階平衡多項式

$$\begin{cases} 1^3\left(\omega^0+\omega^{30}+\cdots+\omega^{180}\right)=0 \\ 10^3\left(\omega^1+\omega^{31}+\cdots+\omega^{181}\right)=0 \\ 17^3\left(\omega^2+\omega^{32}+\cdots+\omega^{182}\right)=0 \\ \quad\vdots \\ \quad\vdots \\ 30^3\left(\omega^{29}+\omega^{59}+\cdots+\omega^{209}\right)=0 \end{cases}$$

$$\begin{cases} (30\times 0)\left(\omega^0+\omega^7+\cdots+\omega^{203}\right)=0 \\ (30\times 1)\left(\omega^1+\omega^8+\cdots+\omega^{204}\right)=0 \\ (30\times 2)\left(\omega^2+\omega^9+\cdots+\omega^{205}\right)=0 \\ \quad\vdots \\ \quad\vdots \\ (30\times 6)\left(\omega^6+\omega^{13}+\cdots+\omega^{209}\right)=0 \end{cases}$$

$$\begin{cases} 3(30\times 0)\left(1^2\omega^0+16^2\omega^7+\cdots+24^2\omega^{203}\right)=0 \\ 3(30\times 1)\left(10^2\omega^1+23^2\omega^8+\cdots+4^2\omega^{204}\right)=0 \\ \quad\vdots \\ \quad\vdots \\ 3(30\times 6)\left(7^2\omega^6+22^2\omega^{13}+\cdots+30^2\omega^{209}\right)=0 \end{cases}$$

$$\begin{cases} 3(30\times 0)^2\left(1\omega^0+16\omega^7+\cdots+24\omega^{203}\right)=0 \\ 3(30\times 1)^2\left(10\omega^1+23\omega^8+\cdots+4\omega^{204}\right)=0 \\ \quad\vdots \\ \quad\vdots \\ 3(30\times 6)^2\left(7\omega^6+22\omega^{13}+\cdots+30\omega^{209}\right)=0 \end{cases}$$

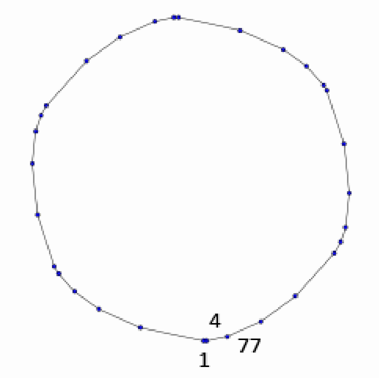
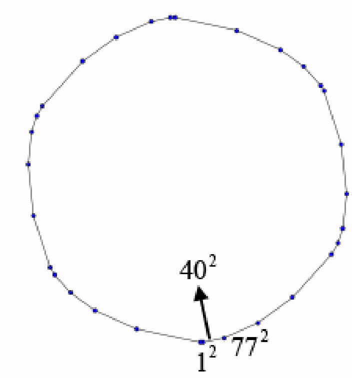
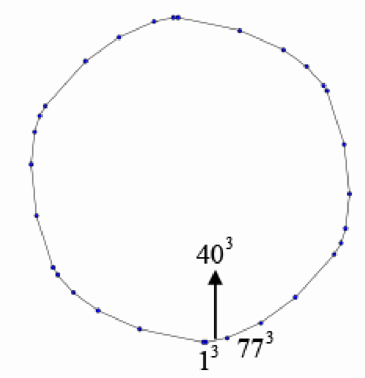
	a^3	$3a^2b$	$3ab^2$	b^3	$(a+b)^3$
ω^0	1^3	$3(1^2)(30\times 0)$	$3(1)(30\times 0)^2$	$(30\times 0)^3$	1^3
ω^1	10^3	$3(10^2)(30\times 1)$	$3(10)(30\times 1)^2$	$(30\times 1)^3$	40^3
ω^2	17^3	$3(17^2)(30\times 2)$	$3(17)(30\times 2)^2$	$(30\times 2)^3$	77^3
			$\vdots$		
ω^7	16^3	$3(16^2)(30\times 0)$	$3(16)(30\times 0)^2$	$(30\times 0)^3$	16^3
			$\vdots$		
ω^{209}	30^3	$3(17^2)(30\times 2)$	$3(17)(30\times 2)^2$	$(30\times 6)^3$	210^3

$$\therefore f_{210}^3 = 1^3 + 40^3 x + 77^3 x^2 + \cdots + 16^3 x^7 + \cdots + 210^3 x^{209}$$

我們更發現

	x^0	x^1	x^2		x^7		x^{209}
f_{210}	1	40	77	$\cdots$	16	$\cdots$	210
${f_{210}}^2$	1^2	40^2	77^2		16^2		210^2
${f_{210}}^3$	1^3	40^3	77^3		16^3		210^3

它的 1 次、2 次及 3 次方的平衡多項式的係數底數其實是一樣的，只差在指數的不同！

<table>
<tr><th>$f_{210}(x)$</th><th>$f_{210}{}^{2}(x)$</th><th>$f_{210}{}^{3}(x)$</th></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

$\because 210 = 2\times3\times5\times7$

$= 30\times7 = 42\times5 = 70\times3 = 105\times2$

且 f_{30}^2 f_{42}^2 f_{70}^2 f_{105}^2 均存在，故可由二階平衡多項導出三階平衡多項式。

$\therefore f_{210}^3$有 C_3^4 種基本型

但若將210分解為6×35，則無法利用 f_6^1 生成 f_{210}^3

$\because$ 6 、35都不具有3個質因子，無法提供$3a^2b$這組關係式

利用這樣的想法，我們可推廣至t階平衡多項式

那當$n=r\times s$ ，$(r,s)=1$時

若r可生成$(t-1)$階平衡多項式，則n必可生成t階平衡多項式

十三、證明當n含有$t+1(t\geq3)$個以上的質因子時，必存在t階平衡多項式，即 $f_n^t(x)=a_0^t+a_1^t x+a_2^t x^2+a_3^t x^3+\cdots+a_{n-1}^t x^{n-1}$ ，其中 $a_0^t,a_1^t,a_2^t,\cdots,a_{n-1}^t$為$1^t,2^t,\cdots,n^t$的一種排列

〈證明〉

設 $n = p_1 p_2 \cdots p_{k+1} = \prod_{i=1}^{k+1} p_i$ ，$(p_i, p_j) = 1$ 、$i \neq j$

又

$n = p_1 p_2$ ，可得 $f_n^1(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$

$n = p_1 p_2 p_3$ ，可得 $f_n^2(x) = a_0^2 + a_1^2 x + a_2^2 x^2 + \cdots + a_{n-1}^2 x^{n-1}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$n = p_1 p_2 p_3 \cdots p_t$ ，可得 $f_n^{t-1}(x) = a_0^{t-1} + a_1^{t-1} x + a_2^{t-1} x^2 + \cdots + a_{n-1}^{t-1} x^{n-1}$

令 $\prod_{i=2}^{t+1} p_i = A$

$$
\text{則}\begin{cases}
\omega^0 + \omega^A + \omega^{2A} + \cdots + \omega^{(p_1-1)A} = 0 - - - - - (1) \\
\omega^0 + \omega^{p_1} + \omega^{2p_2} + \cdots + \omega^{(A-1)p_1} = 0 - - - - - (2) \\
a_0\omega^0 + a_1\omega^{p_1} + a_2\omega^{2p_1} + \cdots + a_{A-1}\omega^{(A-1)p_1} = 0 - - - - - (3) \\
a_0^2\omega^0 + a_1^2\omega^{p_1} + a_2^2\omega^{2p_1} + \cdots + a_{A-1}^2\omega^{(A-1)p_1} = 0 - - - - - (4) \\
\qquad\vdots \\
a_0^{t-1}\omega^0 + a_1^{t-1}\omega^{p_1} + a_2^{t-1}\omega^{2p_1} + \cdots + a_{A-1}^{t-1}\omega^{(A-1)p_1} = 0 - - - - - (t+1)
\end{cases}
$$

$$
\begin{cases}
(1) \times a_{i-1}^t \omega^{(i-1)} \quad i = 1,2,\cdots,A - - - - - (t+2) \\
(2) \times C_t^t [A(m-1)]^t \omega^{(m-1)} \quad m = 1,2,\cdots,p_1 - - - - - (t+3) \\
(3) \times C_{t-1}^t [A(m-1)]^{t-1} \omega^{(m-1)p_1} \quad m = 1,2,\cdots,p_1 - - - - - (t+4) \\
\qquad\vdots \qquad\qquad\qquad\qquad \vdots \\
(t+1) \times C_1^t [A(m-1)] \omega^{(m-1)p_1} \quad m = 1,2,\cdots,p_1 - - - - - 2(t+1)
\end{cases}
$$

C_i^t , $i = 1,2,\cdots,t$

將 $(t+2)$ 式 $+ (t+3)$ 式 $+ \cdots + (2t+2)$ 式　得 $\sum_{i=0}^{n-1} b_i \omega^i$ ，其中

$$b_i = \left\{a_{l-1}^t + C_1^t A(m-1) \times a_{l-1}^{t-1} + C_2^t [A(m-1)]^2 \times a_{l-1}^{t-2} + \cdots + C_t^t [A(m-1)]^t\right\} = [a_{l-1} + A(m-1)]^t$$

$\because (p_i, A) = 1$ 且 $l \in \{1,2,\cdots,A\}$　$m \in \{1,2,\cdots,p_1\}$

又 $i \equiv (l-1)p_1 + (m-1)A \pmod n$ ，其中 $i \in \{1,2,\cdots,n\}$

$l = 1$ 時，$m = 1,2,3,\cdots,p_1$ ，得 i 值：0 、A 、$2A$ 、$\cdots$ 、$(p_1 - 1)A$

$l=2$時，$m=1,2,3,\cdots,p_1$，得i值：p_1、p_1+A、p_1+2A、$\cdots$、$p_1+(p_1-1)A$

$\vdots$

$\vdots$

$l=A$時，$m=1,2,3,\cdots,p_1$，得i值：$(A-1)p_1$、$(A-1)p_1+A$、$(A-1)p_1+2A$

、$\cdots$、$(A-1)p_1+(p_1-1)A$

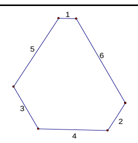
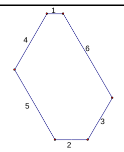
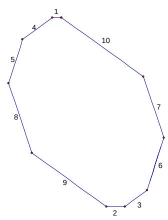
故$\{b_0,b_1,\cdots,b_{n-1}\}$可為$1^t,2^t,3^t,\cdots,n^t$的一個排列

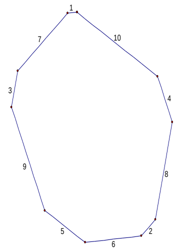
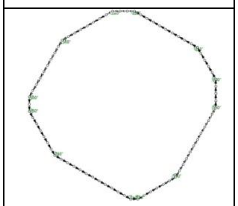
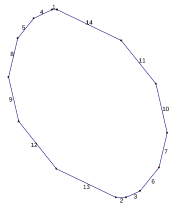
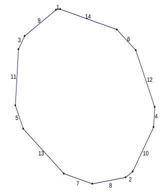
即 $f_n^t(x)=b_0+b_1x+b_2x^2+\cdots+b_{n-1}x^{n-1}$

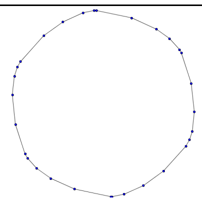
其中$b_0,b_1,\cdots,b_{n-1}$為$1^t,2^t,\cdots,n^t$

伍、研究成果

一、

<table>
<tr><th>n</th><th>f_n與$\Phi_n(x)$</th><th>個數</th><th>圖形</th></tr>
<tr><td rowspan="3">6</td><td>$f_6(x)=1+5x+3x^2+4x^3+2x^4+6x^5$</td><td rowspan="2">2</td><td></td></tr>
<tr><td>$f_6(x)=1+4x+5x^2+2x^3+3x^4+6x^5$</td><td></td></tr>
<tr><td colspan="3">$\Phi_6(x)=x^2-x+1$</td></tr>
<tr><td>10</td><td>$f_{10}(x)=1+4x+5x^2+8x^3+9x^4+2x^5+3x^6+6x^7+7x^8+10x^9$</td><td>24</td><td></td></tr>
</table>

	$f_{10}(x)=1+7x+3x^2+9x^3+5x^4+6x^5+2x^6+8x^7+4x^8+10x^9$		
	$\Phi_{10}(x)=x^4-x^3+x^2-x+1$		
12	$f_{12}(x)=1+5x+9x^2+10x^3+2x^4+6x^5+7x^6+11x^7+3x^8+4x^9+8x^{10}+12x^{11}$	732	
	$f_{12}(x)=1+6x+11x^2+4x^3+5x^4+10x^5+3x^6+8x^7+9x^8+2x^9+7x^{10}+12x^{11}$		
	$\Phi_{12}(x)=x^4-x^2+1$		
14	$f_{14}(x)=1+4x+5x^2+8x^3+9x^4+12x^5+13x^6+2x^7+3x^8+6x^9+7x^{10}+10x^{11}$	720	
	$f_{14}(x)=1+9x+3x^2+11x^3+5x^4+13x^5+7x^6+8x^7+2x^8+10x^9+4x^{10}+12x^{11}+6x^{12}+14x^{13}$		
	$\Phi_{14}(x)=x^6-x^5+x^4-x^3+x^2-1$		

15	$f_{15}(x)=1+7x+13x^2+14x^3+10x^4+11x^5+2x^6+8x^7+14x^8+5x^9+6x^{10}+12x^{11}+3x^{12}+9x^{13}+15x^{14}$	48	
	$f_{15}(x)=1+5x+9x^2+10x^3+14x^4+3x^5+4x^6+8x^7+12x^8+13x^9+2x^{10}+6x^{11}+7x^{12}+11x^{13}+15x^{14}$		
	$\Phi_{15}(x)=x^8-x^7+x^5-x^4+x^3-x^2+x-1$		
30	$f_{30}(x)=1+10x+17x^2+20x^3+27x^4+6x^5+7x^6+16x^7+23x^8+26x^9+3x^{10}+12x^{11}+13x^{12}+22x^{13}+29x^{14}+2x^{15}+9x^{16}+18x^{17}+19x^{18}+28x^{19}+5x^{20}+8x^{21}+15x^{22}+24x^{23}+25x^{24}+4x^{25}+11x^{26}+14x^{27}+21x^{28}+30x^{29}$		
	$\Phi_{30}(x)=x^8+x^7-x^6-x^5-x^4-x^3+x+1$		

二、$f_{2p}=1+4x+5x^2+\cdots+2px^{2p-1}=1+(p+2)x+3x^2+\cdots+2px^{2p-1}$

其中 p 為奇質數，其中 f_{2p} 的個數為 $(p-1)!$。(證明詳見 p.11)

三、$n=p^r$ 時不存在平衡多項式，其中 $r\in Z$ 且 $p\in$ 質數。

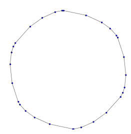
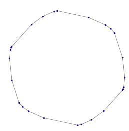
(證明詳見 p.12)

四、當 $n=rs$ ，$(r,s)=1$ 時，必存在平衡多項式。(證明詳見 p.18

五、n 次分圓多項式必整除平衡多項式 f_n ，即 $\Phi_n(x)\mid f_n$ 。(證明詳見

p.21)

六、n含有 3 個以上的質因子時，必存在係數為$1^2,2^2,3^2,\cdots,n^2$的二階平衡多項式，即 $f_n^2=a_0^2+a_1^2x+a_2^2x^2+a_3^2x^3+\cdots+a_{n-1}^2x^{n-1}$ ，並可圍成邊長為$1^2,2^2,3^2,\cdots,n^2$的等角n邊形，其中$a_0^2,a_1^2,a_2^2,\cdots,a_{n-1}^2$為$1^2,2^2,3^2,\cdots,n^2$的一種排列。(證明詳見 p.29，且最小的 f_n^2 為 f_{30}^2，其係數恰有 f_{30}^1係數的平方。

f_{30}^1	f_{30}^2
	

七、n含有$t+1(t\ge 3)$個以上的質因子時，必存在係數$1^t,2^t,3^t,\cdots,n^t$的t階平衡多項式， 即 $f_n^t=a_0^t+a_1^tx+a_2^tx^2+a_3^tx^3+\cdots+a_{n-1}^tx^{n-1}$，並可圍成邊長為$1^t,2^t,3^t,\cdots,n^t$的等角$n$邊形，其中$a_0^t,a_1^t,a_2^t,\cdots,a_{n-1}^t$為$1^t,2^t,\cdots,n^t$的一種排列。(證明詳見 p.34)

八、利用 C++程式語言驗證f_n的個數。(詳見附錄一)

九、1、 f_n^1的係數：$n=rs$，$(r,s)=1$

(1) $a_k\equiv(k_1+1)+rk\ (\mathrm{mod}\ rs)\ \ k_1\equiv k\ (\mathrm{mod}\ r)\ \ k=0,1,2,\cdots,n-1$

(2) $a_k\equiv(k_1+1)+sk\ (\mathrm{mod}\ rs)\ \ k_1\equiv k\ (\mathrm{mod}\ s)\ \ k=0,1,2,\cdots,n-1$

十、 f_n^t的係數：令$n = n_1 \times n_2, (n_1, n_2) = 1$其中$f_{n_1}^{t-1}$存在

則$a_k^t \equiv (s_k)^t$，$s_k \equiv a_{n_1'} + n_1 k \pmod{n}, n_1' \equiv k \pmod{n_1}, k = 0,1,2,\cdots,n-1$其中

a_{n_1}為$f_{n_1}^{t-1}$的係數。

陸、未來應用

本研究可應用在解決天秤上放置$1^t, 2^t, 3^t, \cdots, n^t$砝碼使之平衡的問題，如圖 6

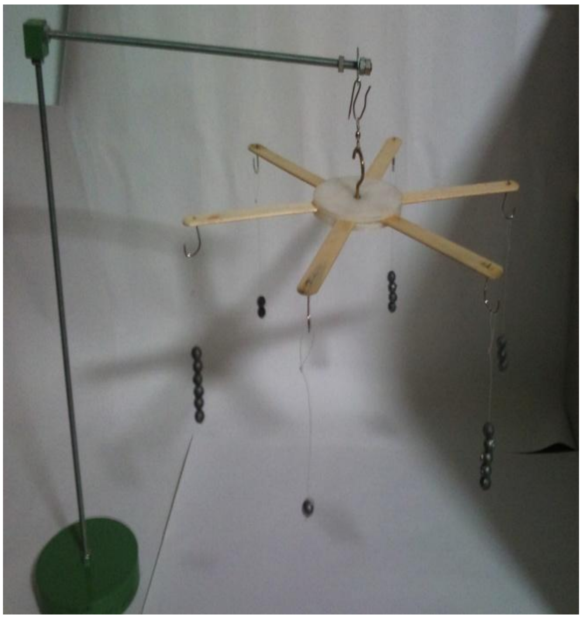
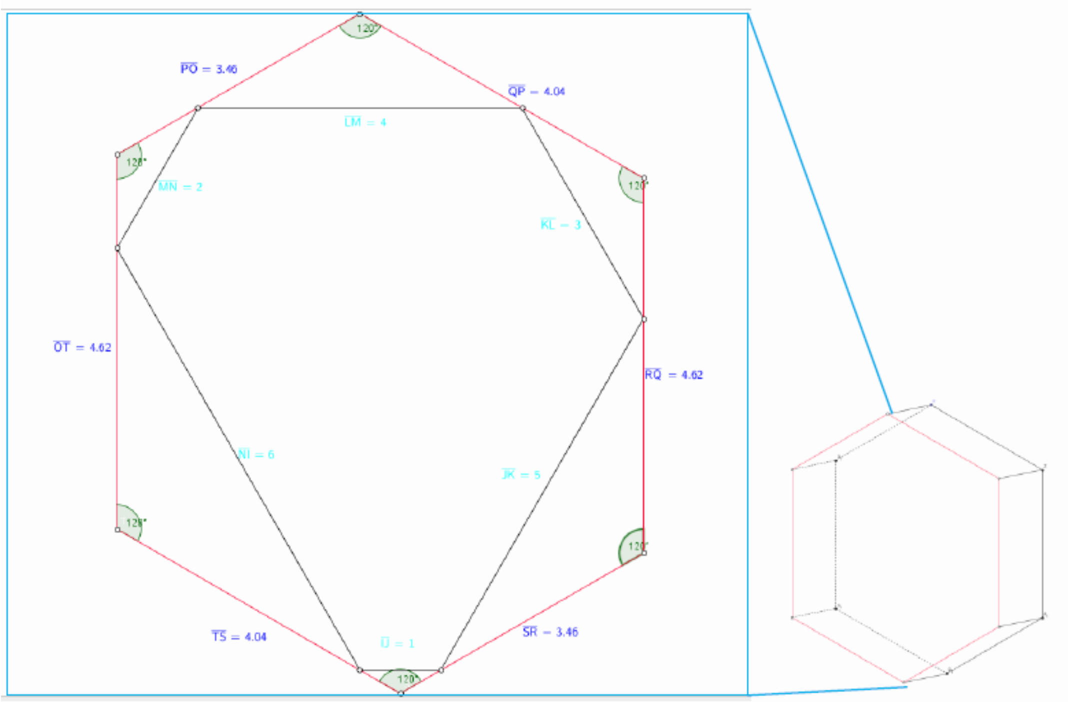


圖 6

還可應用在鏡面反射的問題，由 O 點射出雷射光，若有人入侵，則會擋住入射光線使光線無法回來，且其長度不同，可適用於不同的空間，可有較廣泛的應用，而不反只停留在圖形的鏡面反射



柒、參考資料

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捌、附錄

一、C++程式語言

(1) 此程式是利用和力為零的想法

```
// 20120613 By Prof P-J Lai MATH NKNU
// permu_Cyclotomic

#include <cstdlib>
#include <iostream>
#include <cmath>
// 已下使用 define 定義常量，是不好的用法, ref: S.C. Dewhurst
//#define N 6
//#define Pi 3.1416
//#define eps 0.00001
//#const=常數
//#double=小數點
const int N=6;
const double eps=0.00001;
const double Pi=2.0*asin(1.0);
double a=cos(2*Pi/N),b=sin(2*Pi/N);
double C[N][2];
int index=0;
using namespace std;

//void complexProduct(double, double, int);
void perm ( int* , int );
int main(int argc, char *argv[   ]){
    int num[N+1];
    int   i;

    for(i = 1; i <=N ; i++)
    num[ i ] = i;
    //complexProduct(a, b, N);
    perm(num, 2);
```

```
        cout<<" 符合合力之 case 有"<<index<<"種";
        system("PAUSE");
        return 0;
}

        void perm(int* num, int i) {
                if(i < N) {
                        int j;
                        for(j = i; j <= N; j++) {
                                int tmp = num[j];
                                // 旋轉該區段最右邊數字至最左邊
                                int k;
                                for(k = j; k > i; k--)
                                num[k] = num[k-1];
                                num[i] = tmp;
                                perm(num, i+1);
                                // 還原
                                int m;
                                for(m = i; m < j; m++)
                                num[m] = num[m+1];
                                num[j] = tmp;
                        }
                }
                else {
                                int j;
                                /*
                                 // 顯示此次排列


                                for(j = 1; j <= N; j++)
                               // printf("%d ", num[j]);
                               cout<<num[j];
                               cout<<endl;
                                */


                                // 檢驗此次排列是否達成合力為零
```

```
double sum1=num[1],sum2=0;
//double C[][N];
C[0][0]=0;
C[0][1]=0;
C[1][0]=a;
C[1][1]=b;
int i;
for(i = 2; i <= N-1; i++){
//C[i][0]= (C[i-1][0])^2-(C[i-1][1])^2;
C[i][0]=cos(i*2*Pi/N) ;
C[i][1]=sin(i*2*Pi/N);
}

/* for(j = 1; j <= N-1; j++)
   cout<<num[j+1]<<"    "<<C[j][0]<<"    "<<C[j][1]<<endl;
 */




for(j = 1; j <= N-1; j++){
   sum1=sum1+num[j+1]*C[j][0];
   sum2=sum2+num[j+1]*C[j][1];
}

/* cout<<sum1<<" "<<sum2<<endl;
*/



if(abs(sum1)<=eps && abs(sum2)<=eps){
    //cout<<sum1<<" "<<sum2<<"    ";
   index++;
   for(j = 1; j <= N; j++){

   cout<<num[j];
```

```
                }
                cout<<"    ";
                //cout<<endl;
            }
            //cout<<endl;
        }
    }
```

(2) 此程式為利用分圓多項式整除平衡多項式的想法

```
#include<stdio.h>
#include<time.h>
int p;
int n;
FILE *fp;

void ppm(int te[],int i,int f[],int b[],int nb[]){
  if(i<p-1){
        int j;
        int tmp;
        for(j=i;j<p-1;j++){
              tmp=te[j];
              int k;
              for(k = j; k > i; k--)
                    te[k] = te[k-1];
              te[i] = tmp;
              ppm(te,i+1,f,b,nb);
              int m;
              for(m = i; m < j; m++)
                    te[m] = te[m+1];
              te[j] = tmp;
              }
        }
  else{
        int j;
        if(te[0]<te[p-2]){return;}
```

```
        for(j=0;j<(p-1)/2;j++){
            f[2*j+1]=b[te[j]];
            f[2*j+1+p]=nb[te[j]];
            }
        for(j;j<p-1;j++){
            f[2*j+3]=b[te[j]];
            f[2*j+3-p]=nb[te[j]];
            }
        int l;
            n++;
            for(l=0;l<2*p-1;l++){
                printf("%dx^%d+",f[l],l);
                }
            printf("%dx^%d\n",f[l],l);

        }
    return;
    }

int main(){

    printf("請輸入 f(x)_2p，p=");

    while(scanf("%d",&p)!=EOF){


    clock_t start, end;

    start = clock();
        n=0;
        if(p%2==0){
            printf("共%d 個\n",n);
            printf("請輸入 f(x)_2p，p=");
            continue;
            }
```

```
int k;
k=1;
int b[p-1],nb[p-1];
int i,map[2*p+1];
for(i=1;i<=2*p;i++){
    map[i]=0;
    }
map[1]=1;
//找差值
int t;
int f[2*p],te[p-1];
for(k;k<=p;k+=p-1){

    for(i=2;i<=2*p-k;i++){
    map[i]=0;
    }
    t=-1;
    map[1+k]=1;
    for(i=1;i<=2*p;i++){
        if(map[i]==0){
            t++;
            nb[t]=i;
            b[t]=i+k;
            map[i]=1;
            map[i+k]=1;
            }
        }
    //列式
    for(i=0;i<p-1;i++){
        te[i]=i;
        }
        f[0]=1;
        f[p]=1+k;
    int j;
```

```
            ppm(te,0,f,b,nb);


            }

    end = clock();

    int during = (end-start);
        printf("共%d 個約花%dmS\n",n,during);
        printf("請輸入 f(x)_2p，p=");
        }
    return 0;
    }
```

二、程式成果

(一) $n=6$：

145236　153426　162435　163254

(扣除對稱組後，得到 2 組)

(二) $n=10$：

14589236710 14510723698 14769238510 14710523896 14967231058 14985231076 16
389254710 16310725498 16749258310 16710325894 16947251038 16983251074 1739
5628410 17310462859 17485629310 17410362958 17584621039 17593621048 182956
37410 18210463759 18369274510 18310527496 18475639210 18410263957 18549276
310 18574631029 18592631047 18510327694 18945271036 18963271054 1928564731
0 19210364758 19375648210 19310264857 19573641028 19582641037 11028465739
11029365748 11036729458 11037465829 11038529476 11039265847 11047365928 1
1048265937 11054729638 11058329674 11074529836 11076329854

(扣除對稱組後，得到 24 組)

(三) $n=12$：

126108945371112 126109845371211 126117945381012 126119745381210 126127845391
011 126128745391110 127891034561112 127810934561211 127109638541112 1271161
03459812 127111063459128 127126934510811 127129438561110 127129634510118 12
8791034651112 128791043561211 128710934651211 128115103469712 12811964351012
7 128111053469127 128125934610711 128129534610117 129851074361112 129851163
471012 129105678341112 129106578341211 129114678351012 129116478351210 1291
24578361011 129125478361110 129125674310118 129125763411108 121068945731112
121069845731211 121084116357912 121095678431112 121095687341211 12109657843
1211 121011367845912 121011394578612 121011548736129 121011637845129 121011
934578126 121012357846911 121012384579611 121012476351198 121012537846119 1
21012834579116 121167945831012 121169745831210 121175983461210 1211761034958
12 121171063495128 121183107456912 121185103496712 121181053496127 12119467
8531012 121196478531210 121110367854912 121110394587612 121110563894712 121
110568349127 121110637854129 121110934587126 121112347856910 121112367451098
121112374589610 121112437856109 121112543896710 121112563491078 1211126534
91087 121112734589106 121267845931011 121268745931110 121274983561110 12127
5104396811 121276934105811 121279634105118 121285934106711 121289534106117

121293687541011 121294578631011 121295478631110 121210357864911 121210384597
611 121210468359117 121210537864119 121210834597116 121211347865910 1212113
48756109 121211374598610 121211437865109 121211563410978 121211564391087 12
1211653410987 121211734598106 135108946271112 135109846271211 13511794628101
2 135119728461012 135119746281210 135127846291011 135128746291110 135129628
471011 136891124571012 136811924571210 136910728541112 136107112459812 1361
01172459128 136127924511810 136129724511108 136121042857119 137951164281012
137911528641210 137101142865129 137118521064912 137125864211109 13712842106
5911 138691124751012 138691142571210 138611924751210 138105679241112 138105
112479612 138106579241211 138109742511126 138101152479126 138114679251012 1
38116479251210 138124579261011 138125479261110 138125924711610 1381295247111
06 139751084261211 139751262481011 139115684210127 139115862412107 13911652
1084712 139126421085711 131058946721112 131059846721211 131067112495812 131
061172495128 131074126258911 131071152694128 131085679421112 131085112497612
131086579421211 131081142695127 131081152497126 131092116458712 1310112679
45812 131011294678512 131011472896512 131011486251297 131011627945128 13101
1924678125 131012257946811 131012284679511 131012286451179 131012462897511
131012527946118 131012572491168 131012752491186 131012824679115 131157946821
012 131159746821210 131165108247129 131184679521012 131186479521210 1311957
28104612 131195782410126 131110267954812 131110294687512 131110627954128 13
1110924687125 131112247956810 131112274689510 131112427956108 13111254281076
9 131112724689105 131257846921011 131258746921110 131264108257119 131265114
297810 131267924115810 131269724115108 131272108456911 131279526114108 1312
84579621011 131285479621110 131285924117610 131289426115107 131289524117106
131294782510116 131296528114710 131210257964811 131210284697511 13121052796
4118 131210572411968 131210574291186 131210642811579 131210752411986 131210
824697115 131211247965810 131211268451097 131211274698510 131211427965108 1
31211724698105 146791223581011 146711102358129 146810729531112 1469712235108
11 146911823510127 146117102351289 146119823512107 146121032957118 14769123
2581110 147610113258129 147851263291011 147896310521112 147811529631210 147
961232511810 147910832511126 147103116529812 147106113251289 147108521163912
147109832512116 147101132965128 147115963212108 147123965211810 1471283211
65910 147129231056118 148591223761011 148591232671110 148510113267129 14851
1102376129 148956710231112 148951223710611 148951232611710 148965710231211
148910732611125 148911623710125 148102116539712 148105113261279 148109732612
115 148113671025912 148115102371269 148116371025129 148119623712105 1481229
65311710 148123571026911 148125371026119 149651183271210 149651272381110 14
9731185261012 149856710321112 149865710321211 149105783211126 14910587231211
6 149106521183712 149112671035812 149116271035128 149122571036811 149123685
211107 149125271036118 149126321185710 141057122396811 141051182396127 1410
61152793128 141072118536912 141075689321211 141075122398611 141071162398125
141082126359711 141081132795126 141011296351278 141011528936127 14101158239
1267 141011762391285 141012268531197 141156123297810 141151083297126 141163
127258910 141165123298710 141161073298125 141173810526129 141183671052912 1
41183610725129 141185631092712 141185729103612 141186371052129 1411926710538
12 141193510726128 141193610528127 141196271053128 141110387251296 14111058
3291276 141110673291285 141112237105689 141112327105698 141112523109678 141
112532910768 141256113210789 141257102311689 141259823116107 141259832107116
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3571062911 141285371062119 141286529113710 141289327115106 141292571063811
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37106589 141211327106598 141211328956107 154891037261112 154891126371012 15
4810937261211 154811926371210 154911728361210 154107112639812 15410116283712
9 154101172639128 154116103729812 154118721036912 154111063729128 154126937
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12 158112106431279 158111023769124 158122937610411 158123471126910 15812392
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87411 159124271136108 151038124276119 151039114276128 151047112693812 15104
1172693128 151062128437911 151061142893127 151071132894126 151083671142912
151083112697412 151083124271169 151086371142129 151089642711123 151081132697

124 151092671143812 151093114271268 151096271143128 151098642712113 1510112
78431296 151012237114689 151012327114698 151012372691148 151012732691184 15
1137124286109 151139104286127 151146103792812 151141063792128 15116391042712
8 151173610824129 151173124281069 151179642810123 151182103796412 151183710
429126 151181023796124 151193410826127 151193104281267 151197642812103 1511
12263791048 151112623791084 151237114296108 151238104296117 151246937102811
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3107 151272610834119 151273114291068 151278642910113 151279328114106 151282
710439116 151282937106411 151283471162910 151283926117410 151283104291167 1
51284371162109 151286421011379 151287642911103 151289237106114 1512893261171
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1011478 151210237116489 151210327116498 151210362811749 151210372611948 151
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11 161075481132129 161071142598123 161082124371159 161083571242911 16108472
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85109 161129104385127 161135987421210 161143107852912 161145103892712 16114
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8 161228104395117 161234987521110 161235104792811 161237102511489 161239825
114107 161242108753911 161245938102711 161249538102117 161249823117105 1612
52810734119 161259723118104 161272510934118 161272938105411 161272114391058
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176491128531012 176411928531210 176589410321112 176598410321211 176810421253
119 176910321254118 176102114839512 176103112859412 176101132859124 1761129
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561249 178109346511122 178109435612112 179351168421012 179351186241210 1794
31286251011 179103621285411 179105486211123 179113521286410 179113586212104
179125268411103 171034116852912 171031182496125 171042128635911 17104511289
3612 171045118239126 171041152893126 171051162498123 171063112895412 171065
982311124 171061132895124 171084621293511 171011258631294 171011432129658 1
71012352891146 171012426851193 171012532891164 171125108643129 1711261235948
10 171121083594126 171143910625128 171153941082612 171153910428126 17115934
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109865 184591227361011 184591236271110 184510113627129 184511102736129 1846
11102539127 184711925310126 184951227310611 184951236211710 184910736211125
184911627310125 184105921137612 184105113621279 184109736212115 18411510273
1269 184119627312105 184125721139610 185679411231012 185697411231210 185691
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評語

1. 將砝碼分置於圓盤上的問題化成多項式的研究，相當不錯。

2. 1 階甚至 K 階平衡多項式的存在性多用建構的方式來證明，不存在的情形，則用反証法。

3. 當平衡多項式 f_n 存在時，記明了分圓多項式 ψ_n 整除 f_n，是非常好的結果。

4. 從 1 階 f_{12} 的總數遠遠超過本研究所能建構的 1 階 f_{12} 數目，很顯然還有非常多的 f_n 不能用本研究的方法來建構,應找出原因或用新方法來建構。

5. 證明相當嚴謹、困難度夠，是一篇相當不錯的作品。

6. 報告中有一些誤植及說明不清楚的地方，宜改進，結構上亦可再加強。。

Scale Balanced Polynomial

Project ID: MA303

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Scale Balanced Polynomial

Abstract

A Balanced Polynomial is an integral polynomial having a zero at some primitive root of unity. A scale balanced polynomial (SBP) of degree n-1 is a Balanced Polynomial having all of 1,2,...,n as coefficients with constant term 1. These properties of SBP's are explored:

1. Algebraic Properties
   a. SBP of degree n-1 exists precisely for all integers n except powers of primes.
   b. Due to the Chinese Remainder Theorem, we give a method to construct SBP's of degree n-1 whenever n is a product of two relatively prime integers.
   c. Every integral polynomial divisible by a cyclotomic polynomial is balanced. Conversely, each SBP of degree n-1 has the nth cyclotomic polynomial as a factor.
   d. If p and q are distinct primes, then there are at least 2*(p-1)!*(q-1)! SBP's of degree pq-1.
2. Linking between Algebra and Physics

   Through the eyes of Physics, a Balanced Polynomial represents a system of n point particles being placed on entire roots of unity, each with weight given by the coefficient and having 0 as center of gravity.
3. Linking between Physics and Geometry

   Since the torque is the vector product of the lever-arm distance and the force, it follows that each Balanced Polynomial is associated with an equiangular n-sided polygon with vertices formed by the partial sums of the torques. In case of SBP's, the equiangular polygon has edges of the length
   1,2,…,n in some order.
4. Linking between Equiangular Polygons

   If n has m distinct prime factors, then to every equiangular polygon P defined by SBP of degree n-1, there is a finite sequence of n-sided equiangular polygons with edge lengths k-th powers of those of P for any positive integer k less than m.

Contents

Introduction

This project is originated from a puzzle asking for the possibility of arranging weights 1,2,3,4,5,6 units on the vertices of a regular hexagon (Fig. 1) in some order making the system balanced, (Fig. 2). In solving the physical problem, we are lead to explore the principle of superposition, cyclotomic polynomial, torque, cross product and Chinese Remainder Theorem.

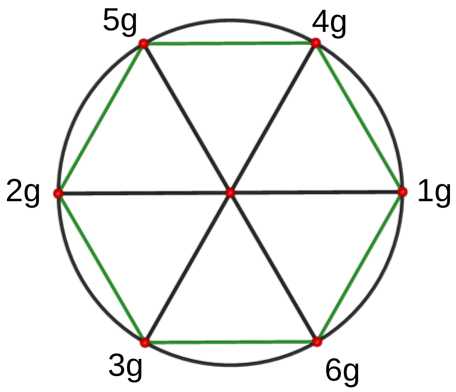


Fig. 1 Weight be put on the vertices of a regular hexagon

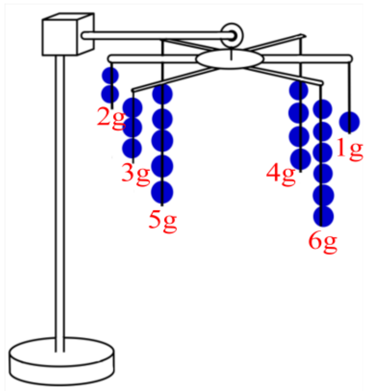


Fig. 2 SBP Realized

Definitions

a. A polynomial $p(\omega_n)=0$ is said to be balanced if $p(\omega_n)=0$, where

$n=\deg(p)+1$ and $\omega_n=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$.

b. The scale balanced polynomial (SBP) refers to a balanced polynomial $p(x)$ of the form $p(x)=1+\pi(2)x+\pi(3)x^2+\cdots+\pi(n)x^{n-1}$ where π is a permutation of $\{2,3,\cdots,n\}$.

c. A balanced polynomial of the form $1+\pi(2)^t x+\pi(3)^t x+\cdots+\pi(n)^t x^{n-1}$ is called t-BP.

d. Recall that a cyclotomic polynomial [3] is a polynomial taking the form

$$\Phi_n(x)=\prod_{\substack{1\le k<n\\ \gcd(k,n)=1}}(x-\omega_n^k).$$

Research Purposes

a. Investigate the generation of SBP.
b. Investigate the relation between existence of SBP and nth roots of unity.
c. Investigate the geometric interpretation of SBP in physical model.
d. Investigate the relation between SBP and cyclotomic polynomial.
e. Establish the recursive relationship between SBP and t-BP.

Methods and Results

Consider the problem of putting weights 1,2,3,4,5,6 units on the vertices of a regular hexagon makes the system balanced. There are seven hundred and twenty ossibility arranging weights. Among them, there are only twenty four can achieve alanced. The possibility of getting correct is one out of thirty. By equivalence, we mean if you consider having one situation the same as its rotations, there are only four.

Try to solve the problem by the system of polynomial equations.

$$\because \begin{cases} \omega^0 + \omega^3 = 0 \\ \omega^1 + \omega^4 = 0 \\ \omega^2 + \omega^5 = 0 \end{cases} \text{ and } \begin{cases} \omega^0 + \omega^2 + \omega^4 = 0 \\ \omega^1 + \omega^3 + \omega^5 = 0 \end{cases}$$

To get the permutation $\{2,3,\cdots,n\}$, so we multiply a number on each system of polynomial equations.

$$\therefore \begin{cases} A_1(\omega^0 + \omega^3) = 0 \\ A_2(\omega^1 + \omega^4) = 0 \\ A_3(\omega^2 + \omega^5) = 0 \end{cases} \text{ and } \begin{cases} B_1(\omega^0 + \omega^2 + \omega^4) = 0 \\ B_2(\omega^1 + \omega^3 + \omega^5) = 0 \end{cases}$$

The sum is given by the following equations:

$$1\omega^0 + (A_2 + B_2)\omega^1 + (A_3 + B_1)\omega^2 + (A_1 + B_2)\omega^3 + (A_2 + B_1)\omega^4 + (A_3 + B_2)\omega^5 .$$

After solving the system of polynomial equations, we get the result. However, because of the complicated process, so we apply the Chinese Remainder Theorem.

Primal Algorithm for SBP of Degree 5

Motivated by the linking between Physics and Algebra, the following algorithm is created (Table 1) for the case $n = 6 = 2 \cdot 3$ using Chinese Remainder Theorem [2].

Write $n = 2 \cdot 3$.

Assign $A_1 = 0$, $A_2 = 2$, $A_3 = 4$, $B_1 = 0$, $B_2 = 1$.

Row 1: 2*mod (k,3), k=0,1,…,5

Row 2: mod (k,2), k=0,1,…,5

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

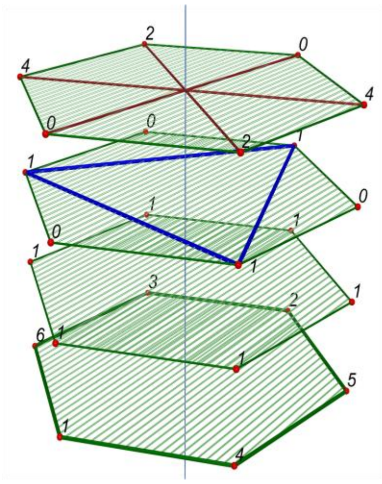


Fig. 3 SBP of Degree 5

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5
Row 1	A_1	A_2	A_3	A_1	A_2	A_3
	0	2	4	0	2	4
Row 2	B_1	B_2	B_1	B_2	B_1	B_2
	0	1	0	1	0	1
Row 3	1	1	1	1	1	1
Row 4	1	4	5	2	3	6

Table 1 Primal Algorithm for SBP of Degree 5

The numbers in Row 4, listed counterclockwisely (Fig. 3), give the required weights.

Dual Algorithm for SBP of Degree 5

Write $n = 3 \cdot 2$.

Assign $A_1 = 0$, $A_2 = 1$, $A_3 = 2$, $B_1 = 0$, $B_2 = 3$.

Row 1: 3*mod (k,2), k=0,1,…,5

Row 2: mod (k,3), k=0,1,…,5

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

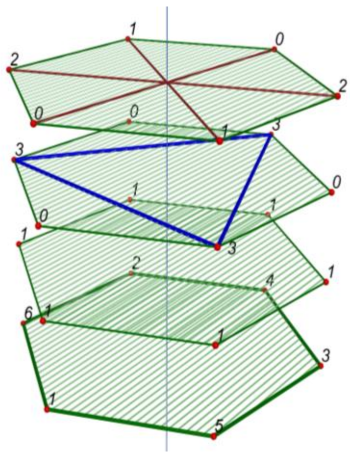


Fig. 4 SBP of Degree 5

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5
Row 1	A_1	A_2	A_3	A_1	A_2	A_3
	0	1	2	0	1	2
Row 2	B_1	B_2	B_1	B_2	B_1	B_2
	0	3	0	3	0	3
Row 3	1	1	1	1	1	1
Row 4	1	5	3	4	2	6

Table 2 Dual Algorithm for SBP of Degree 5

Algorithm be Compressed Interns Algebra

Such an algorithm can be compressed interns of algebra.

From Table 1, we are able to construct the SBP of degree 5:

$$p_6(x) = 1 + 5x + 3x^2 + 4x^3 + 2x^4 + 6x^5$$

$$= 3 \cdot \left(\sum_{k=0}^{1} kx^k \right) \left(\sum_{k=0}^{2} x^{2k} \right) + \left(\sum_{k=0}^{2} kx^k \right) \left(\sum_{k=0}^{1} x^{3k} \right) + \sum_{k=0}^{5} x^k .$$

From Table 2, we are able to construct the SBP of degree 5:

$$p_6(x) = 1 + 4x + 5x^2 + 2x^3 + 3x^4 + 6x$$

$$= 2 \cdot \left(\sum_{k=0}^{2} kx^k \right) \left(\sum_{k=0}^{1} x^{3k} \right) + \left(\sum_{k=0}^{1} kx^k \right) \left(\sum_{k=0}^{2} x^{2k} \right) + \sum_{k=0}^{5} x^k .$$

We observed the $p_6(x) = 1 + 4x + 5x^2 + 2x^3 + 3x^4 + 6x^5$ and $p_6(x) = 1 + 5x + 3x^2 + 4x^3 + 2x^4 + 6x^5$ all have a sixth cyclotomic polynomial of the form $x^2 - x + 1$.

Primal Algorithm for SBP of degree 9

Write $n = 10 = 2 \cdot 5$

We use the same algorithm to create SBP of degree 9.

$\because \omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 = 0$

$$\because \begin{cases} \omega^0 + \omega^5 = 0 \\ \omega^1 + \omega^6 = 0 \\ \omega^2 + \omega^7 = 0 \\ \omega^3 + \omega^8 = 0 \\ \omega^4 + \omega^9 = 0 \end{cases} \text{ and } \begin{cases} \omega^0 + \omega^2 + \omega^4 + \omega^6 + \omega^8 = 0 \\ \omega^1 + \omega^3 + \omega^5 + \omega^7 + \omega^9 = 0 \end{cases}$$

$$\therefore \begin{cases} A_1(\omega^0 + \omega^5) = 0 \\ A_2(\omega^1 + \omega^6) = 0 \\ A_3(\omega^2 + \omega^7) = 0 \\ A_4(\omega^3 + \omega^8) = 0 \\ A_5(\omega^4 + \omega^9) = 0 \end{cases} \text{ and } \begin{cases} B_1(\omega^0 + \omega^2 + \omega^4 + \omega^6 + \omega^8) = 0 \\ B_2(\omega^1 + \omega^3 + \omega^5 + \omega^7 + \omega^9) = 0 \end{cases}$$

Assign $A_1 = 0$, $A_2 = 2$, $A_3 = 4$, $A_4 = 6$, $A_5 = 8$, $B_1 = 0$, $B_2 = 1$.

Row 1: 2*mod (k,5), k=0,1,…,9

Row 2: mod (k,2), k=0,1,…,9

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9
Row 1	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	2	4	6	8	0	2	4	6	8
Row 2	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
	0	1	0	1	0	1	0	1	0	1
Row 3	1	1	1	1	1	1	1	1	1	1
Row 4	1	4	5	8	9	2	3	6	7	10

Table 3 Primal Algorithm for SBP of Degree 9

From the numbers in Row 4, we are able to construct the SBP of degree 9. It can be compressed interns of algebra.

$$p_{10}(x) = 1 + 4x + 5x^2 + 8x^3 + 9x^4 + 2x^5 + 3x^6 + 6x^7 + 7x^8 + 10x^9$$

$$= 2 \cdot \left(\sum_{k=0}^{4} kx^k \right) \left(\sum_{k=0}^{1} x^{5k} \right) + \left(\sum_{k=0}^{1} kx^k \right) \left(\sum_{k=0}^{4} x^{2k} \right) + \sum_{k=0}^{9} x^k$$

Dual Algorithm for SBP of degree 9

Using the dual algorithm, we write $n = 5 \cdot 2$ and

assign $A_1 = 0$, $A_2 = 1$, $A_3 = 2$, $A_4 = 3$, $A_5 = 4$, $B_1 = 0$, $B_2 = 5$.

Row 1: 5*mod (k,2), k=0,1,…,9

Row 2: mod (k,5), k=0,1,…,9

Row 3: All 1

Row 4: Row 1 + Row 2 + Row

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9
Row 1	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	1	2	3	4	0	1	2	3	4
Row 2	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
	0	5	0	5	0	5	0	5	0	5
Row 3	1	1	1	1	1	1	1	1	1	1
Row 4	1	7	3	9	5	6	2	8	4	10

Table 4 Dual Algorithm for SBP of Degree 9

$$p_{10}(x) = 1 + 7x + 3x^2 + 9x^3 + 5x^4 + 6x^5 + 2x^6 + 8x^7 + 4x^8 + 10x^9$$

$$= 5\cdot\left(\sum_{k=0}^{1} kx^k\right)\left(\sum_{k=0}^{4} x^{2k}\right) + \left(\sum_{k=0}^{4} kx^k\right)\left(\sum_{k=0}^{1} x^{5k}\right) + \sum_{k=0}^{9} x^k$$

We observed $p_{10}(x) = 1 + 4x + 5x^2 + 8x^3 + 9x^4 + 2x^5 + 3x^6 + 6x^7 + 7x^8 + 10x^9$ and $p_{10}(x) = 1 + 7x + 3x^2 + 9x^3 + 5x^4 + 6x^5 + 2x^6 + 8x^7 + 4x^8 + 10x^9$ all have

$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$ as a factor.

Theorem 1. *If* $n = 2\cdot p$ *where* p *is a prime, then there exist a SBP.*

proof.

$\because \omega^0 + \omega^1 + \omega^2 + \cdots + \omega^{2p-2} + \omega^{2p-1} = 0$

$$\because \begin{cases} \omega^0 + \omega^p = 0 \\ \omega^1 + \omega^{p+1} = 0 \\ \omega^2 + \omega^{p+2} = 0 \\ \quad\vdots \\ \omega^{p-1} + \omega^{2p-1} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \omega^0 + \omega^2 + \omega^4 + \cdots + \omega^{2p-2} = 0 \\ \omega^1 + \omega^3 + \omega^5 + \cdots + \omega^{2p-1} = 0 \end{cases}$$

$$\therefore \begin{cases} A_1(\omega^0 + \omega^p) = 0 \\ A_2(\omega^1 + \omega^{p+1}) = 0 \\ A_3(\omega^2 + \omega^{p+2}) = 0 \\ \quad\vdots \\ A_p(\omega^{p-1} + \omega^{2p-1}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} B_1(\omega^0 + \omega^2 + \omega^4 + \cdots + \omega^{2p-2}) = 0 \\ B_2(\omega^1 + \omega^3 + \omega^5 + \cdots + \omega^{2p-1}) = 0 \end{cases}$$

We write $n = 2 \cdot p$ and assign $A_1 = 0$, $A_2 = 2$, $A_3 = 4$, $\cdots$, $A_{p-1} = 2p-2$, $B_1 = 0$, $B_2 = 1$.

Row 1: 2*mod (k,p), k=0,1,…,(2*p-1)

Row 2: mod (k,2), k=0,1,…,(2*p-1)

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

	ω^0	ω^1	ω^2	ω^3		ω^{p-1}	ω^p	ω^{p+1}	ω^{p+2}		ω^{2p-1}
Row 1	A_1	A_2	A_3	A_4		A_p	A_1	A_2	A_3		A_p
	0	2	4	6		$2p-2$	0	2	4		$2p-2$
Row 2	B_1	B_2	B_1	B_2	$\cdots$	B_1	B_2	B_1	B_2	$\cdots$	B_2
	0	1	0	1		0	1	0	1		1
Row 3	1	1	1	1		1	1	1	1		1
Row 4	1	4	5	8		$2p-1$	2	3	6		$2p$

Table 5 Primal Algorithm for SBP of Degree 2p-1

$$p_{2p}(x) = 1 + 4x + 5x^2 + \cdots + 2px^{2p-1}$$

$$= 2 \cdot \left(\sum_{k=0}^{p-1} kx^k\right)\left(\sum_{k=0}^{1} x^{pk}\right) + \left(\sum_{k=0}^{1} kx^k\right)\left(\sum_{k=0}^{p-1} x^{2k}\right) + \sum_{k=0}^{2p-1} x^k$$

We write $n = p \cdot 2$ and assign $A_1 = 0$, $A_2 = 1$, $A_3 = 2$, $\cdots$, $A_{p-1} = p-1$, $B_1 = 0$, $B_2 = p$.

Row 1: p*mod (k,2), k=0,1,…,(2*p-1)

Row 2: mod (k,p), k=0,1,…,(2*p-1)

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

	ω^0	ω^1	ω^2	ω^3		ω^{p-1}	ω^p	ω^{p+1}	ω^{p+2}		ω^{2p-1}
Row 1	A_1	A_2	A_3	A_4		A_p	A_1	A_2	A_3		A_p
	0	1	2	3		$p-1$	0	1	2		$p-1$
Row 2	B_1	B_2	B_1	B_2	$\cdots$	B_1	B_2	B_1	B_2	$\cdots$	B_2
	0	p	0	p		0	p	0	p		p
Row 3	1	1	1	1		1	1	1	1		1
Row 4	1	$2+p$	3	$4+p$		p	$1+p$	2	$3+p$		$2p$

Table 6 Dual Algorithm for SBP of Degree 2p-1

$$p_{2p}(x)=1+(p+2)x+3x^2+\cdots+2px^{2p-1}$$

$$=p\cdot\left(\sum_{k=0}^{1}kx^k\right)\left(\sum_{k=0}^{p-1}x^{2k}\right)+\left(\sum_{k=0}^{p-1}kx^k\right)\left(\sum_{k=0}^{1}x^{pk}\right)+\sum_{k=0}^{2p-1}x^k$$

We observed $p_{2p}(x)=1+4x+5x^2+\cdots+2px^{2p-1}$ and $p_{2p}(x)=1+(p+2)x+3x^2+\cdots+2px^{2p-1}$

all have $\Phi_{2p}(x)=x^{p-1}-x^{p-2}+x^{p-3}+\cdots+1$ as a factor.

Case $n=15$. Write $n=15=3\cdot 5$

We assign $A_1=0$, $A_2=3$, $A_3=6$, $A_4=9$, $A_5=12$, $B_1=0$, $B_2=1$, $B_3=2$.

Row 1: 3*mod (k,5), k=0,1,…,11

Row 2: mod (k,5), k=0,1,…,11

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}
Row 1	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
Row 2	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
Row 3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Row 4	1	5	9	10	14	3	4	8	12	13	2	6	7	11	15

Table 7 Primal Algorithm for SBP of Degree 14

$$p_{15}(x)=1+5x+9x^2+10x^3+14x^4+3x^5+4x^6+8x^7+12x^8+13x^9+2x^{10}+6x^{11}+7x^{12}+11x^{13}+15x^{14}$$

$$=3\cdot\left(\sum_{k=0}^{4}kx^k\right)\left(\sum_{k=0}^{2}x^{5k}\right)+\left(\sum_{k=0}^{2}kx^k\right)\left(\sum_{k=0}^{4}x^{3k}\right)+\sum_{k=0}^{14}x^k$$

Write $n = 15 = 5 \cdot 3$.

We assign $A_1 = 0$, $A_2 = 1$, $A_3 = 2$, $A_4 = 3$, $A_5 = 4$, $B_1 = 0$, $B_2 = 5$, $B_3 = 10$.

Row 1: 5*mod (k,3), k=0,1,…,11

Row 2: mod (k,3), k=0,1,…,11

Row 3: All 1

Row 4: Row 1 + Row 2 + Row 3

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}
Row 1	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5	A_1	A_2	A_3	A_4	A_5
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
Row 2	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10
Row 3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Row 4	1	7	13	4	10	11	2	8	14	5	6	12	3	9	15

Table 8 Dual Algorithm for SBP of Degree 14

$$p_{15}(x) = 1 + 7x + 13x^2 + 4x^3 + 10x^4 + 11x^5 + 2x^6 + 8x^7 + 14x^8 + 5x^9 + 6x^{10} + 12x^{11} + 3x^{12} + 9x^{13} + 15x^{14}$$

$$= 5 \cdot \left(\sum_{k=0}^{2} kx^k \right) \left(\sum_{k=0}^{4} x^{3k} \right) + \left(\sum_{k=0}^{4} kx^k \right) \left(\sum_{k=0}^{2} x^{5k} \right) + \sum_{k=0}^{14} x^k$$

We observed

$$p_{15}(x) = 1 + 5x + 9x^2 + 10x^3 + 14x^4 + 3x^5 + 4x^6 + 8x^7 + 12x^8 + 13x^9 + 2x^{10} + 6x^{11} + 7x^{12} + 11x^{13} + 15x^{14}$$

and

$$p_{15}(x) = 1 + 7x + 13x^2 + 4x^3 + 10x^4 + 11x^5 + 2x^6 + 8x^7 + 14x^8 + 5x^9 + 6x^{10} + 12x^{11} + 3x^{12} + 9x^{13} + 15x^{14}$$

all have $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$ as a cyclotomic factor.

Create an algorithm to generate solutions of more general problems

Theorem 2. *If* $n = r \cdot s$ *where* $\gcd(r,s) = 1$, *then there exist a SBP.*

proof.

For the remaining of this section, fix the pair (r,s) with $n = r \cdot s$ and $\gcd(r,s) = 1$.

Primal Algorithm

Write $n = r \cdot s$

We assign $A_1 = 0$, $A_2 = r$, $A_3 = 2r, \cdots$, $A_r = (s-1)r$, $B_1 = 0$, $B_2 = 1$, $B_3 = 2, \cdots$, $B_r = r - 1$.

```
Row 1: r*mod (k,s), k=0,1,…,rs-1
Row 2: mod (k,r), k=0,1,…,rs-1
Row 3: All 1
Row 4: Row 1 + Row 2 + Row 3
```

	ω^0	ω^1	ω^2	ω^3		ω^{rs-3}	ω^{rs-2}	ω^{rs-1}
Row 1	A_1	A_2	A_3	A_4		A_{r-2}	A_{r-1}	A_r
	0	r	$2r$	$3r$	$\cdots$	$(s-3)r$	$(s-2)r$	$(s-1)r$
Row 2	B_1	B_2	B_1	B_2		B_2	B_1	B_2
	0	1	2	3		$r-3$	$r-2$	$r-1$
Row 3	1	1	1	1		1	1	1
Row 4	1	$2+r$	$3+2r$	$4+3r$		$rs-2r-2$	$rs-r-1$	rs

Table 9 Primal Algorithm for SBP of Degree rs

$$p_{rs}(x) = 1 + (2+r)x + (3+2r)x^2 + \cdots + (rs-r-1)x^{rs-2} + rsx^{rs-1}$$

$$= r \cdot \left(\sum_{k=0}^{s-1} kx^k \right) \left(\sum_{k=0}^{r-1} x^{sk} \right) + \left(\sum_{k=0}^{r-1} kx^k \right) \left(\sum_{k=0}^{s-1} x^{rk} \right) + \sum_{k=0}^{rs-1} x^k$$

Dual Algorithm

Write $n = s \cdot r$

We assign $A_1 = 0$, $A_2 = s$, $A_3 = 2s, \cdots$, $A_r = (r-1)s$, $B_1 = 0$, $B_2 = 1$, $B_3 = 2, \cdots$, $B_r = s - 1$.

```
Row 1: s*mod (k,r), k=0,1,…,rs-1
Row 2: mod (k,s), k=0,1,…,rs-1
Row 3: All 1
Row 4: Row 1 + Row 2 + Row 3
```

	ω^0	ω^1	ω^2	ω^3		ω^{rs-3}	ω^{rs-2}	ω^{rs-1}
Row 1	A_1	A_2	A_3	A_4	$\cdots$	A_{r-2}	A_{r-1}	A_r
	0	s	$2s$	$3s$		$(r-3)s$	$(r-2)s$	$(r-1)s$
Row 2	B_1	B_2	B_1	B_2		B_2	B_1	B_2
	0	1	2	3		$s-3$	$s-2$	$s-1$
Row 3	1	1	1	1		1	1	1
Row 4	1	$2+s$	$3+2s$	$4+3s$		$sr-2s-2$	$sr-s-1$	sr

Table 10 Dual Algorithm for SBP of Degree rs

$$p_{sr}(x)=1+(2+s)x+(3+2s)x^2+\cdots+(sr-s-1)x^{rs-2}+srx^{rs-1}$$

$$=s\cdot\left(\sum_{k=0}^{r-1}kx^k\right)\left(\sum_{k=0}^{s-1}x^{rk}\right)+\left(\sum_{k=0}^{s-1}kx^k\right)\left(\sum_{k=0}^{r-1}x^{sk}\right)+\sum_{k=0}^{sr-1}x^k$$

We observed $p_{rs}(x)=1+(2+r)x+(3+2r)x^2+\cdots+(rs-r-1)x^{rs-2}+rsx^{rs-1}$ and

$p_{sr}(x)=1+(2+s)x+(3+2s)x^2+\cdots+(sr-s-1)x^{rs-2}+srx^{rs-1}$ all have

$\Phi_{rs}(x)=\prod_{\substack{1\le k<rs\\ \gcd(k,rs)=1}}(x-\omega_{rs}^k)$ as a factor.

Investigate the Relation between cyclotomic polynomial and SBP.

Lemma 1. *Let n be an arbitrary positive integer. Then the cyclotomic polynomial* $\Phi_n(x)$ *is irreducible* [4].

Lemma 2. *Let* $p(x)$ 、$\Phi_n(x)\in F[x]$. *If* $\Phi_n(x)$ *is irreducible, then* $\Phi_n(x)\,|\,p(x)$ *or*

$(p(x),\Phi_n(x))=1$.

proof.

Let $(p(x), \Phi_n(x)) = d(x)$.

Then $d(x) \mid \Phi_n(x)$ ， $\Phi_n(x)$ is irreducible

$\therefore d(x) = 1$ or $\alpha\, d(x) = \Phi_n(x)$, $\alpha \in F$

If $\therefore d(x) = 1$, then $(p(x), \Phi_n(x)) = 1$

If $\alpha\, d(x) = \Phi_n(x)$, then $\alpha\, d(x) \mid p(x) \Rightarrow \Phi_n(x) \mid p(x)$.

Theorem 3. $p_n(x)$*,* $\Phi_n(x) \in F(x)$*, then* $\Phi_n(x) \mid p_n(x)$.

proof.

By the Lemma 1. and Lemma 2. and we see the fact of $p_n(x)$ and $\Phi_n(x)$ is not coprime, so we get $\Phi_n(x) \mid p_n(x)$.

Theorem 4. *When* $n = p^k$ *,* p *is prime number, and* $k \in Z$ *, the SBP does not exist.*

proof.

When $k = t$,

let SBP exist when $n = p^t$.

Thus, SBP must include the factor $\Phi_n(x)$.

$$\Rightarrow f_{p^t}(x) = \Phi_{p^t}(x) \cdot q(x) = (x^{p^{t-1}(p-1)} + x^{p^{t-1}(p-2)} + \cdots + 1)q(x)$$

Also, the degree of $f_{p^t}(x)$ is $p^t - 1$.

$\therefore$ The degree of $q(x)$ is $p^{t-1} - 1$.

The definitions of

$$f_{p^t}(x) = (x^{p^{t-1}(p-1)} + x^{p^{t-1}(p-2)} + \cdots + 1)(ax^{p^{t-1}-1} + bx^{p^{t-1}-2} + \cdots + c) = ax^{p^t-1} + bx^{p^t-2} + \cdots + ax^{p^t - p^{t-1} - 2} + bx^{p^t - p^{t-1} - 3} + \cdots + c$$

and SBP thus contradict, and for this reason, SBP does not exist when $n = p^t$.

Linking between Physics and Algebra

Recall that if n point particles are located at $z_0, z_1, \cdots, z_{n-1}$ in the complex plane [1], carrying weights $m_0, m_1, \cdots, m_{n-1}$ respectively, then their center of gravity is located at

$$\frac{z_0 m_0 + z_1 m_1 + \cdots + z_{n-1} m_{n-1}}{m_0 + m_1 + \cdots + m_{n-1}}$$

Therefore, a polynomial is balanced if and only if the n point particles located at $\omega_n^0, \omega_n^1, \cdots, \omega_n^{n-1}$, carrying weights $1, a_1, a_2, \cdots, a_{n-1}$ has the center of gravity at 0.

Investigate the Geometric interpretation of SBP

Take $n = 6 = 2 \cdot 3$ for example. (Fig. 5)

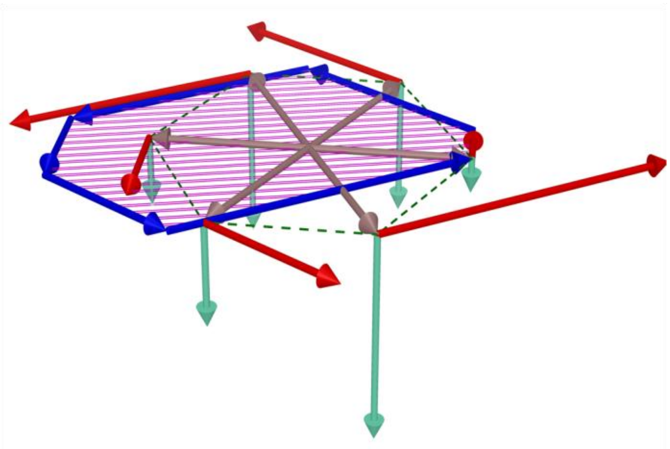


Fig. 5 SBP Becomes Geometric

The green vector represents the weight.
The pink vector represents the lever-arm.
The red vector represents the torque.
Since the torque is the vector product of the lever-arm distance and the force and since the sum of the torques (Fig. 6) is 0 (system is balanced) it follows that to each balanced polynomial there is an equiangular n-sided polygon with vertices formed by the partial sums of the torques. In case of SBP's, the equiangular polygon has edges of the length $1, 2, \cdots, n$ in some order, (Fig. 7).

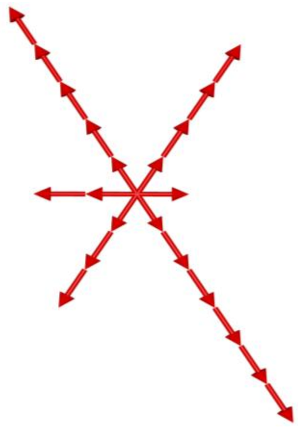


Fig. 6 Vector Diagram of Torques

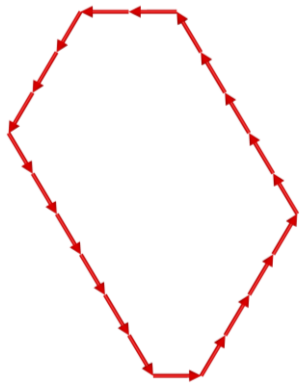


Fig. 7 Equiangular Polygon Formed by Torques

Investigate 2-BP of the form $1+\pi(2)^2 x+\pi(3)^2 x^2+\cdots+\pi(n)^2 x^{n-1}$

where π is a permutation of $\{2^2,3^2,\cdots,n^2\}$.

Construct 2-BP of degree 29 form SBP of degree 5. (Table 5)

Write $n=30=6\cdot 5$

$$\begin{cases} 1(\omega^0+\omega^6+\omega^{12}+\omega^{18}+\omega^{24})=0 \\ 4(\omega^1+\omega^7+\omega^{13}+\omega^{19}+\omega^{25})=0 \\ 5(\omega^2+\omega^8+\omega^{14}+\omega^{20}+\omega^{26})=0 \\ 2(\omega^3+\omega^9+\omega^{15}+\omega^{21}+\omega^{27})=0 \\ 3(\omega^4+\omega^{10}+\omega^{16}+\omega^{22}+\omega^{28})=0 \\ 6(\omega^5+\omega^{11}+\omega^{17}+\omega^{23}+\omega^{29})=0 \end{cases} \text{ and } \begin{cases} (6\times 0)(\omega^0+\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25})=0 \\ (6\times 1)(\omega^1+\omega^6+\omega^{11}+\omega^{16}+\omega^{21}+\omega^{26})=0 \\ (6\times 2)(\omega^2+\omega^7+\omega^{12}+\omega^{17}+\omega^{22}+\omega^{27})=0 \\ (6\times 3)(\omega^3+\omega^8+\omega^{13}+\omega^{18}+\omega^{23}+\omega^{28})=0 \\ (6\times 4)(\omega^4+\omega^9+\omega^{14}+\omega^{19}+\omega^{24}+\omega^{29})=0 \end{cases}$$

	ω^0	ω^1	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7	ω^8	ω^9	ω^{10}	ω^{11}	ω^{12}	ω^{13}	ω^{14}
A	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3
	1	4	5	2	3	6	1	4	5	2	3	6	1	4	5
B	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
	0	6	12	18	24	0	6	12	18	24	0	6	12	18	24
A+B	1	10	17	20	27	6	7	16	23	26	3	12	13	22	29

	ω^{15}	ω^{16}	ω^{17}	ω^{18}	ω^{19}	ω^{20}	ω^{21}	ω^{22}	ω^{23}	ω^{24}	ω^{25}	ω^{26}	ω^{27}	ω^{28}	ω^{29}
A	A_4	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2	A_3	A_4	A_5	A_6	A_1	A_2
	2	3	6	1	4	5	2	3	6	1	4	5	2	3	6
B	B_1	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4
	0	6	12	18	24	0	6	12	18	24	0	6	12	18	24
A+B	2	9	18	19	28	5	8	15	24	25	4	11	14	21	30

Table 11 SBP of degree 29

$$p_{30}(x) = 1+10x+17x^2+20x^3+27x^4+6x^5+7x^6+16x^7+23x^8+26x^9+3x^{10}+12x^{11}+13x^{12}+22x^{13}$$
$$+\ 29x^{14}+2x^{15}+9x^{16}+18x^{17}+19x^{18}+28x^{19}+5x^{20}+8x^{21}+15x^{22}+24x^{23}+25x^{24}+4x^{25}$$
$$+\ 11x^{26}+14x^{27}+21x^{28}+30x^{29}$$

$$= \left[2\cdot\left(\sum_{k=0}^{2} kx^k\right)\left(\sum_{k=0}^{1} x^{3k}\right)+\left(\sum_{k=0}^{1} kx^k\right)\left(\sum_{k=0}^{2} x^{2k}\right)\right]\left(\sum_{k=0}^{4} x^{6k}\right)+6\cdot\left(\sum_{k=0}^{4} kx^k\right)\left(\sum_{k=0}^{5} x^{5k}\right)+\sum_{k=0}^{29} x^k$$

Construct 2-BP:

$$\begin{cases} 1^2(\omega^0+\omega^6+\omega^{12}+\omega^{18}+\omega^{24})=0 \\ 4^2(\omega^1+\omega^7+\omega^{13}+\omega^{19}+\omega^{25})=0 \\ 5^2(\omega^2+\omega^8+\omega^{14}+\omega^{20}+\omega^{26})=0 \\ 2^2(\omega^3+\omega^9+\omega^{15}+\omega^{21}+\omega^{27})=0 \\ 3^2(\omega^4+\omega^{10}+\omega^{16}+\omega^{22}+\omega^{28})=0 \\ 6^2(\omega^5+\omega^{11}+\omega^{17}+\omega^{23}+\omega^{29})=0 \end{cases}$$

$$=(1^2+4^2\omega+5^2\omega^2+2^2\omega^3+3^2\omega^4+6^2\omega^5)(\omega^0+\omega^6+\omega^{12}+\omega^{18}+\omega^{24})$$

$$\begin{cases} (6\times 0)^2(\omega^0+\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25})=0 \\ (6\times 1)^2(\omega^1+\omega^6+\omega^{11}+\omega^{16}+\omega^{21}+\omega^{26})=0 \\ (6\times 2)^2(\omega^2+\omega^7+\omega^{12}+\omega^{17}+\omega^{22}+\omega^{27})=0 \\ (6\times 3)^2(\omega^3+\omega^8+\omega^{13}+\omega^{18}+\omega^{23}+\omega^{28})=0 \\ (6\times 4)^2(\omega^4+\omega^9+\omega^{14}+\omega^{19}+\omega^{24}+\omega^{29})=0 \end{cases}$$

$$=(0^2+6^2\omega+12^2\omega^2+18^2\omega^3+24^2\omega^4)(1+\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25})$$

$$
\begin{cases}
2(6\times 0)(1\omega^{0}+6\omega^{5}+3\omega^{10}+2\omega^{15}+5\omega^{20}+4\omega^{25})=0\\
2(6\times 1)(4\omega^{1}+1\omega^{6}+6\omega^{11}+3\omega^{16}+2\omega^{21}+5\omega^{26})=0\\
2(6\times 2)(5\omega^{2}+4\omega^{7}+1\omega^{12}+6\omega^{17}+3\omega^{22}+2\omega^{27})=0\\
2(6\times 3)(4\omega^{3}+5\omega^{8}+4\omega^{13}+1\omega^{18}+6\omega^{23}+3\omega^{28})=0\\
2(6\times 4)(3\omega^{4}+2\omega^{9}+5\omega^{14}+4\omega^{19}+1\omega^{24}+6\omega^{29})=0
\end{cases}
$$

$$
=2(1+6\omega^{5}+3\omega^{10}+2\omega^{15}+5\omega^{20}+4\omega^{25})(0+6\omega^{6}+12\omega^{12}+18\omega^{18}+24\omega^{24})\,.
$$

A^2	1^2	4^2	5^2	2^2	3^2	6^2	1^2	4^2	5^2	2^2
B^2	0^2	6^2	12^2	18^2	24^2	0^2	6^2	12^2	18^2	24^2
$2\cdot A\cdot B$	$2\cdot1\cdot0$	$2\cdot4\cdot6$	$2\cdot5\cdot12$	$2\cdot2\cdot18$	$2\cdot3\cdot24$	$2\cdot6\cdot0$	$2\cdot1\cdot6$	$2\cdot4\cdot12$	$2\cdot5\cdot18$	$2\cdot2\cdot24$
$(A+B)^2$	1^2	10^2	17^2	20^2	27^2	6^2	7^2	16^2	23^2	26^2

A^2	3^2	6^2	1^2	4^2	5^2	2^2	3^2	6^2	1^2	4^2
B^2	0^2	6^2	12^2	18^2	24^2	0^2	6^2	12^2	18^2	24^2
$2\cdot A\cdot B$	$2\cdot3\cdot0$	$2\cdot6\cdot6$	$2\cdot1\cdot12$	$2\cdot4\cdot18$	$2\cdot5\cdot24$	$2\cdot2\cdot0$	$2\cdot3\cdot6$	$2\cdot6\cdot12$	$2\cdot1\cdot18$	$2\cdot4\cdot24$
$(A+B)^2$	3^2	12^2	13^2	22^2	29^2	2^2	9^2	18^2	19^2	28^2

A^2	5^2	2^2	3^2	6^2	1^2	4^2	5^2	2^2	3^2	6^2
B^2	0^2	6^2	12^2	18^2	24^2	0^2	6^2	12^2	18^2	24^2
$2\cdot A\cdot B$	$2\cdot5\cdot0$	$2\cdot2\cdot6$	$2\cdot3\cdot12$	$2\cdot6\cdot18$	$2\cdot1\cdot24$	$2\cdot4\cdot0$	$2\cdot5\cdot6$	$2\cdot2\cdot12$	$2\cdot3\cdot18$	$2\cdot6\cdot24$
$(A+B)^2$	5^2	8^2	15^2	24^2	25^2	4^2	11^2	14^2	21^2	30^2

Table 12 2-BP of degree 29

$$
\begin{aligned}
p_{30}^{2}(x)=\;&1^2+10^2x+17x^2+20^2x^3+27^2x^4+6^2x^5+7^2x^6+16^2x^7+23^2x^8+26^2x^9+3^2x^{10}+12^2x^{11}+\\
&13^2x^{12}+22^2x^{13}+29^2x^{14}+2^2x^{15}+9^2x^{16}+18^2x^{17}+19^2x^{18}+28^2x^{19}+5^2x^{20}+8^2x^{21}+15^2x^{22}+\\
&24^2x^{23}+25^2x^{24}+4^2x^{25}+11^2x^{26}+14^2x^{27}+21^2x^{28}+30^2x^{29}\\
=\;&(900x^{21}-500x^{20}+756x^{19}+225x^{18}+200x^{17}+1581x^{16}-524x^{15}+1001x^{14}+705x^{13}+305x^{12}+\\
&1613x^{11}-495x^{10}+1049x^{9}+581x^{8}+381x^{7}+809x^{6}-155x^{5}+417x^{4}+380x^{3}+105x^{2}+\\
&+120x+1)(x^8+x^7-x^5-x^4-x^3+x+1)
\end{aligned}
$$

Write $n = 30 = 10 \cdot 3$

The algorithm can be compressed as follows:

$$(1^2 + 8^2\omega + 5^2\omega^2 + 7^2\omega^3 + 4^2\omega^4 + 6^2\omega^5 + 3^2\omega^6 + 10^2\omega^7 + 2^2\omega^8 + 9^2\omega^9)(\omega^0 + \omega^{10} + \omega^{20}) = 0$$

$$(1 + 10^2\omega + 20^2\omega^2)(1 + \omega^3 + \omega^6 + \omega^9 + \omega^{12} + \omega^{15} + \omega^{18} + \omega^{21} + \omega^{24} + \omega^{27}) = 0$$

$$(1^2 + 7^2\omega^3 + 3^2\omega^6 + 9^2\omega^9 + 5^2\omega^{12} + 6^2\omega^{15} + 2^2\omega^{18} + 8^2\omega^{21} + 4^2\omega^{24} + 10^2\omega^{27})(0 + 10\omega^{10} + 20\omega^{20}) = 0$$

Then we are able to construct the 2-BP of the form:

$$p_{30}^2(x) = 1^2 + 17^2 x + 23x^2 + 9^2 x^3 + 15^2 x^4 + 26^2 x^5 + 2^2 x^6 + 18^2 x^7 + 24^2 x^8 + 10^2 x^9 + 11^2 x^{10} + 27^2 x^{11} + 3^2 x^{12} + 19^2 x^{13} + 25^2 x^{14} + 6^2 x^{15} + 12^2 x^{16} + 28^2 x^{17} + 4^2 x^{18} + 20^2 x^{19} + 21^2 x^{20} + 7^2 x^{21} + 13^2 x^{22} + 29^2 x^{23} + 5^2 x^{24} + 16^2 x^{25} + 22^2 x^{26} + 8^2 x^{27} + 14^2 x^{28} + 30^2 x^{29}.$$

Write $n = 15 \cdot 2$

$$(1^2 + 12^2\omega + 5^2\omega^2 + 13^2\omega^3 + 9^2\omega^4 + 2^2\omega^5 + 10^2\omega^6 + 6^2\omega^7 + 14^2\omega^8 + 7^2\omega^9 + 3^2\omega^{10} + 11^2\omega^{11} + 4^2\omega^{12} + 15^2\omega^{13} + 8^2\omega^{14})(1 + \omega^{15}) = 0$$

$$(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} + \omega^{11} + \omega^{12} + \omega^{13} + \omega^{14})(0 + 15^2\omega^{15}) = 0$$

$$(1 + 5\omega^2 + 9\omega^4 + 10\omega^6 + 14\omega^8 + 3\omega^{10} + 4\omega^{12} + 8\omega^{14} + 12\omega^{16} + 13\omega^{18} + 2\omega^{20} + 6\omega^{22} + 7\omega^{24} + 11\omega^{26} + 15\omega^{28})(0 + 15\omega^{15}) = 0$$

Then we are able to construct the 2-BP of the form:

$$p_{30}^2(x) = 1^2 + 20^2 x + 9x^2 + 25^2 x^3 + 14^2 x^4 + 18^2 x^5 + 4^2 x^6 + 23^2 x^7 + 12^2 x^8 + 28^2 x^9 + 2^2 x^{10} + 21^2 x^{11} + 7^2 x^{12} + 26^2 x^{13} + 15^2 x^{14} + 16^2 x^{15} + 5^2 x^{16} + 24^2 x^{17} + 10^2 x^{18} + 29^2 x^{19} + 3^2 x^{20} + 19^2 x^{21} + 8^2 x^{22} + 27^2 x^{23} + 13^2 x^{24} + 17^2 x^{25} + 6^2 x^{26} + 22^2 x^{27} + 11^2 x^{28} + 30^2 x^{29}.$$

Investigate 3-BP of the form $1+\pi(2)^3 x+\pi(3)^3 x^2+\cdots+\pi(n)^3 x^{n-1}$

where π is a permutation of $\{2^3, 3^3, \cdots, n^3\}$.

A	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
	1	10	17	20	27	6	7	16	23	26
B	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_1	B_2	B_3
	0	30	60	90	120	150	180	0	30	60
$A+B$	1	40	77	110	147	156	187	16	53	86

A	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}	A_{20}
	3	12	13	22	29	2	9	18	19	28
B	B_4	B_5	B_6	B_7	B_1	B_2	B_3	B_4	B_5	B_6
	90	120	150	180	0	30	90	120	150	90
$A+B$	93	132	163	202	29	32	69	108	139	178

⋮

A	A_{21}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}	A_{27}	A_{28}	A_{29}	A_{30}
	5	8	15	24	25	4	11	14	21	30
B	B_4	B_5	B_6	B_1	B_2	B_3	B_4	B_5	B_6	B_7
	120	150	180	0	30	60	90	120	150	180
$A+B$	125	158	195	24	55	64	101	134	171	210

Table 13 Algorithm for SBP of degree 209

$\therefore p_{210}(x)=1+40x+77x^2+\cdots+16x^7+\cdots+210x^{209}$

A^2	1^2	10^2	17^2	20^2	27^2	6^2	7^2	16^2
B^2	0^2	30^2	60^2	90^2	120^2	150^2	180^2	0^2
$2\cdot A\cdot B$	$2\cdot 1\cdot 0$	$2\cdot 10\cdot 30$	$2\cdot 17\cdot 60$	$2\cdot 20\cdot 90$	$2\cdot 27\cdot 120$	$2\cdot 6\cdot 150$	$2\cdot 7\cdot 180$	$\cdot 16\cdot 0$
$(A+B)^2$	1^2	40^2	77^2	110^2	147^2	156^2	187^2	16^2

A^2	23^2	26^2	3^2	12^2	13^2	22^2	29^2	2^2
B^2	30^2	60^2	90^2	120^2	150^2	180^2	0^2	30^2
$2\cdot A\cdot B$	$2\cdot 23\cdot 30$	$2\cdot 26\cdot 60$	$2\cdot 3\cdot 90$	$2\cdot 12\cdot 120$	$2\cdot 13\cdot 150$	$2\cdot 22\cdot 180$	$2\cdot 29\cdot 0$	$2\cdot 2\cdot 30$
$(A+B)^2$	53^2	86^2	93^2	132^2	163^2	202^2	29^2	32^2

$\vdots$

A^2	15^2	24^2	25^2	4^2	11^2	14^2	21^2	30^2
B^2	180^2	0^2	30^2	60^2	90^2	120^2	150^2	180^2
$2\cdot A\cdot B$	$2\cdot 15\cdot 180$	$2\cdot 24\cdot 0$	$2\cdot 25\cdot 30$	$2\cdot 4\cdot 60$	$2\cdot 11\cdot 90$	$2\cdot 14\cdot 120$	$2\cdot 21\cdot 150$	$2\cdot 30\cdot 180$
$(A+B)^2$	195^2	24^2	55^2	64^2	101^2	134^2	171^2	210^2

Table 14 Algorithm for 2-BP of degree 209

$$\therefore p_{210}^{2}(x) = 1^2 + 40^2 x + 77^2 x^2 + \cdots + 16^2 x^7 + \cdots + 210^2 x^{209}$$

$$\begin{cases} 1^3\left(\omega^0 + \omega^{30} + \cdots + \omega^{180}\right) = 0 \\ 10^3\left(\omega^1 + \omega^{31} + \cdots + \omega^{181}\right) = 0 \\ 17^3\left(\omega^2 + \omega^{32} + \cdots + \omega^{182}\right) = 0 \\ \quad\vdots \\ \quad\vdots \\ 30^3\left(\omega^{29} + \omega^{59} + \cdots + \omega^{209}\right) = 0 \end{cases}$$

$$= (1^3 + 10^3\omega + 17^3\omega^2 + \cdots + 30^3\omega^{29})(1 + \omega^{30} + \omega^{60} + \omega^{90} + \omega^{120} + \omega^{150} + \omega^{180}) = 0$$

$$\begin{cases}(30\times 0)^3\left(\omega^0+\omega^7+\cdots+\omega^{203}\right)=0\\(30\times 1)^3\left(\omega^1+\omega^8+\cdots+\omega^{204}\right)=0\\(30\times 2)^3\left(\omega^2+\omega^9+\cdots+\omega^{205}\right)=0\\\quad\vdots\\\quad\vdots\\(30\times 6)^3\left(\omega^6+\omega^{13}+\cdots+\omega^{209}\right)=0\end{cases}$$

$$=(0^3+30^3\omega+60^3\omega^2+\cdots+180^3\omega^6)(1+\omega^7+\omega^{14}+\omega^{21}+\cdots+\omega^{203})=0$$

$$\begin{cases}3(30\times 0)\left(1^2\omega^0+16^2\omega^7+\cdots+24^2\omega^{203}\right)=0\\3(30\times 1)\left(10^2\omega^1+23^2\omega^8+\cdots+4^2\omega^{204}\right)=0\\\quad\vdots\\\quad\vdots\\3(30\times 6)\left(7^2\omega^6+22^2\omega^{13}+\cdots+30^2\omega^{209}\right)=0\end{cases}$$

$$=3(1^2+16^2\omega+\cdots+24^2\omega^{203})(0+30\omega^{30}+60\omega^{60}+90\omega^{90}+120\omega^{120}+150\omega^{150}+180\omega^{180})=0$$

$$\begin{cases}3(30\times 0)^2\left(1\omega^0+16\omega^7+\cdots+24\omega^{203}\right)=0\\3(30\times 1)^2\left(10\omega^1+23\omega^8+\cdots+4\omega^{204}\right)=0\\\quad\vdots\\\quad\vdots\\3(30\times 6)^2\left(7\omega^6+22\omega^{13}+\cdots+30\omega^{209}\right)=0\end{cases}$$

$$=3(1+16\omega+\cdots+24\omega^{203})(0^2+30^2\omega^{30}+60^2\omega^{60}+90^2\omega^{90}+120^2\omega^{120}+150^2\omega^{150}+180^2\omega^{180})=0$$

A^3	1^3	10^3	17^3	20^3	27^3	6^3	7^3	16^3
B^3	0^3	30^3	60^3	90^3	120^3	150^3	180^3	0^3
$3\cdot A^2\cdot B$	$3\cdot 1^2\cdot 0$	$3\cdot 10^2\cdot 30$	$3\cdot 17^2\cdot 60$	$3\cdot 20^2\cdot 90$	$3\cdot 27^2\cdot 120$	$3\cdot 6^2\cdot 150$	$3\cdot 7^2\cdot 180$	$3\cdot 16^2\cdot 0$
$3\cdot A\cdot B^2$	$3\cdot 1\cdot 0^2$	$3\cdot 10\cdot 30^2$	$3\cdot 17\cdot 60^2$	$3\cdot 20\cdot 90^2$	$3\cdot 27\cdot 120^2$	$3\cdot 6\cdot 150^2$	$3\cdot 7\cdot 180^2$	$3\cdot 16\cdot 0^2$
$(A+B)^3$	1^3	40^3	77^3	110^3	147^3	156^3	187^3	16^3

A^3	23^3	26^3	3^3	12^3	13^3	22^3	29^3	2^3
B^3	30^3	60^3	90^3	120^3	150^3	180^3	0^3	30^3
$3\cdot A^2\cdot B$	$3\cdot 23^2\cdot 30$	$3\cdot 26^2\cdot 60$	$3\cdot 3^2\cdot 90$	$3\cdot 12^2\cdot 120$	$3\cdot 13^2\cdot 150$	$3\cdot 22^2\cdot 180$	$3\cdot 29^2\cdot 0$	$3\cdot 2^2\cdot 30$
$3\cdot A\cdot B^2$	$3\cdot 23\cdot 30^2$	$3\cdot 26\cdot 60^2$	$3\cdot 3\cdot 90^2$	$3\cdot 12\cdot 120^2$	$3\cdot 13\cdot 150^2$	$3\cdot 22\cdot 180^2$	$3\cdot 29\cdot 0^2$	$3\cdot 2\cdot 30^2$
$(A+B)^3$	53^3	86^3	93^3	132^3	163^3	202^3	29^3	32^3

⋮

A^3	15^3	24^3	25^3	4^3	11^3	14^3	21^3	30^3
B^3	180^3	0^3	30^3	60^3	90^3	120^3	150^3	180^3
$3\cdot A^2\cdot B$	$3\cdot 15^2\cdot 180$	$3\cdot 24^2\cdot 0$	$3\cdot 25^2\cdot 30$	$3\cdot 4^2\cdot 60$	$3\cdot 11^2\cdot 90$	$3\cdot 14^2\cdot 120$	$3\cdot 21^2\cdot 150$	$3\cdot 30^2\cdot 180$
$3\cdot A\cdot B^2$	$3\cdot 15\cdot 180^2$	$3\cdot 24\cdot 0^2$	$3\cdot 25\cdot 30^2$	$3\cdot 4\cdot 60^2$	$3\cdot 11\cdot 90^2$	$3\cdot 14\cdot 120^2$	$3\cdot 21\cdot 150^2$	$3\cdot 30\cdot 180^2$
$(A+B)^3$	195^3	24^3	55^3	64^3	101^3	134^3	171^3	210^3

Table 15 Algorithm for 3-BP of degree 209

$$\therefore p_{210}^3(x) = 1^3 + 40^3 x + 77^3 x^2 + \cdots + 16^3 x^7 + \cdots + 210^3 x^{209}$$

	x^0	x^1	x^2		x^7		x^{209}
$p_{210}(x)$	1	40	77	$\cdots$	16	$\cdots$	210
$p_{210}^2(x)$	1^2	40^2	77^2		16^2		210^2
$p_{210}^3(x)$	1^3	40^3	77^3		16^3		210^3

Table 16 Compare the Coefficients of SBP, 2-BP and 3-BP

When $n = 210$. The polynomial of the form $1 + \pi(2)^k x + \pi(3)^k x^2 + \cdots + \pi(n)^k x^{n-1}$ are balanced for $1 \le k \le 3$.

Theorem 6. *If* n *has at least* $t+1$ *distinct prime factors, then there exists an SBP of the form* $1+\pi(2)x+\pi(3)x+\cdots+\pi(n)x^{n-1}$ *such that polynomials of the form*

$$1+\pi(2)^k x+\pi(3)^k x^2+\cdots+\pi(n)^k x^{n-1}$$

are balanced for $1\le k\le t$.

proof.

Let $n = p_1 p_2 \cdots p_{k+1} = \prod_{i=1}^{k+1} p_i$ ， $(p_i, p_j) = 1$ 、 $i \neq j$

If $n = p_1 p_2$, we are able to get the polynomial of the form

$$p_n^1(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

If $n = p_1 p_2 p_3$, we are able to get the polynomial of the form

$$p_n^2(x) = a_0^2 + a_1^2 x + a_2^2 x^2 + \cdots + a_{n-1}^2 x^{n-1}$$

If $n = p_1 p_2 p_3 \cdots p_t$, we are able to get the polynomial of the form

$$p_n^{t-1}(x) = a_0^{t-1} + a_1^{t-1} x + a_2^{t-1} x^2 + \cdots + a_{n-1}^{t-1} x^{n-1}$$

Let $\prod_{i=2}^{t+1} p_i = A$

$$\begin{cases} \omega^0 + \omega^A + \omega^{2A} + \cdots + \omega^{(p_1-1)A} = 0 & (1) \\ \omega^0 + \omega^{p_1} + \omega^{2p_2} + \cdots + \omega^{(A-1)p_1} = 0 & (2) \\ a_0\omega^0 + a_1\omega^{p_1} + a_2\omega^{2p_1} + \cdots + a_{A-1}\omega^{(A-1)p_1} = 0 & (3) \\ a_0^2\omega^0 + a_1^2\omega^{p_1} + a_2^2\omega^{2p_1} + \cdots + a_{A-1}^2\omega^{(A-1)p_1} = 0 & (4) \\ \vdots & \\ a_0^{t-1}\omega^0 + a_1^{t-1}\omega^{p_1} + a_2^{t-1}\omega^{2p_1} + \cdots + a_{A-1}^{t-1}\omega^{(A-1)p_1} = 0 & (t+1) \end{cases}$$

$$\begin{cases} (1)\times a_{i-1}^t \omega^{(i-1)},\ i=1,2,\cdots,A & \text{(t+2)} \\ (2)\times C_t^t\left[A(m-1)\right]^t \omega^{(m-1)},\ m=1,2,\cdots,p_1 & \text{(t+3)} \\ (3)\times C_{t-1}^t\left[A(m-1)\right]^{t-1} \omega^{(m-1)p_1},\ m=1,2,\cdots,p_1 & \text{(t+4)} \\ \vdots & \vdots \\ (t+1)\times C_1^t\left[A(m-1)\right]\omega^{(m-1)p_1},\ m=1,2,\cdots,p_1 & \text{2(t+1)} \end{cases}$$

The sum of eq. $(t+2)$, eq. $(t+3)$, and eq. $(2t+2)$ is $\sum_{i=0}^{n-1} b_i\omega^i$, where

$$b_i = \left\{a_{l-1}^t + C_1^t A(m-1)\times a_{l-1}^{t-1} + C_2^t\left[A(m-1)\right]^2 \times a_{l-1}^{t-2} + \cdots + C_t^t\left[A(m-1)\right]^t\right\} = \left[a_{l-1} + A(m-1)\right]^t$$

$\because (p_i, A)=1$ and $l\in\{1,2,\cdots,A\}$ $m\in\{1,2,\cdots,p_1\}$

$i\equiv(l-1)p_1+(m-1)A \pmod n,\ i\in\{1,2,\cdots,n\}$

When $l=1,\ m=1,2,3,\cdots,p_1 \Rightarrow i=0,A,2A,\cdots,(p_1-1)A$

When $l=2,\ m=1,2,3,\cdots,p_1 \Rightarrow i=p_1,p_1+A,p_1+2A,\cdots,p_1+(p_1-1)A$

$\vdots$

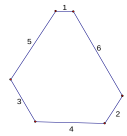
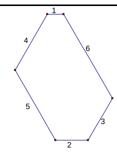
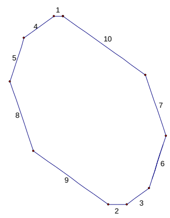
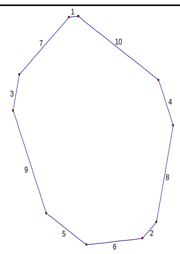
When $l=A,\ m=1,2,3,\cdots,p_1$

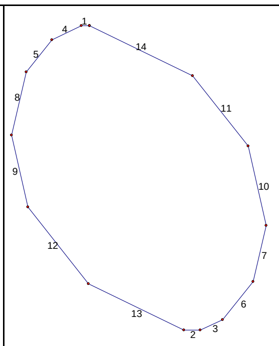
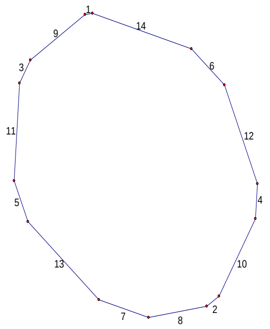
$\Rightarrow i=(A-1)p_1,(A-1)p_1+A,(A-1)p_1+2A,\cdots,(A-1)p_1+(p_1-1)A$

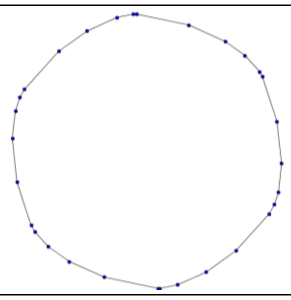
Then the t-BP of the form $p_n^t(x)=b_0+b_1x+b_2x^2+\cdots+b_{n-1}x^{n-1}$ is constructed,

where $\{b_0,b_1,\cdots,b_{n-1}\}$ is a permutation of $\{1^t,2^t,3^t,\cdots,n^t\}$.

Results

n	The relation between $p_n(x)$ and $\Phi_n(x)$	Equiangular polygon
6	$p_6(x) = 1 + 5x + 3x^2 + 4x^3 + 2x^4 + 6x^5$	
	$p_6(x) = 1 + 4x + 5x^2 + 2x^3 + 3x^4 + 6x^5$	
	$\Phi_6(x) = x^2 - x + 1$	
10	$p_{10}(x) = 1 + 4x + 5x^2 + 8x^3 + 9x^4 + 2x^5 + 3x^6 + 6x^7 + 7x^8 + 10x^9$	
	$p_{10}(x) = 1 + 7x + 3x^2 + 9x^3 + 5x^4 + 6x^5 + 2x^6 + 8x^7 + 4x^8 + 10x^9$	
	$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$	
12	$p_{12}(x) = 1 + 5x + 9x^2 + 10x^3 + 2x^4 + 6x^5 + 7x^6 + 11x^7 + 3x^8 + 4x^9 + 8x^{10} + 12x^{11}$	
	$p_{12}(x) = 1 + 6x + 11x^2 + 4x^3 + 5x^4 + 10x^5 + 3x^6 + 8x^7 + 9x^8 + 2x^9 + 7x^{10} + 12x^{11}$	

<table>
<tr><td></td><td colspan="2">$\Phi_{12}(x) = x^4 - x^2 + 1$</td></tr>
<tr><td rowspan="3">14</td><td>$p_{14}(x) = 1 + 4x + 5x^2 + 8x^3 + 9x^4 + 12x^5 + 13x^6 + 2x^7 + 3x^8 + 6x^9 + 7x^{10} + 10x^{11} + 11x^{12} + 14x^{13}$</td><td>
</td></tr>
<tr><td>$p_{14}(x) = 1 + 9x + 3x^2 + 11x^3 + 5x^4 + 13x^5 + 7x^6 + 8x^7 + 2x^8 + 10x^9 + 4x^{10} + 12x^{11} + 6x^{12} + 14x^{13}$</td><td>
</td></tr>
<tr><td colspan="2">$\Phi_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - 1$</td></tr>
<tr><td rowspan="3">15</td><td>$p_{15}(x) = 1 + 7x + 13x^2 + 14x^3 + 10x^4 + 11x^5 + 2x^6 + 8x^7 + 14x^8 + 5x^9 + 6x^{10} + 12x^{11} + 3x^{12} + 9x^{13} + 15x^{14}$</td><td></td></tr>
<tr><td>$p_{15}(x) = 1 + 5x + 9x^2 + 10x^3 + 14x^4 + 3x^5 + 4x^6 + 8x^7 + 12x^8 + 13x^9 + 2x^{10} + 6x^{11} + 7x^{12} + 11x^{13} + 15x^{14}$</td><td></td></tr>
<tr><td colspan="2">$\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x^2 + x - 1$</td></tr>
</table>

30	$p_{30}(x)=1+10x+17x^2+20x^3+27x^4+6x^5+7x^6+16x^7+23x^8+26x^9+3x^{10}+12x^{11}$ $+13x^{12}+22x^{13}+29x^{14}+2x^{15}+9x^{16}+18x^{17}+19x^{18}+28x^{19}+5x^{20}+8x^{21}$ $+15x^{22}+24x^{23}+25x^{24}+4x^{25}+11x^{26}+14x^{27}+21x^{28}+30x^{29}$	
	$\Phi_{30}(x)=x^8+x^7-x^6-x^5-x^4-x^3+x+1$	

Theorem 1. *If* $n=2\cdot p$ *where* p *is a prime, then there exist a SBP.*

Theorem 2*. If* $n=r\cdot s$ *where* $\gcd(r,s)=1$*, then there exist a SBP.*

Theorem 3. *If* $p_n(x)$ *is SBP and* $\Phi_n(x)$ *is cyclotomic polynomial, then* $\Phi_n(x)|p_n(x)$.

Theorem 4. *When* $n=p^k$, *p is prime number, and* $k\in Z$ *, the SBP does not exist.*

Theorem 5. *If* n *has at least* $t+1$ *distinct prime factors, then there exists an SBP of the form* $1+\pi(2)x+\pi(3)x+\cdots+\pi(n)x^{n-1}$ *such that polynomials of the form* $1+\pi(2)^k x+\pi(3)^k x^2+\cdots+\pi(n)^k x^{n-1}$ *are balanced for* $1\le k\le t$.

We also find the recursive relationship of SBP.

The following is the example of constructing SBP of degree rstu from SBP of degree rs.

$$p_{rs}(x)=\left(\sum_{k=0}^{r-1}kx^k\right)\left(\sum_{k=0}^{s-1}x^{rk}\right)+r\left(\sum_{k=0}^{s-1}kx^k\right)\left(\sum_{k=0}^{r-1}x^{sk}\right)$$

$$p_{rst}(x)=p_{rs}(x)\left(\sum_{k=0}^{t-1}x^{rsk}\right)+rs\left(\sum_{k=0}^{t-1}kx^k\right)\left(\sum_{k=0}^{rs-1}x^{tk}\right)$$

$$p_{rstu}(x)=p_{rst}(x)\left(\sum_{k=0}^{u-1}x^{rstk}\right)+rst\left(\sum_{k=0}^{u-1}kx^k\right)\left(\sum_{k=0}^{rst-1}x^{uk}\right)$$

Applications

Generation of Puzzles

Regular triangles with edges lengths 9 and 12 can be superimposed to enclose a hexagon with edge length 1, 4, 5, 2, 3, 6, the magic permutation appearing in SBP of degree 5 (Fig. 8). Applying the Dual Algorithm we obtain the dual puzzle (Fig. 9) in the same manner. A family of new puzzles asking for regular triangles enclosing equiangular polygon (Fig. 10) with edge lengths forming an arithmetic progression, can be created.

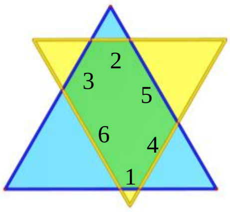


Fig. 8 Puzzle generated from 1,4,5,2,3,6

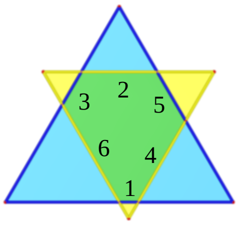


Fig. 9 Dual Puzzle generated from 1,5,3,4,2,6

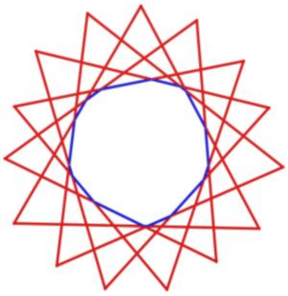


Fig. 10 Puzzle generated from SBP of deg. 14

Static Tire Balancing

No tire is ever completely balanced, no matter how expertly engineered and installed. Balanced tires provide smoother ride and durable tires, (Fig. 11). Assume that 6 weights forming arithmetic progression be fixed on the vertices of a regular hexagon concentric with the wheel to achieve the static balance. Each of the four SBP's of degree 5 solves the problem. The 2-BP given above shows that an unevenly worn weights could still maintain balancing, (Fig. 12).

Fig. 11 Tire Balancing

Photo Taken by the Authors

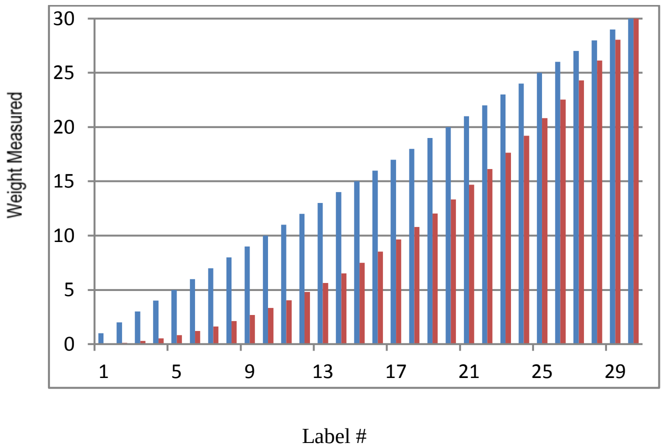


Fig. 12 Amount of Weight Remaining

Old Fashioned Merry-Go-Round Seating Problem

Assume that 6 people whose weights forming an arithmetic progression, be seated on the vertices of a regular hexagon concentric with the axis of a merry-go-round. Each of the four SBP's of degree 5 makes the ride comfortable (Fig. 13).

Fig. 13　Old Fashioned Merry-Go-Round

Infrared rays with equiangular polygon form

The balance polynomial can make up an equiangular polygon. Furthermore, we can design an alarm system make up of infrared rays with equiangular polygon form. If one beam cannot be reflected to the origin, the alarm will be activated. This system can be used in spaces with diverse form (Fig. 14).

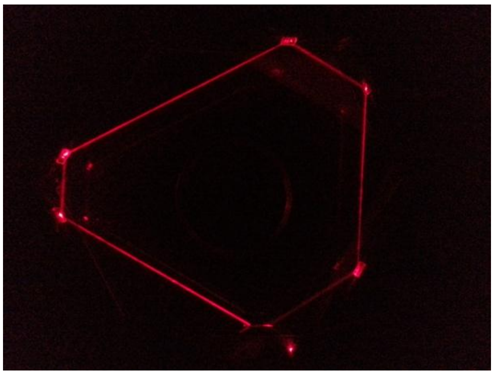


Fig. 14　Infrared Rays with Equiangular Polygon Form

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Appendix

The following is the software we used in our project:

1. We use Maple mainly for polynomial factorization.

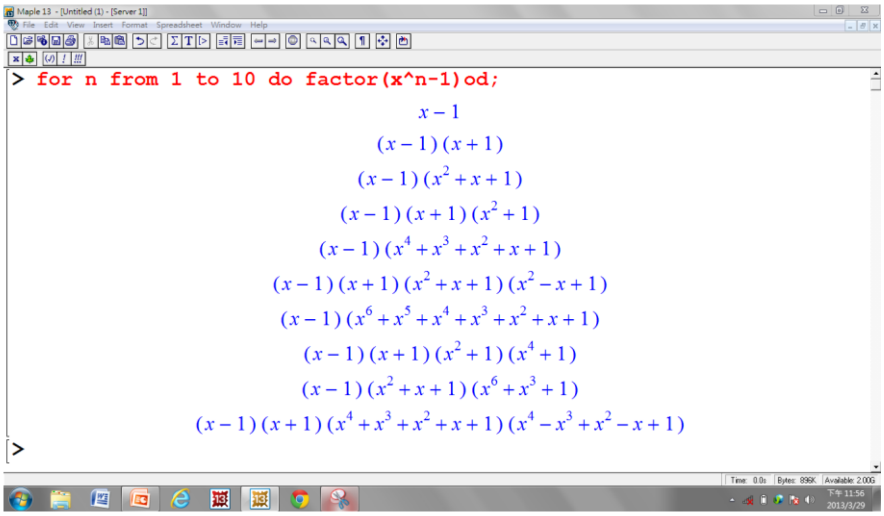


Fig. 15 Polynomial Factorization

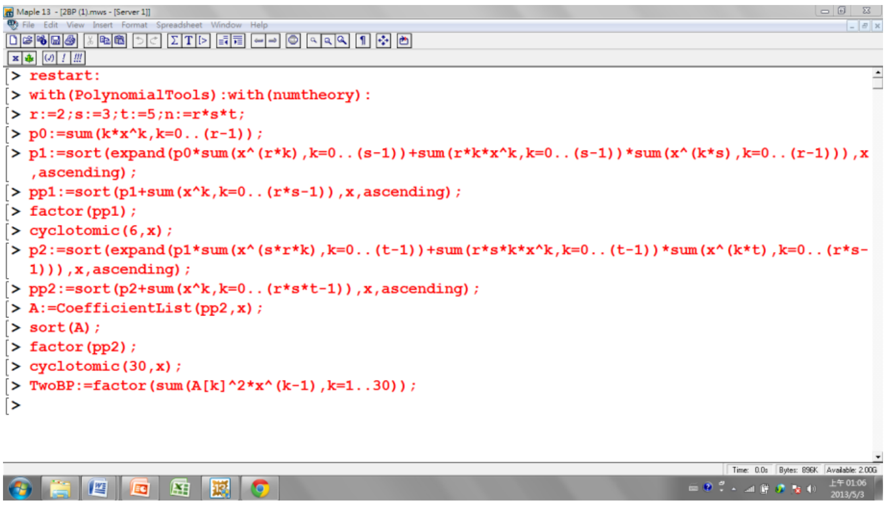
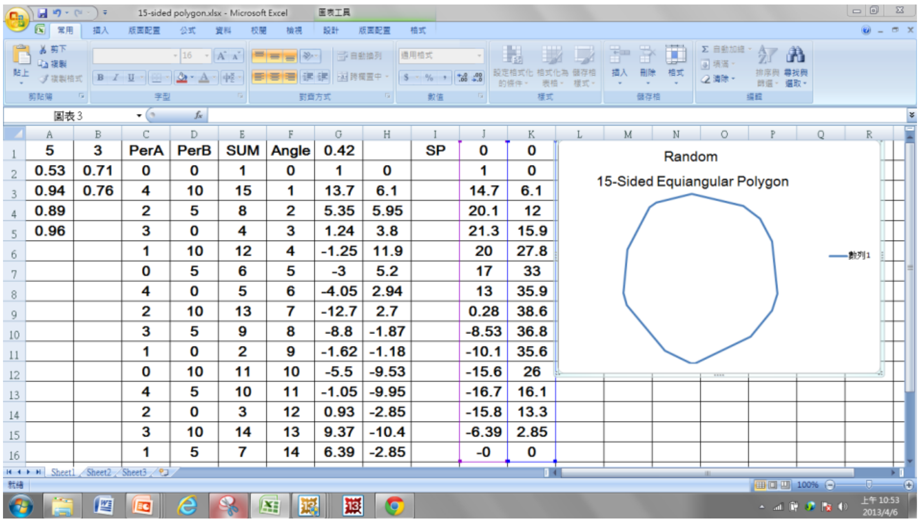


Fig. 16 Recursive Relationship Between SBP and t-BP

2. With the Excel, we can randomly generate equiangular polygon.



A	B	C	D	E	F	G	H	I	J	K
5	3	PerA	PerB	SUM	Angle	0.42		SP	0	0
0.53	0.71	0	0	1	0	1	0		1	0
0.94	0.76	4	10	15	1	13.7	6.1		14.7	6.1
0.89		2	5	8	2	5.35	5.95		20.1	12
0.96		3	0	4	3	1.24	3.8		21.3	15.9
		1	10	12	4	-1.25	11.9		20	27.8
		0	5	6	5	-3	5.2		17	33
		4	0	5	6	-4.05	2.94		13	35.9
		2	10	13	7	-12.7	2.7		0.28	38.6
		3	5	9	8	-8.8	-1.87		-8.53	36.8
		1	0	2	9	-1.62	-1.18		-10.1	35.6
		0	10	11	10	-5.5	-9.53		-15.6	26
		4	5	10	11	-1.05	-9.95		-16.7	16.1
		2	0	3	12	0.93	-2.85		-15.8	13.3
		3	10	14	13	9.37	-10.4		-6.39	2.85
		1	5	7	14	6.39	-2.85		-0	0

Fig. 17 Random 15-sided Equiangular Polygon

3. With Cabri 3D, we can draw the figure of geometric interpretation of SBP and superposition of SBP.

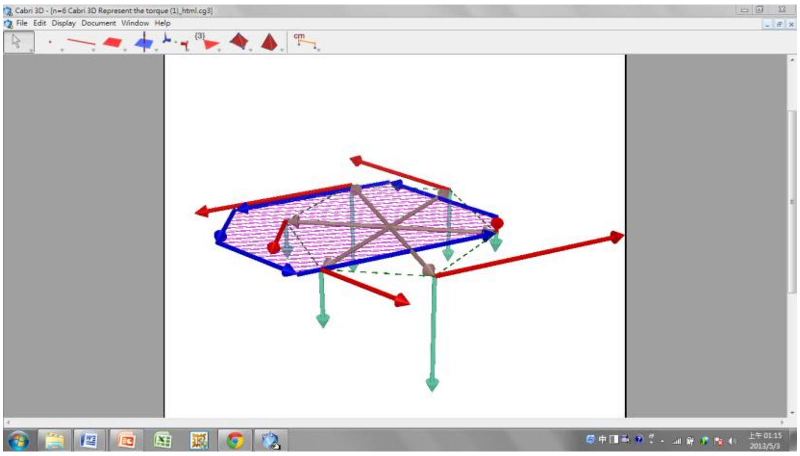


Fig. 18 Geometric Interpretation of SBP

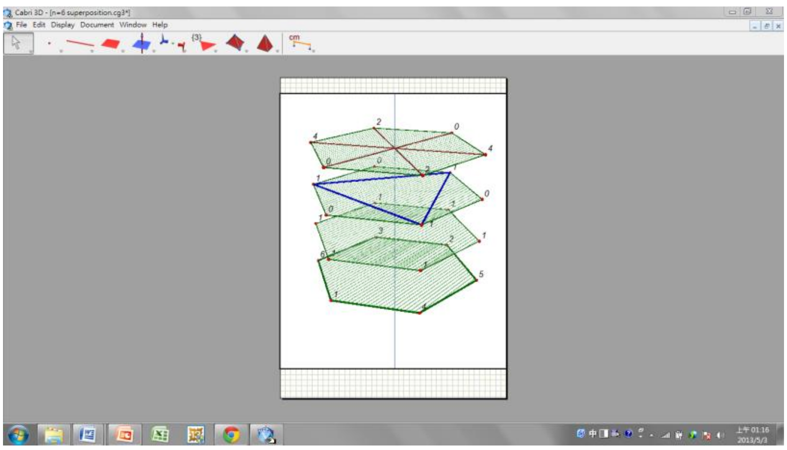


Fig. 19 Superposition of SBP

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