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峰迴路轉-等比繞行的秘密

得獎獎項

大會獎：一等獎

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關鍵字：等比級數、複數、圓錐曲線

作者簡介

我是**孫宏奇**，目前就讀於**國立新竹高商**，熱愛數學的高三生。

國中起，對數學就泛起了極大的興趣，曾經多次參與數學相關競賽以及相關研習營；在 2010 年，政大舉辦的「數學資優研習營」中，探索數學更深層的領域後，更確定了對數學的熱誠；並且在各項的活動中學習每一次的經驗。

對我來說，數學研究並不是研究，而是為了更接近數學、更熱愛數學，了解數學之美，讓我確實的體悟數學的奧妙。

摘要

對於轉向次數$k\to\infty$且轉向角θ為任意角時，各收斂點P於坐標平面上恰形成圓C：$\left|\frac{1}{1-z}-\frac{1}{1-r^2}\right|=\frac{r}{1-r^2}$。已知$U$為$x$軸上任一點且坐標為$(u,0)$，當$\theta$改變時，$\overleftrightarrow{P_1P_2}$與$\overleftrightarrow{UP}$之交點$S$的軌跡為圓錐曲線（點、直線、拋物線、橢圓、雙曲線）。當$U=C$時，交點S的軌跡為橢圓，此橢圓的長軸長為圓C半徑（$\frac{r}{1-r^2}$），且焦點為$P_1(1,0)$與$C\left(\frac{1}{1-r^2},0\right)$。各轉向點$P_n(n\in\mathbb{N})$位於一個方程式為$R=m\cdot r^{\frac{\theta-\pi}{\alpha}}$，$m=\overline{OP}=\sqrt{\frac{1}{1-2r\cos\alpha+r^2}}$，定角為$\cot^{-1}(\frac{\ln r}{\alpha})$之等角螺線上；同時繪出轉向次數$k$在不同值時，瓢蟲行進終點之軌跡，以驗證當$k$愈來愈大時，各終點形成的軌跡會趨近於一個圓。當$k=2$時，圖形為蚶線並證明其經平移後之極坐標方程式為$R=r+2r^2\cos\theta$。最後我們展示行進公比$r\to1^-$，$r=1$，$r\to1^+$時所呈現的終點軌跡，並對此軌跡所呈現出的意象與自然界連結，而其實質關聯性則有待未來研究。

Abstract

The research focuses on that the machinery ladybug crawls on the way if there are some properties under some different independent variables. For the turning times k （$k \to \infty$） and any turning angle θ ,We get all convergent points form a circle *C*.

We choose a point $U(u,0)$ in the x axis，the locus by the intersection S of the lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{UP}$ is a conic section（A point,line,parabola,ellipse or hyperbola）.When $U = C$, the locus by the intersection S is an ellipse of which the major axis is the radius of the circle C and of which the foci are points $P_1(1,0)$ and the center of the circle C

All turning points $P_{\mathrm{n}}(n \in \mathbb{N})$ locate on a logarithmic spiral , the equation of the logarithmic spiral is $R = m \cdot r^{\frac{\theta-\pi}{\alpha}}$ （$m = \overline{OP} = \sqrt{\dfrac{1}{1\text{-}2r\cos\alpha + r^2}}$）, and the pitch angle is $\cot^{-1}(\dfrac{\ln r}{\alpha})$.

We get different locuses in different k to match the fact that all convergent points form a circle *C*,when $k \to \infty$

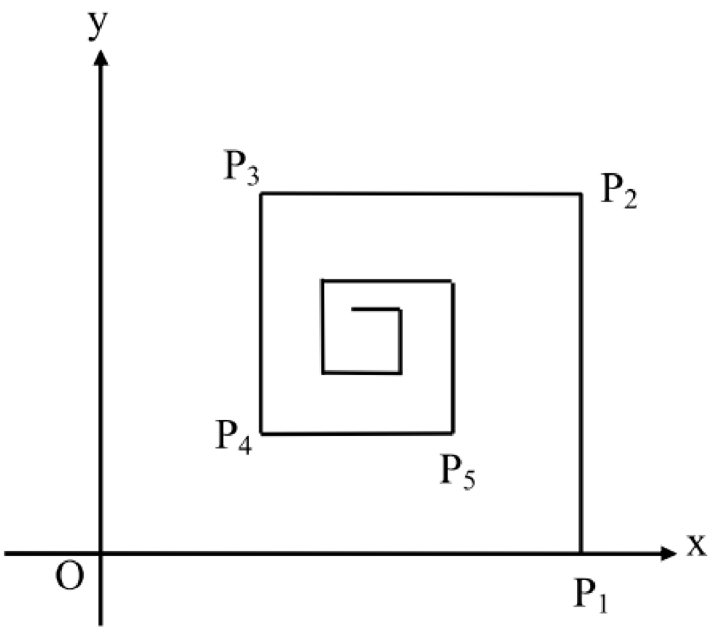
When the turning times k eguals to 2, the locus is a limacon , We prove that the polar coordinates is $R = r + 2r^2\cos\theta$.Finally, we show the route of the convergent point P in the common ratios $r \to 1^-$, $r = 1$ and $r \to 1^+$ .We connect the feeling of the route with the nature,but the real relationship is our future work.

目次

壹、研究動機

於高中數學第一冊第二單元 — 無窮等比級數中，我們曾學過以下題目：

坐標平面上一機器瓢蟲由原點出發，先向東移動 1 單位，停在 P_1 點，再往北移動 $\frac{4}{5}$ 單位，停在 P_2 點，接著又向西移動 $\left(\frac{4}{5}\right)^2$ 單位，停在 P_3 點，再向南移動 $\left(\frac{4}{5}\right)^3$ 單位，停在 P_4 點，再依序向東、向北、向西、向南、...繼續不斷（如下圖），每次移動距離都是前一次的 $\frac{4}{5}$，則這隻機器瓢蟲最後幾乎停止於 P 點，則 P 點的坐標為何？



上述問題中我們已知瓢蟲於 P_1、P_2、P_3、⋯時均需轉變行進方向，故瓢蟲轉變行進方向 k 次後會來到點 P_{k+1}，當 $k\to\infty$ 且此瓢蟲行進公比 $r=\frac{\overline{P_1P_2}}{\overline{OP_1}}=\frac{\overline{P_2P_3}}{\overline{P_1P_2}}=\cdots$ $=\frac{\overline{P_kP_{k+1}}}{\overline{P_{k\text{-}1}P_k}}=\cdots$，$0<r<1$ 時，此瓢蟲最終會幾乎停止於某一個點 P。而當瓢蟲來到點 P_1、P_2、P_3、⋯時，原始行進方向與下個行進方向的夾角(以下簡稱轉向角)均為 $90°$，倘若轉向角改成 $120°$(正三角形的一個外角)、$72°$(正五邊形的一個外角)，$60°$(正六邊形的一個外角)、$\left(\frac{360°}{7}\right)$(正七邊形的一個外角)，⋯，$\left(\frac{360°}{N}\right)$(正 N 邊形的

一個外角)，則此瓢蟲最後也均會停止於各點(以下簡稱收斂點)，而這些收斂點與收斂點間有什麼關係呢？又若改變瓢蟲的轉向角θ為任意角抑或轉向次數k為有限次時，則瓢蟲的終點所成的軌跡探討，促成了本次的研究。

貳、研究目的

一、探討當機器瓢蟲的轉向角為$\dfrac{360^\circ}{N}\left(N\in\mathbb{N},N\geq 3\right)$（正$N$邊形的一個外角）時，各轉向角所對應的收斂點於坐標平面上的關係。

二、探討當機器瓢蟲的轉向角θ為任意角時，各轉向角θ所對應的收斂點間的關係。

三、探討當機器瓢蟲的轉向角θ為任意角時，收斂點與圓錐曲線的關聯性。

四、探討各轉向點$P_n\left(n\in\mathbb{N}\right)$之關聯性。

五、探討當機器瓢蟲的轉向次數為k次$\left(k\in\mathbb{N},k\geq 3\right)$且轉向角$\theta$為任意角時，瓢蟲之終點$P_{k+1}$於坐標平面上所成之軌跡。

六、探討轉向次數$k=2$所成之軌跡。

七、當機器瓢蟲行進公比$\mathrm{r}\to 1^-,r=1,r\to 1^+$，轉向角$\theta$為任意角，轉向次數為$k$時，展示瓢蟲之終點$P_{k+1}$於坐標平面上所成之圖形。

參、研究設備及器材

紙、筆、電腦、GSP 幾何畫板、MathType5、Microsoft Office Excel、Microsoft Office Word

肆、研究過程或方法

一、轉向角為 $\dfrac{360^\circ}{N}\left(N\in\mathbb{N}, N\geq 3\right)$ 時，收斂點的猜想。

定理一：當瓢蟲的轉向角為 $\dfrac{360^\circ}{N}\left(N\in\mathbb{N}, N\geq 3\right)$ 時，各收斂點均位於圓 C 上。

$$\left(\text{圓C}:\left|\frac{1}{1-z}-\frac{1}{1-r^2}\right|=\frac{r}{1-r^2}\right)$$

（一）當瓢蟲的轉向角為 $\dfrac{360^\circ}{N}\left(N\in\mathbb{N}, N\geq 3\right)$ 時，收斂點坐標為

$$\begin{cases} x=\displaystyle\sum_{k=1}^{N} S_k\cos\left(\frac{360^\circ}{N}(k-1)\right) \\ y=\displaystyle\sum_{k=1}^{N} S_k\sin\left(\frac{360^\circ}{N}(k-1)\right) \end{cases}$$ **其中**

$$S_k=r^{k-1}+r^{N+k-1}+r^{2N+k-1}+\cdots+r^{(n-1)N+k-1}+\cdots=\frac{r^{k-1}}{1-r^N}\text{。}$$

【證明】如圖一，透過觀察，我們定義 $S_1,S_2,S_3,\cdots,S_N$ 如下

$$S_1=\overline{OP_1}+\overline{P_NP_{N+1}}+\overline{P_{2N}P_{2N+1}}+\cdots=1+r^N+r^{2N}+\cdots$$

$$S_2=\overline{P_1P_2}+\overline{P_{N+1}P_{N+2}}+\overline{P_{2N+1}P_{2N+2}}+\cdots=r+r^{N+1}+r^{2N+1}+\cdots$$

$$\vdots$$

$$S_{\mathrm{N}}=\overline{P_{\mathrm{N}-1}P_{\mathrm{N}}}+\overline{P_{2N-1}P_{2N}}+\overline{P_{3N-1}P_{3N}}+\cdots=r^{N-1}+r^{2N-1}+r^{3N-1}+\cdots\text{，}$$

由各式歸納得

$$S_{\mathrm{k}}=\overline{P_{\mathrm{k}-1}P_{\mathrm{k}}}+\overline{P_{N+\mathrm{k}-1}P_{N+\mathrm{k}}}+\overline{P_{2N+\mathrm{k}-1}P_{2N+\mathrm{k}}}+\cdots$$

$$=r^{k-1}+r^{N+k-1}+r^{2N+k-1}+\cdots\text{（}1\leq k\leq N\text{）}$$

故收斂點的 x 坐標

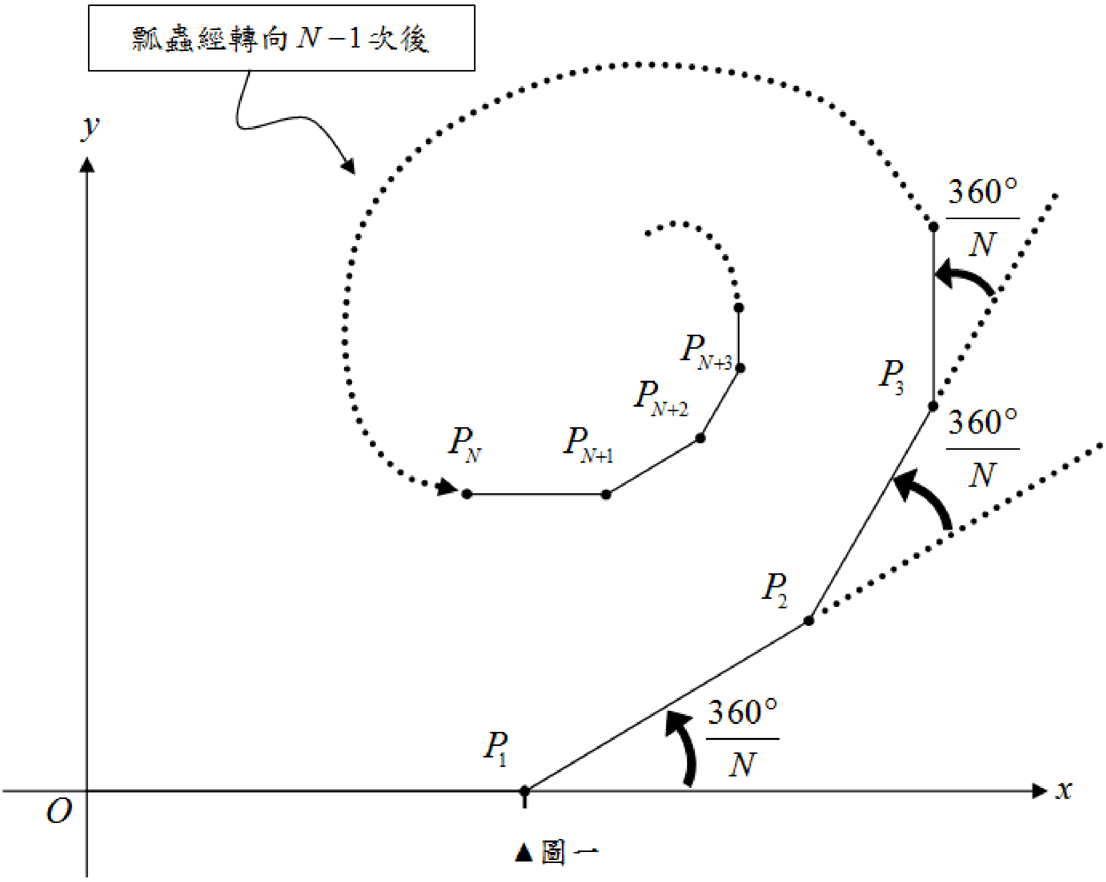
$$=S_1+S_2\cos\frac{360^\circ}{N}+S_3\cos\left(\frac{360^\circ}{N}\times 2\right)+\ldots+S_N\cos\left(\frac{360^\circ}{N}\times(N-1)\right)$$

$$=\sum_{k=1}^{N}S_k\cos\left[\frac{360^\circ}{N}\times(k-1)\right]$$

故收斂點的 y 坐標

$$=S_2\sin\frac{360^\circ}{N}+S_3\sin\left(\frac{360^\circ}{N}\times 2\right)+\ldots+S_N\sin\left(\frac{360^\circ}{N}\times(N-1)\right)$$

$$=\sum_{k=1}^{N}S_k\sin\left[\frac{360^\circ}{N}\times(k-1)\right]$$



▲圖一

（二）當瓢蟲的轉向角為 θ $\left(\theta=\frac{360^\circ}{N}, N\in\mathbb{N}, N\geq 3\right)$ 時，收斂點坐標在複數平面上可化簡為

$$\begin{cases} x=\sum_{n=1}^{\infty}r^{n-1}\cos(n-1)\theta=\dfrac{1-r\cos\theta}{1-2r\cos\theta+r^2} \\ y=\sum_{n=1}^{\infty}r^{n-1}\sin(n-1)\theta=\dfrac{r\sin\theta}{1-2r\cos\theta+r^2} \end{cases}$$ 。

【證明】

1. $x = \sum_{k=1}^{N} S_k \cos\left(\frac{360°}{N}(k-1)\right)$

$= S_1 + S_2\cos\theta + S_3\cos 2\theta + S_4\cos 3\theta + \cdots$

$= \left(1 + r^N + r^{2N} + \cdots\right) + \left(r + r^{N+1} + r^{2N+1} + \cdots\right)\cos\theta$

$+ \left(r^2 + r^{N+2} + r^{2N+2} + \cdots\right)\cos 2\theta + \cdots\cdots$

$= 1 + r(\cos\theta) + r^2(\cos 2\theta) + r^3(\cos 3\theta) + \cdots = \sum_{n=1}^{\infty} r^{n-1}\cos(n-1)\theta$

同理可得：$y = r(\sin\theta) + r^2(\sin 2\theta) + r^3(\sin 3\theta) + \cdots = \sum_{n=1}^{\infty} r^{n-1}\sin(n-1)\theta$

2.透過複數的極式可得 $x + yi$

$= 1 + r(\cos\theta + i\sin\theta) + r^2(\cos\theta + i\sin\theta)^2 + \cdots + r^{n-1}(\cos\theta + i\sin\theta)^{n-1} + \cdots$

$= \frac{1}{1 - r(\cos\theta + i\sin\theta)} = \frac{1}{(1 - r\cos\theta) - i(r\sin\theta)}$

$= \frac{(1 - r\cos\theta) + i(r\sin\theta)}{\left[(1 - r\cos\theta) - i(r\sin\theta)\right]\left[(1 - r\cos\theta) + i(r\sin\theta)\right]}$

$= \frac{(1 - r\cos\theta) + i(r\sin\theta)}{(1 - r\cos\theta)^2 - (ir\sin\theta)^2} = \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2}$

3.故瓢蟲的收斂點坐標可化簡為 $\begin{cases} x = \dfrac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} \\ y = \dfrac{r\sin\theta}{1 - 2r\cos\theta + r^2} \end{cases}$

$\left(\theta = \frac{360°}{N}, N \in \mathbb{N}, N \geq 3\right)$

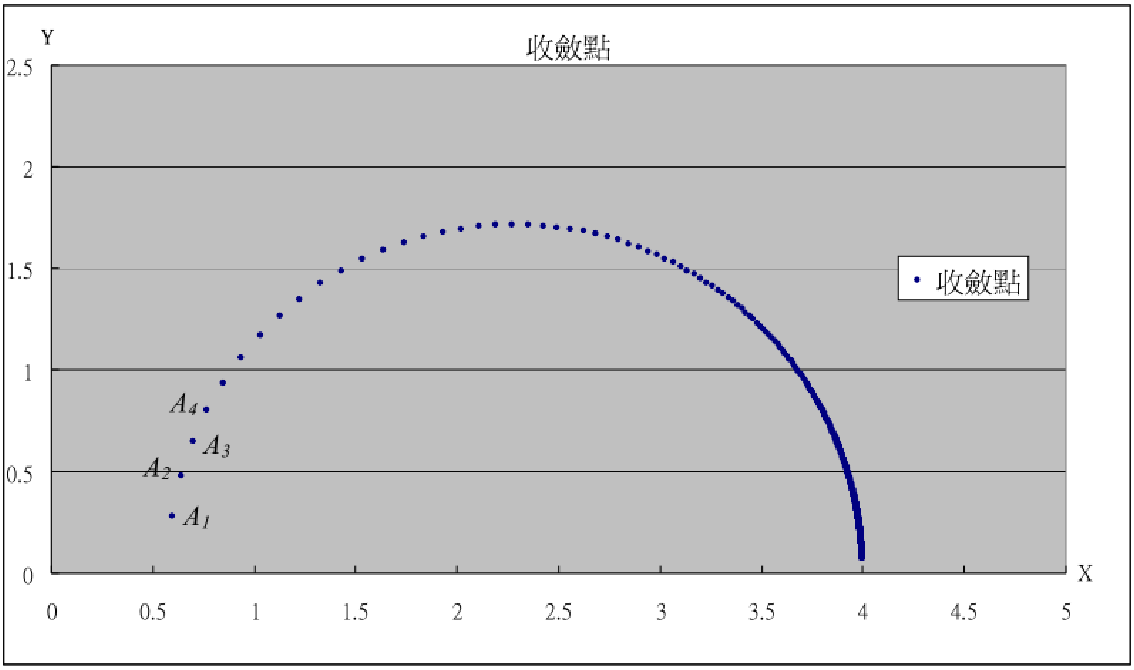
（三）我們輔以 Excel 算出 $N = 3, 4, 5, \cdots, 500$ 所對應的各收斂點坐標，並繪於坐標平面上。

1. N 的取值為 3 至 500 的正整數時，各收斂點的 (x,y) 坐標

N 值	x 座標	y 座標	N 值	x 座標	y 座標
3	0.594595	0.280873	60	3.593309	1.10859
4	0.64	0.48	70	3.691634	0.9809
5	0.699049	0.649053	80	3.758894	0.876651
6	0.769231	0.799408	90	3.806682	0.790843
7	0.848736	0.93481	100	3.84174	0.719416
8	0.935896	1.056772	110	3.868165	0.659261
9	1.029106	1.166066	120	3.888547	0.608033
10	1.126837	1.263241	130	3.904581	0.563955
11	1.227664	1.348816	140	3.917413	0.525674
12	1.330291	1.423356	150	3.927837	0.492147
13	1.433568	1.487489	160	3.936416	0.462559
14	1.5365	1.5419	170	3.943558	0.436267
15	1.638245	1.587312	180	3.949567	0.412759
16	1.738109	1.62447	190	3.954668	0.391622
17	1.835538	1.654121	200	3.959036	0.372518
18	1.930102	1.676996	210	3.962804	0.35517
19	2.021484	1.6938	220	3.966076	0.339351
20	2.109459	1.705201	230	3.968936	0.324868
21	2.193888	1.711825	240	3.97145	0.311561
22	2.274696	1.71425	250	3.973671	0.299293
23	2.351866	1.713009	260	3.975644	0.287947
24	2.425425	1.708583	270	3.977403	0.277425
25	2.495434	1.701409	280	3.978979	0.26764
26	2.561982	1.691878	290	3.980395	0.258517
27	2.625175	1.68034	300	3.981674	0.249993
28	2.685137	1.667104	310	3.982831	0.24201
29	2.741999	1.652447	320	3.983883	0.234519
30	2.795899	1.636609	330	3.984841	0.227476
40	3.201702	1.449049	400	3.989668	0.187931
50	3.443038	1.264665	500	3.99338	0.150507

▲由於數據過多，僅以部分數據代表

2. 我們根據上表各收斂點坐標的數據，以 Excel 軟體描繪出以下圖形，根據其收斂點的軌跡（A_1為轉向角$\theta=\dfrac{360^\circ}{3}$之收斂點、$A_2$為轉向角$\theta=\dfrac{360^\circ}{4}$之收斂點、$A_3$為轉向角$\theta=\dfrac{360^\circ}{5}$之收斂點…以此類推），我們推測這些收斂點應位於某個圓上，以下我們將給予證明。



▲圖二：此圖為 N 取值於 3 到 1000 的正整數所繪製

（四）各收斂點的軌跡方程式。

我們令 $z=r(\cos\theta+i\sin\theta)$，並以複數改寫收斂點坐標可得收斂點坐標可寫成：

$$P=1+z+z^2+\cdots+z^n+\cdots=\frac{1}{1-z}=\frac{1}{1-z}\times\frac{1+z}{1+z}=\frac{1+z}{1-|z|^2}=\frac{1+z}{1-r^2}$$

$$\Rightarrow\frac{1}{1-z}=\frac{1+z}{1-r^2}\Rightarrow\frac{1}{1-z}-\frac{1}{1-r^2}=\frac{z}{1-r^2}\Rightarrow\left|\frac{1}{1-z}-\frac{1}{1-r^2}\right|=\left|\frac{z}{1-r^2}\right|=\frac{r}{1-r^2}$$

（五）結論：各收斂點存在於一個圓心在$\left(\dfrac{1}{1-r^2},0\right)$，且半徑為$\left(\dfrac{r}{1-r^2}\right)$的圓 C 上。

二、轉向角 θ 為任意角 $\left(-180^\circ < \theta \le 180^\circ\right)$ 時之收斂點探討。

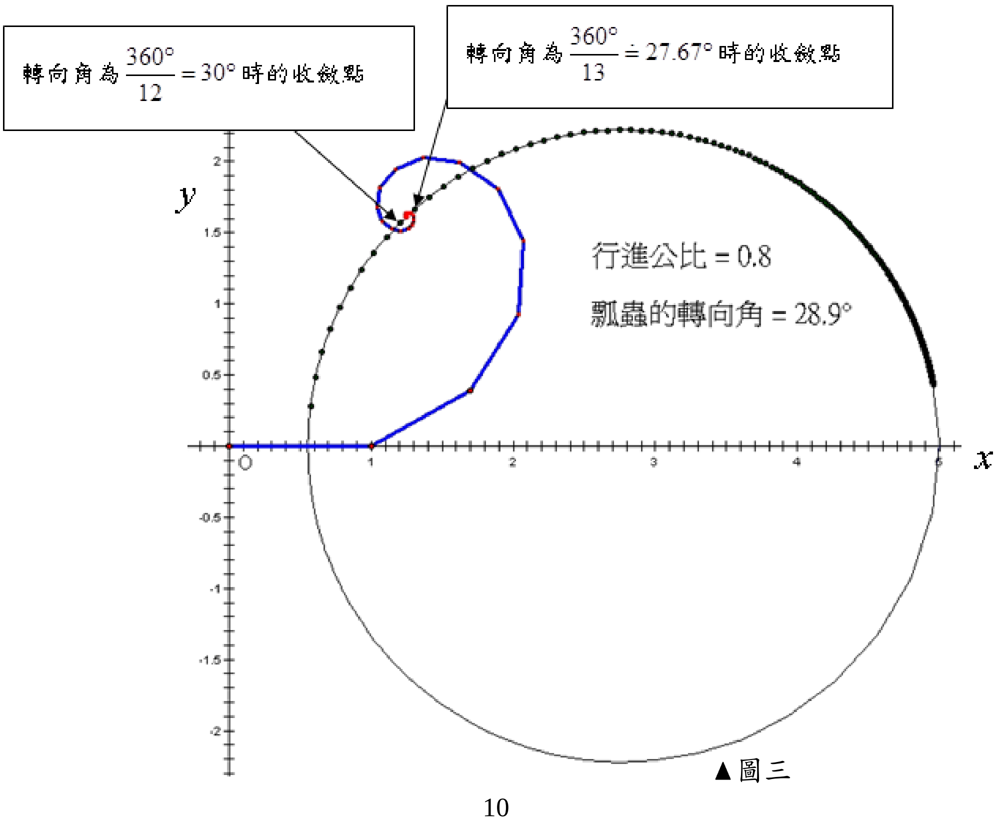
定理二：瓢蟲的轉向角 θ $\left(-180^\circ < \theta \le 180^\circ\right)$ 所對應的各收斂點於坐標平面上恰可形成圓 C ：$\left|\dfrac{1}{1-z}-\dfrac{1}{1-r^2}\right|=\dfrac{r}{1-r^2}$

（一）給定轉向角 θ $\left(-180^\circ < \theta \le 180^\circ\right)$，說明其收斂點的位置。

當瓢蟲的轉向角 θ 為任意角度時，收斂點坐標仍為 $\dfrac{1}{1-z}$ ，透過定理一的相關推導可得：當瓢蟲的轉向角 θ 為任意角度時，各收斂點均位於圓 C ：$\left|\dfrac{1}{1-z}-\dfrac{1}{1-r^2}\right|=\dfrac{r}{1-r^2}$ 上。

（二）以 GSP 軟體繪圖。

當瓢蟲的轉向角 θ 為任意角（EX： 28.9° ），行進公比 $r=0.8$ ，其收斂點也是位於圓 C 上。此外，這個點也會如預期的介於轉向角 $\dfrac{360^\circ}{12}=30^\circ$ 與 $\dfrac{360^\circ}{13}\doteq 26.67^\circ$ 之間的弧上。(如圖三)



▲圖三

（三）給定圓 C 上任一點 P，試證明可找到轉向角 θ $\left(-180^\circ < \theta \le 180^\circ\right)$，使得瓢蟲之收斂點恰為 P。

【證明】

已知 P 為圓 C 上一點，故可得：$\left|P-\dfrac{1}{1-r^2}\right|=\dfrac{r}{1-r^2}$

$$\Rightarrow \left|\left(1-r^2\right)P-1\right|^2=r^2 \Rightarrow \left[\left(1-r^2\right)P-1\right]\times\left[\left(1-r^2\right)\overline{P}-1\right]=r^2$$

$$\Rightarrow \left(1-r^2\right)^2 P\times\overline{P}-\left(1-r^2\right)\left(P+\overline{P}\right)+1=r^2$$

$$\Rightarrow \left(1-r^2\right)^2 P\times\overline{P}-\left(1-r^2\right)\left(P+\overline{P}\right)=r^2-1$$

$$\Rightarrow \left(r^2-1\right)P\times\overline{P}+\left(P+\overline{P}\right)=1 \Rightarrow \left(r^2-1\right)P\times\overline{P}=1-P-\overline{P}$$

$$\Rightarrow r^2=\frac{1}{P\times\overline{P}}-\frac{P}{P\times\overline{P}}-\frac{\overline{P}}{P\times\overline{P}}+1 \Rightarrow r^2=\left(1-\frac{1}{P}\right)\left(1-\frac{1}{\overline{P}}\right)$$

$\Rightarrow r^2=\left|1-\dfrac{1}{P}\right|^2 \Rightarrow r=\left|1-\dfrac{1}{P}\right|$，故可令 $1-\dfrac{1}{P}=r\left(\cos\theta+i\sin\theta\right)$，則 θ 即為所求。

（四）結論：瓢蟲的轉向角 θ $\left(-180^\circ < \theta \le 180^\circ\right)$ 所對應的收斂點於坐標平面上恰形成圓 C: $\left|\dfrac{1}{1-z}-\dfrac{1}{1-r^2}\right|=\dfrac{r}{1-r^2}$ 。

三、收斂點與圓錐曲線的探討。

定理三：已知 U 為 x 軸上任意點且坐標為 $(u,0)$，假設瓢蟲轉向角為 θ 時，其收斂點為 P。當 θ 改變時，$\overleftrightarrow{P_1P_2}$ 與 $\overleftrightarrow{UP}$ 之交點 S 的軌跡為圓錐曲線（點、直線、拋物線、橢圓、雙曲線）。

（一）交點 S 的軌跡方程式及圖形。

【證明】

設 $\overleftrightarrow{P_1P_2}$ 與 $\overleftrightarrow{CP}$ 之交點 S 的坐標為 z'，由上圖知

$$\begin{cases} \overleftrightarrow{P_1P_2}: z' = 1 + tz \\ \overleftrightarrow{UP}: z' = u + \left(\dfrac{1}{1-z} - u\right)s \end{cases}$$，則：

$$\Rightarrow 1 + zt = u + \left(\frac{1}{1-z} - u\right)s \Rightarrow zt - \left(\frac{1}{1-z} - u\right)s = u - 1 \cdots ①$$

取共軛 $\Rightarrow \bar{z}t - \left(\dfrac{1}{1-\bar{z}} - u\right)s = u - 1 \cdots ②$

①－②

$$\Rightarrow t\left(z-\bar{z}\right) = s\left(\frac{1}{1-z} - \frac{1}{1-\bar{z}}\right) \Rightarrow t\left(z-\bar{z}\right) = s\left(\frac{z-\bar{z}}{(1-z)\left(1-\bar{z}\right)}\right)$$

$$\Rightarrow t = s\left(\frac{1}{(1-z)\left(1-\bar{z}\right)}\right)$$代入 $\overleftrightarrow{P_1P_2}$ 、$\overleftrightarrow{CP}$

$$\Rightarrow \begin{cases} z' = 1 + s\left(\dfrac{z}{(1-z)\left(1-\bar{z}\right)}\right) \\ z' = u + s\left(\dfrac{1}{1-z} - u\right) \end{cases} \Rightarrow s\left(\frac{z}{(1-z)\left(1-\bar{z}\right)} - \frac{1}{1-z} + u\right) = u - 1$$

$$\Rightarrow s\left(\frac{z+\bar{z}-1}{(1-z)\left(1-\bar{z}\right)} + u\right) = u - 1 \Rightarrow s = \frac{(1-z)\left(1-\bar{z}\right)(u-1)}{\left(r^2 + (1-z)\left(1-\bar{z}\right)(u-1)\right)}$$

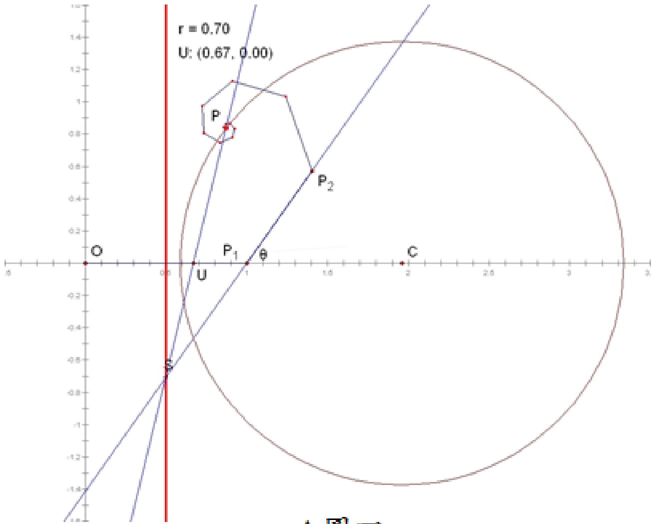
$$\therefore z' = 1 + \frac{(1-z)\left(1-\bar{z}\right)(u-1)}{\left(r^2 + (1-z)\left(1-\bar{z}\right)(u-1)\right)}\left(\frac{z}{(1-z)\left(1-\bar{z}\right)}\right)$$

$$=1+\frac{(u-1)z}{r^2+(1-z)(1-\bar{z})(u-1)}$$

$$\Rightarrow z'-1=\frac{(u-1)z}{r^2u+(u-1)(1-2r\cos\theta)}$$

$$\Rightarrow |z'-1|=\left|\frac{(u-1)r}{r^2u+u-1-2(u-1)r\cos\theta}\right|$$

1. 若 $r^2u+u-1=0 \Rightarrow u=\frac{1}{1+r^2}$，此時交點 S 的軌跡為一直線，方程式為 $R=-\frac{1}{2\cos\theta}$。(如圖四)



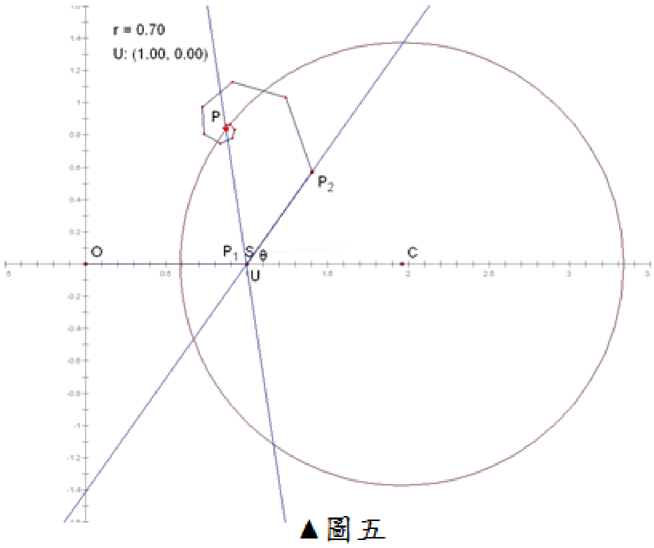
▲圖四

2. 若 $r^2u+u-1\neq 0$，則 $|z'-1|=\left|\frac{\frac{z(u-1)}{r^2u+u-1}}{1-2\frac{r(u-1)}{r^2u+u-1}\cos\theta}\right|=\left|\frac{\frac{r(u-1)}{r^2u+u-1}}{1-2\frac{r(u-1)}{r^2u+u-1}\cos\theta}\right|$，

以 1 為極點時，交點 S 的極坐標方程式為：$R=\frac{M\times\frac{1}{2}}{1-M\cos\theta}$（$M=\frac{2r(u-1)}{r^2u+u-1}$），

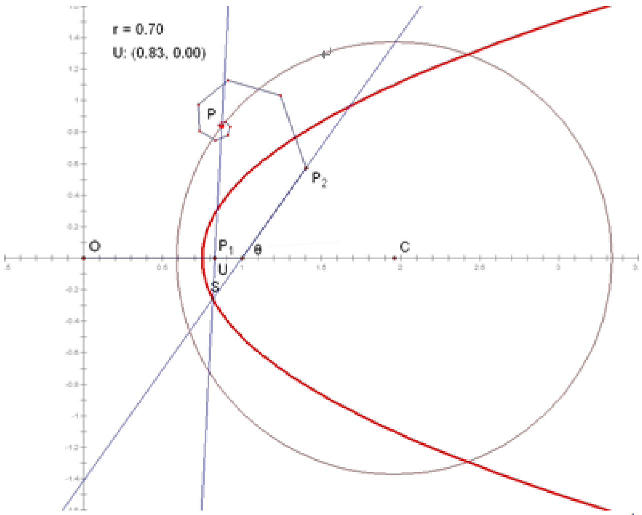
離心率 $e=|M|$，

（1）當 $e=0$ 時，$u=1$，此時 S 的軌跡為一點。(如圖五)



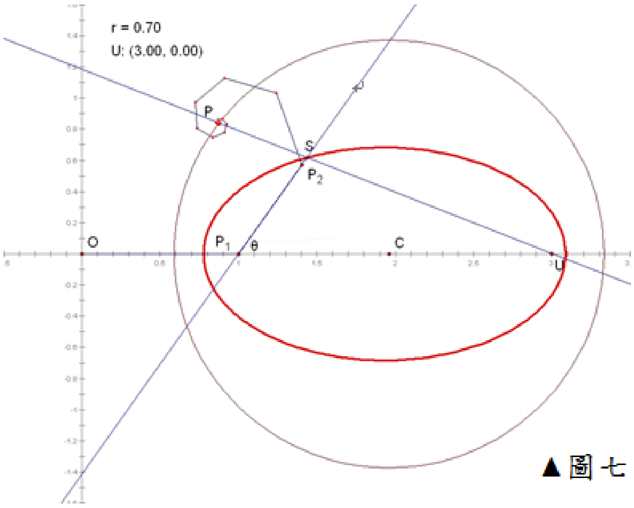
▲圖五

（2）當 $e=1$ 時，$u=\dfrac{1-2r}{(1-r)^2}\ or\ \dfrac{1+2r}{(1+r)^2}$ ，此時 S 的軌跡為拋物線。(如圖六)



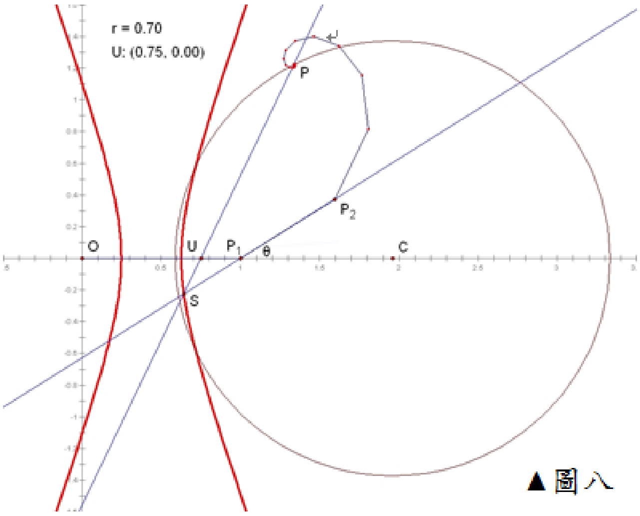
▲圖六

（3）當 $0<e<1$ 時，$u\in\left(-\infty,\dfrac{1-2r}{(1-r)^2}\right)or\left(\dfrac{1+2r}{(1+r)^2},\infty\right), u\neq 1$，此時 S 的軌跡為橢圓。(如圖七)



▲圖七

（4）$e>1$時，$u\in\left(\dfrac{1-2r}{(1-r)^2},\dfrac{1+2r}{(1+r)^2}\right)$，此時 S 的軌跡為雙曲線。(如圖八)



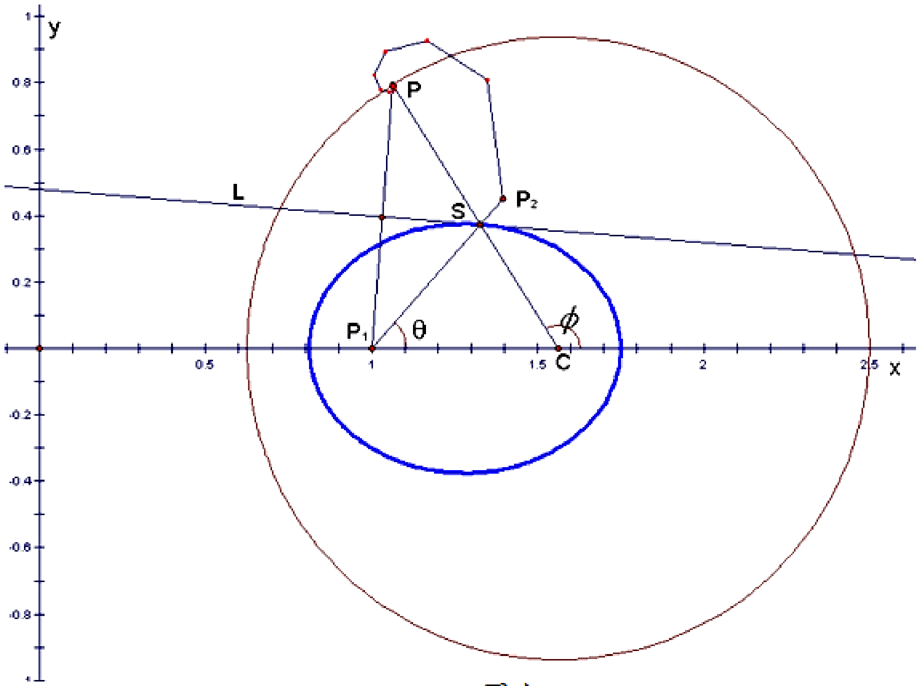
▲圖八

(二)結論：已知 U 為 x 軸上任意點且坐標為$(u,0)$，假設瓢蟲轉向角為θ時，其收斂點為P。當θ改變時，$\overleftrightarrow{P_1P_2}$與$\overleftrightarrow{UP}$之交點 S 的軌跡為圓錐曲線，且 u 點在各範圍時所對應的圖形如下表：

u 值	離心率 e	S 的軌跡	S 的軌跡方程式
$u=\dfrac{1}{1+r^2}$	不存在	一直線	$R=-\dfrac{1}{2\cos\theta}$
$u=1$	0	一點	$R=0$ (極點)
$u=\dfrac{1-2r}{(1-r)^2}\ or\ \dfrac{1+2r}{(1+r)^2}$	1	拋物線	$R=\dfrac{M\times\frac{1}{2}}{1-M\cos\theta}$ ($M=\dfrac{2r(u-1)}{r^2u+u-1}$)
$u<\dfrac{1-2r}{(1-r)^2}\ or\ u>\dfrac{1+2r}{(1+r)^2}, u\neq 1$	$0<e<1$	橢圓	
$\dfrac{1-2r}{(1-r)^2}<u<\dfrac{1+2r}{(1+r)^2}$	$e>1$	雙曲線	

（三）特殊情形討論

當 U=收斂圓圓心 $C\left(\dfrac{1}{1-r^2}\right)$ 時，由於 $\dfrac{1}{1-r^2}>\dfrac{1-2r}{(1-r)^2}$，故此時 $\overleftrightarrow{P_1P_2}$ 與 $\overleftrightarrow{UP}$ 之交點 S 的軌跡為橢圓，我們令此橢圓為 Γ，且方程式為 $R=\dfrac{\frac{r}{2}}{1-r\cos\theta}$，由此可知此橢圓的焦點為 $(1,0)$ 及 $C\left(\dfrac{1}{1-r^2}\right)$，且長軸長為收斂圓 C 的半徑 $\dfrac{r}{1-r^2}$。此與以直尺及圓規繪出橢圓軌跡的方法類似，意即；若給定收斂圓 C 連接 $\overline{P_1P}$ 並作其中垂線 L，則 L 與 $\overline{CP}$ 的交點坐標即為 S，當我們改變 θ 時，交點 S 的軌跡就是上述的橢圓 Γ，中垂線 L 即為橢圓 Γ 的切線（如圖九）。



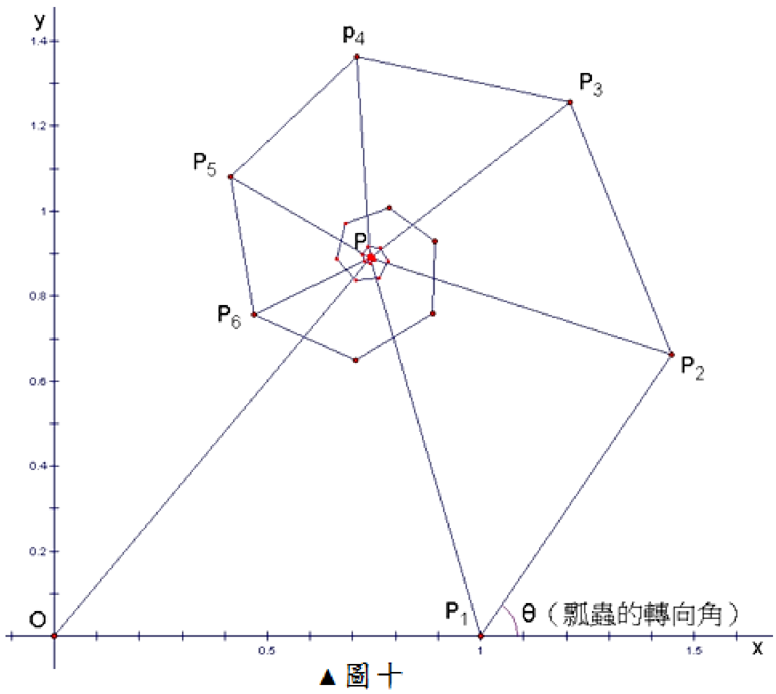
▲圖九

四、各轉向點 $P_n\left(n\in\mathbb{N}\right)$ 之關聯性的探討。

> **定理四：當瓢蟲以行進公比 *r*、轉向角為 α 行進時，各轉向點 $P_n\left(n\in\mathbb{N}\right)$ 位於一個極坐標方程式為 $R=m\cdot r^{\frac{\theta-\pi}{\alpha}},m=\overline{OP}=\sqrt{\dfrac{1}{1-2r\cos\alpha+r^2}}$，定角為 $\cot^{-1}(\dfrac{\ln r}{\alpha})$ 之等角螺線上。**

（一）將瓢蟲的收斂點 *P* 與 *O* 點及各轉向點 $P_n\left(n\in\mathbb{N}\right)$ 連線，

則 $\angle OPP_1=\angle P_1PP_2=\angle P_2PP_3=\cdots=\angle P_nPP_{n+1}=\cdots=$ 轉向角 θ。（如圖十）



▲圖十

【證明】

由圖可推得各轉向點 $P_n\left(n\in\mathbb{N}\right)$ 坐標為

$$P_n=1+z+z^2+\cdots+z^{n-1}=\frac{1-z^n}{1-z}$$

$$\overline{PP_n}=\left|\frac{z^n}{1-z}\right|=\left|\frac{1}{1-z}\right|\times|z|^n=r^n\times\left|\frac{1}{1-z}\right|$$

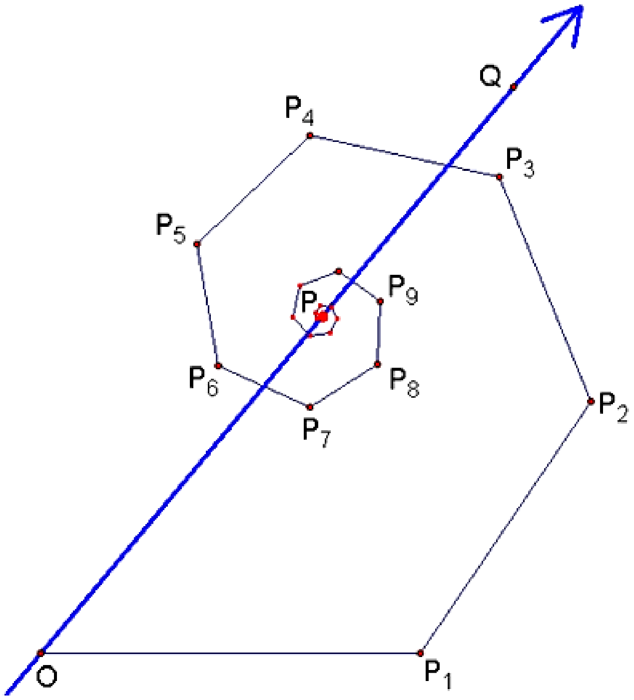
$$\Rightarrow\triangle OPP_1\sim\triangle P_1PP_2\sim\triangle P_2PP_3\sim\cdots\sim\triangle P_nPP_{n+1}\sim\cdots(SSS\text{相似})$$

又 $\angle OPP_1 + \angle POP_1 = \angle PP_1P_2 +$ 轉向角 $\theta \Rightarrow \angle OPP_1 = \theta$

$\therefore \angle OPP_1 = \angle P_1PP_2 = \angle P_2PP_3 = \cdots = \angle P_nPP_{n+1}\left(n \in \mathbb{N}\right) = \cdots =$ 轉向角 θ

（三）探究各轉向點 $P_n\left(n \in \mathbb{N}\right)$ 所在的曲線。

【證明】將原點 O 與瓢蟲之收斂點 P 連線，以 P 為極點，$\overrightarrow{PQ}$ 為極軸（如圖十一），

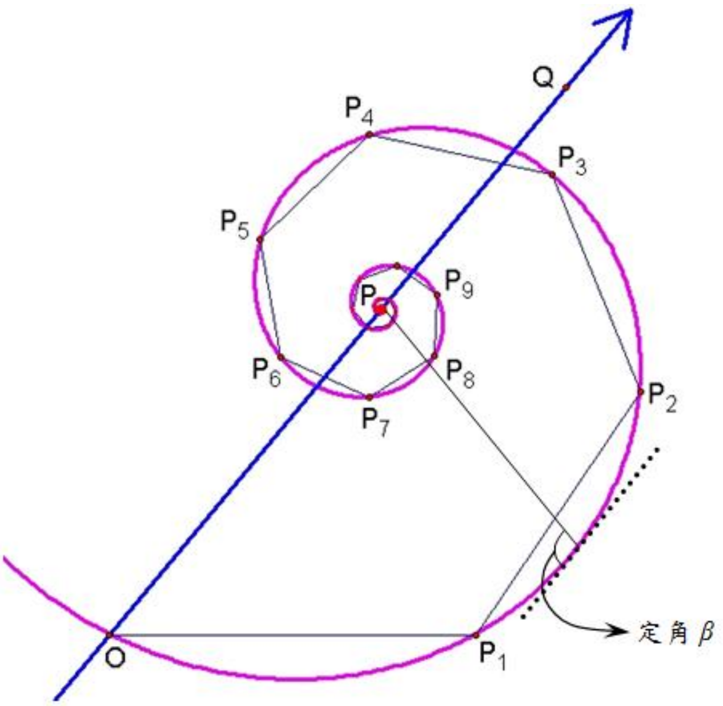


▲圖十一

※為避免混淆，此處的轉向角 θ 改以 α 表示。

由（二）可得：$O\left(m,\pi\right)$、$P_1\left(mr,\pi+\alpha\right)$、$P_2\left(mr^2,\pi+2\alpha\right)$、⋯、$P_n\left(mr^n,\pi+n\alpha\right)$、⋯，其中 $m = \overline{OP} = \sqrt{\dfrac{1}{1-2r\cos\alpha+r^2}}$，

我們發現上述各點恰位於一個極坐標方程式為 $R = m \cdot r^{\frac{\theta-\pi}{\alpha}}$ 之等角螺線上。



▲圖十二

由等角螺線的標準式$R=ae^{\theta\cot\beta}$（β為螺線的定角）及點$O(m,\pi)$可知$a=me^{-\pi\cot\beta}$，故$R=me^{(\theta-\pi)\cot\beta}=m\cdot r^{\frac{\theta-\pi}{\alpha}}\Rightarrow e^{(\theta-\pi)\cot\beta}=r^{\frac{\theta-\pi}{\alpha}}$

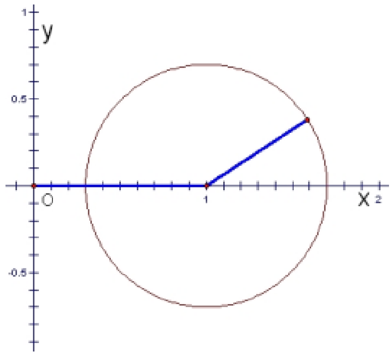
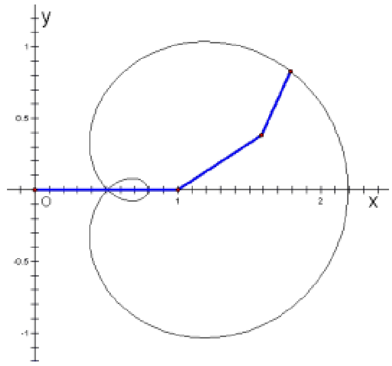
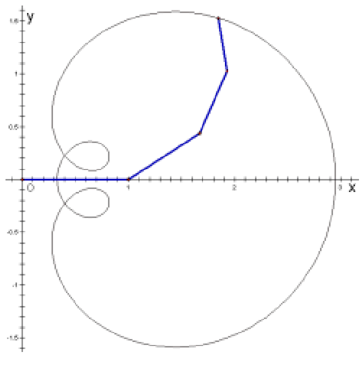
$$\Rightarrow(\theta-\pi)\cot\beta=\frac{\theta-\pi}{\alpha}\ln r$$

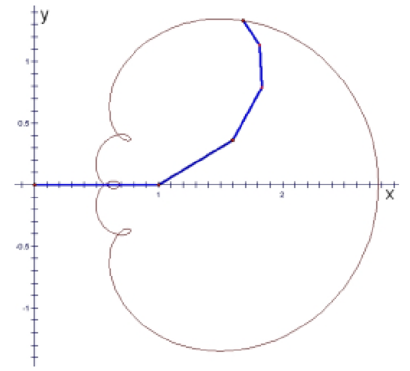
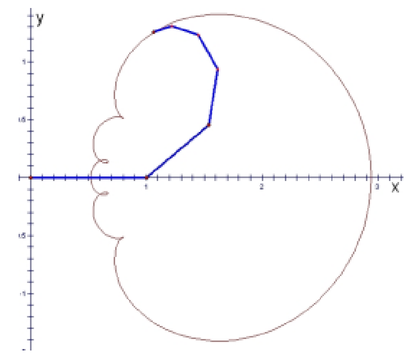
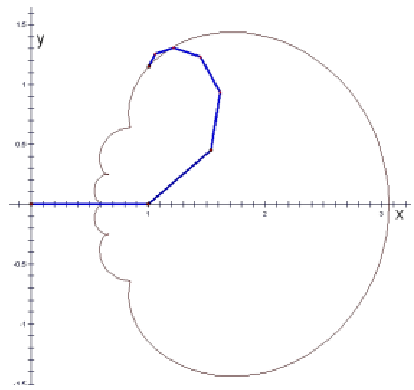
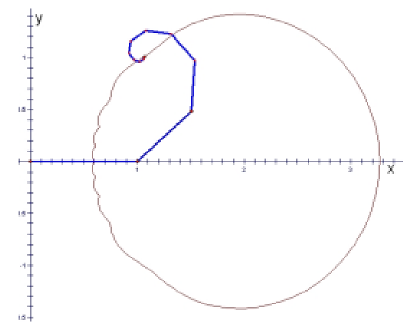
$\Rightarrow$此等角螺線的定角$\beta=\cot^{-1}(\frac{\ln r}{\alpha})$

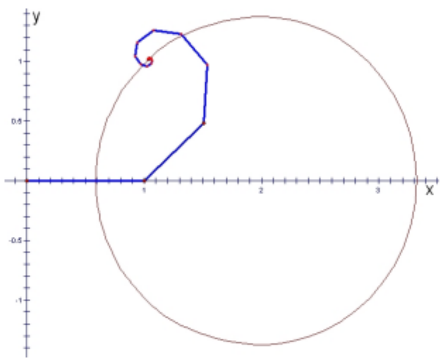
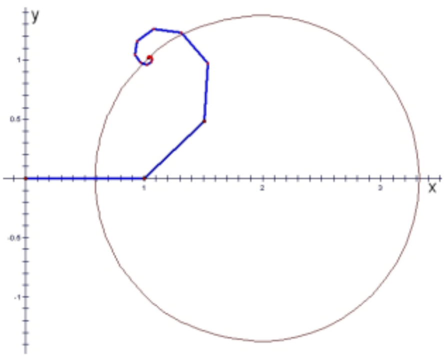
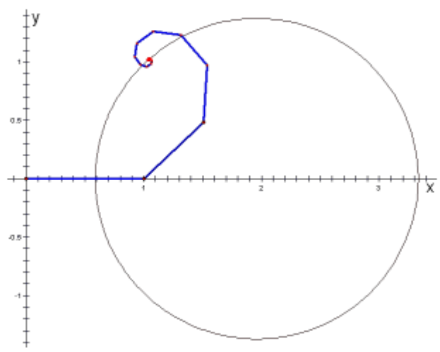
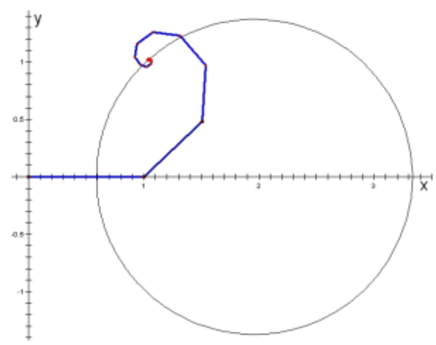
（四）結論：當瓢蟲以行進公比 r、轉向角為 α 行進時，各轉向點 $P_n\,(n\in\mathbb{N})$ 位於一個極坐標方程式為 $R=m\cdot r^{\frac{\theta-\pi}{\alpha}},m=\overline{OP}=\sqrt{\dfrac{1}{1-2r\cos\alpha+r^2}}$，定角為 $\cot^{-1}(\frac{\ln r}{\alpha})$ 之等角螺線上。

五、當機器瓢蟲轉向的次數為k次$\left(k \in \mathbb{N}\right)$且轉向角$\theta$為任意角時之終點軌跡 P_{k+1}的探討。

（一）瓢蟲的轉向次數$k=1,2,3,4,\cdots,1000$時對應的圖形如下。

k 值	參數方程式	圖形
$k=1$	$P_2=1+z$	
$k=2$	$P_3=1+z+z^2$	
$k=3$	$P_4=\dfrac{1-z^4}{1-z}$	

$k=4$	$P_5=\dfrac{1-z^5}{1-z}$	
$k=5$	$P_6=\dfrac{1-z^6}{1-z}$	
$k=6$	$P_7=\dfrac{1-z^7}{1-z}$	
$k=10$	$P_{11}=\dfrac{1-z^{11}}{1-z}$	

k 值	參數方程式	圖形
$k=15$	$P_{16}=\dfrac{1-z^{16}}{1-z}$	
$k=20$	$P_{21}=\dfrac{1-z^{21}}{1-z}$	
$k=100$	$P_{101}=\dfrac{1-z^{101}}{1-z}$	
$k=1000$	$P_{1001}=\dfrac{1-z^{1001}}{1-z}$	

（二）由上述各圖可看出當瓢蟲的轉向次數 k 越來越大時，其終點所形成的軌跡越來越趨近於一個圓。

當 $k \to \infty$ 時，瓢蟲的終點坐標 $P = \lim\limits_{n\to 0} P_n = \lim\limits_{n\to 0} \dfrac{1-z^n}{1-z} = \dfrac{1}{1-z}$ ，

其圖形也就是先前所推導的圓 $C: \left|\dfrac{1}{1-z} - \dfrac{1}{1-r^2}\right| = \dfrac{r}{1-r^2}$

（三）當轉向次數 $k=2$ 時瓢蟲所行進的軌跡探討。

定理五：當轉向次數 $k=2$ 時瓢蟲行進之終點坐標 $P_3 \begin{cases} x = 1 + r\cos\theta + r^2\cos 2\theta \\ y = r\sin\theta + r^2\sin 2\theta \end{cases}$ 的軌跡為一蚶線，且其經平移後的極坐標方程式為 $R = r + 2r^2\cos\theta$。

【證明】

當 $k=2$ 時由圖十一可知 $P_3 = 1 + z + z^2$

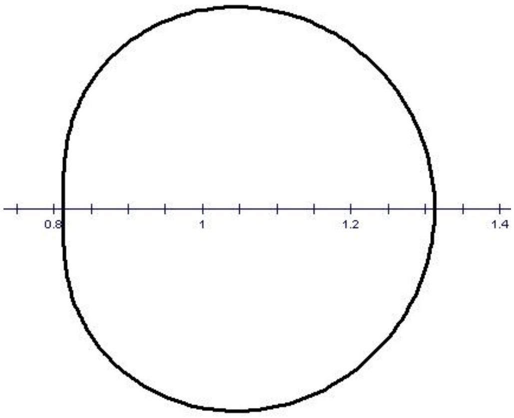
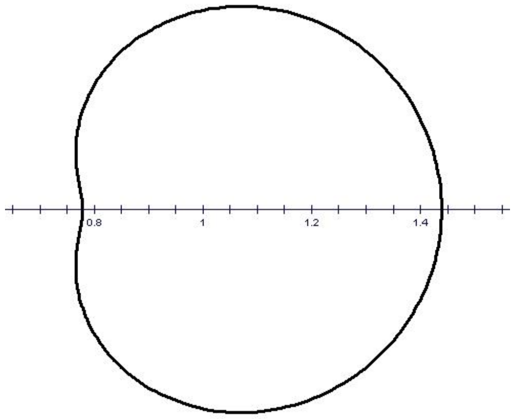
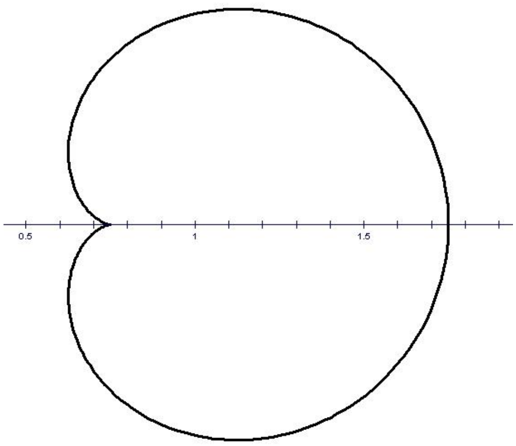
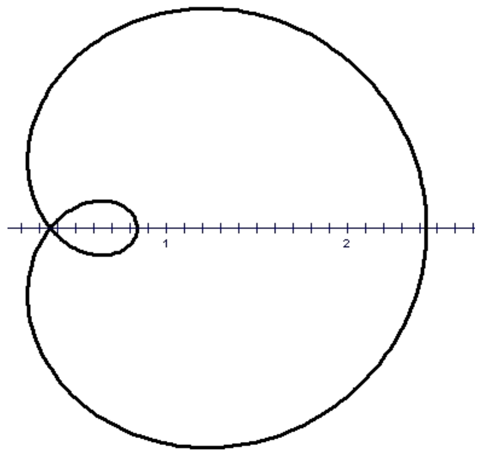
$\Rightarrow P_3 - \left(1-r^2\right) = 1 + z + z^2 - \left(1-r^2\right)$

$\left|z''\right| = \left|z + z^2 + r^2\right| = |z| \times \left|1 + z + \overline{z}\right| = r\left|1 + 2r\cos\theta\right| = \left|r + 2r^2\cos\theta\right|$ ，

故極坐標方程式為 $R = r + 2r^2\cos\theta$

（四）由上述可得當轉向次數 $k=2$ 時經平移後之蚶線方程式為：

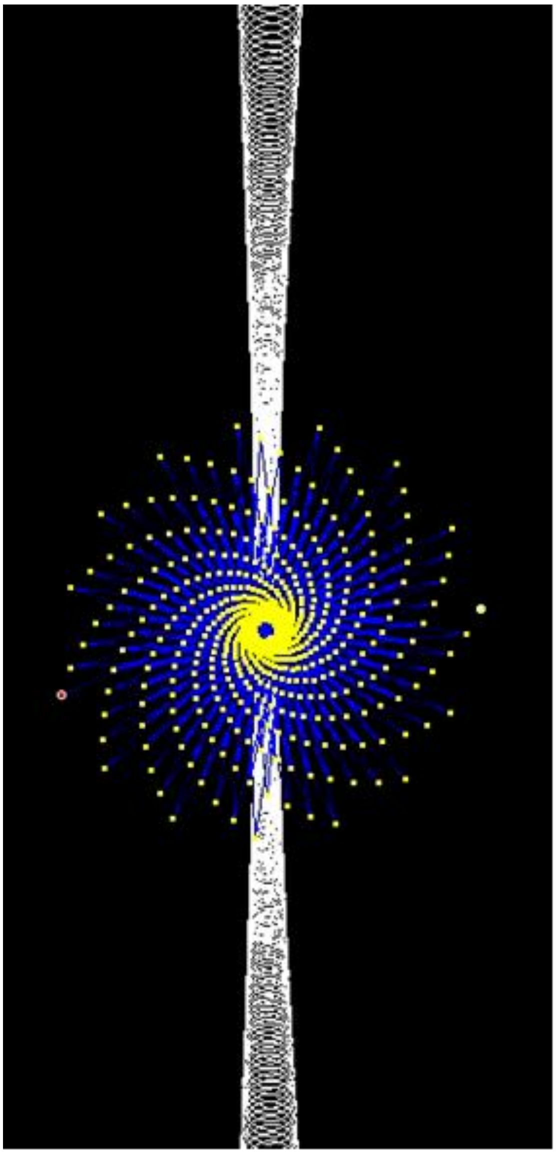
$R=r+2r^2\cos\theta$ **以下將討論當 r 於各區間所對應的圖形：**

r 值	$0<r\le\frac{1}{4}$	$\frac{1}{4}<r<\frac{1}{2}$
此時蚶線所代表的圖形	 ▲凸蚶線	 ▲凹蚶線
r 值	$r=\frac{1}{2}$ （此時極坐標方程式：$R=\frac{1}{2}+\frac{1}{2}\cos\theta$ ）	$\frac{1}{2}<r<1$
此時蚶線所代表的圖形	 ▲心臟線	 ▲蚶線

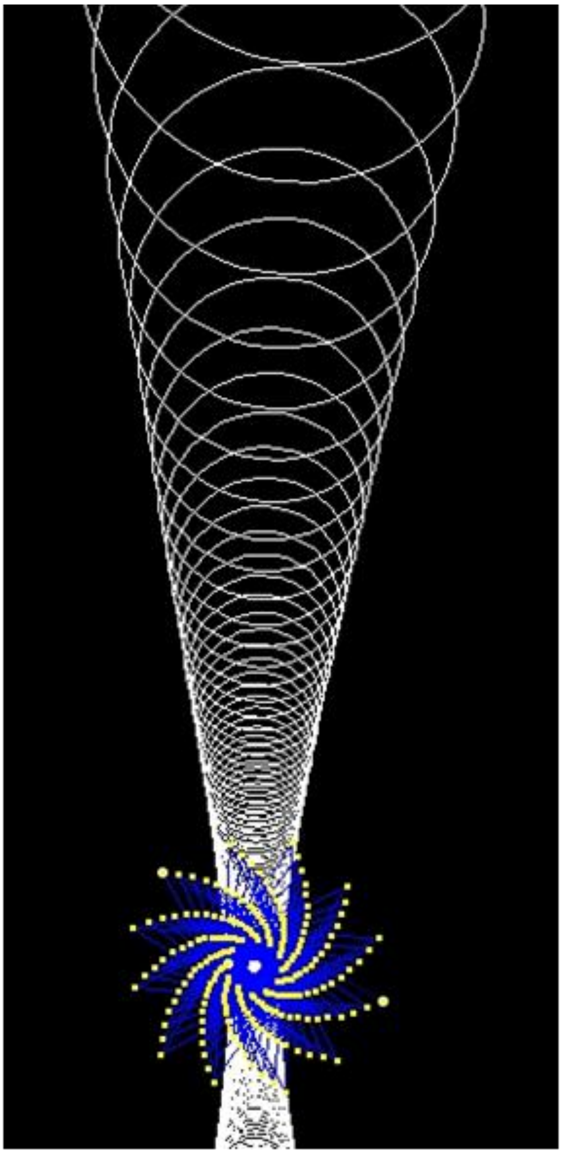
六、展示當機器瓢蟲行進公比 $r \to 1^-$，$r=1$，$r \to 1^+$，轉向角 θ 為任意角，且轉向次數為 k 次時（$k \in \mathbb{N}$），瓢蟲的終點坐標所形成之軌跡。

研究的歷程中真的讓我們理解所謂「數學乃科學之母」，在本研究中我們發現如下：

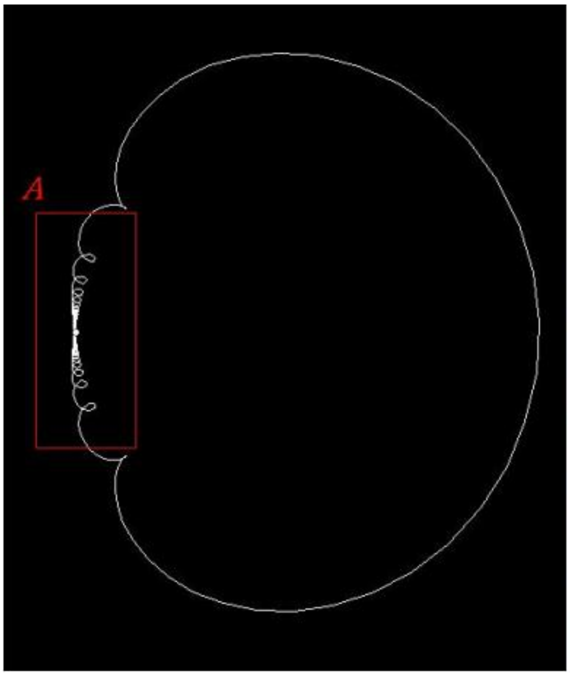
（一）瓢蟲的行進公比 $r \to 1^-$，我們取轉向次數為 $k=500$ 次時，其瓢蟲於各 θ 角所對應的終點坐標所形成的軌跡圖形呈現如下：當 $r=0.996$，如圖十三與圖十四之黃色部分為各轉向點，白色漩渦部分為瓢蟲於各轉向角 θ 所對應的終點軌跡，其皆截自於圖十五中的 A 部分，你是否覺得這些圖形像似物理學上的黑洞（如圖十六）呢？



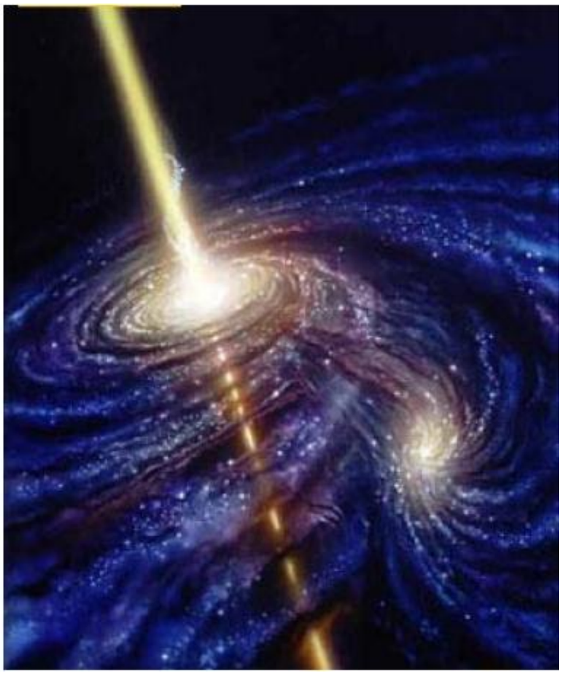
▲圖十三



▲圖十四

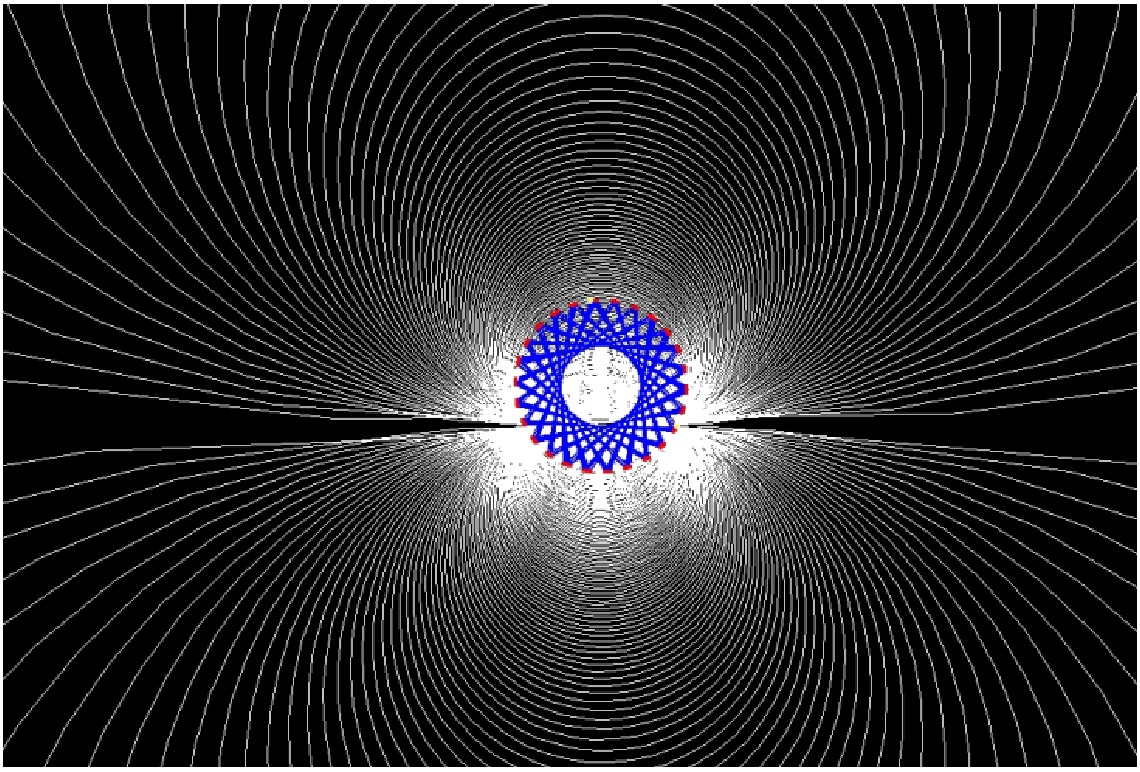


▲圖十五

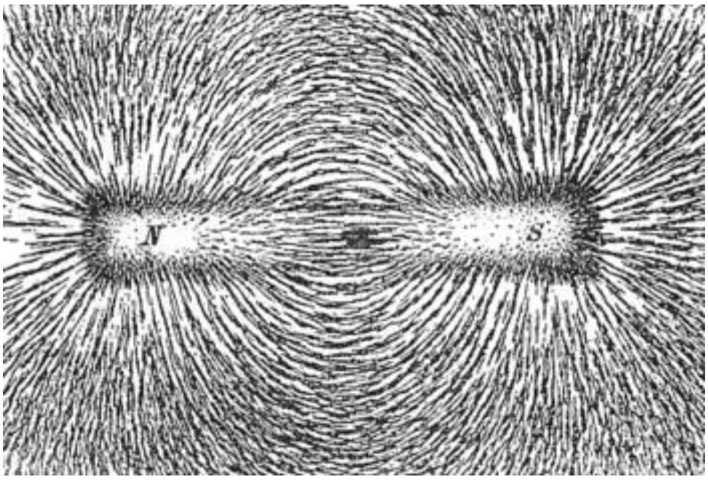


▲圖十六 黑洞圖形

（二）瓢蟲的行進公比 r＝1，我們取轉向次數為 $k=200$ 次時，各 θ 角所對應的終點坐標所形成的軌跡圖形（如圖十七），你是否覺得其圖形像似物理學上的磁力線（如圖十八）呢？

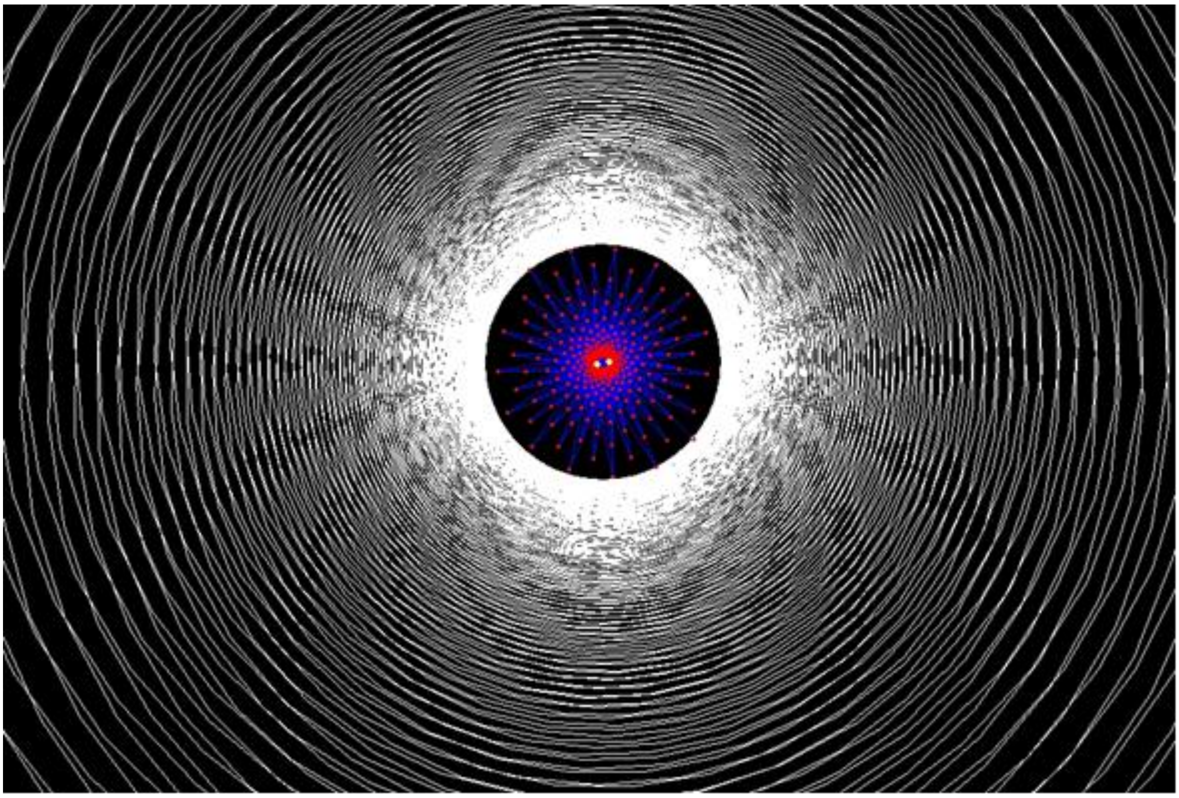


▲圖十七



▲圖十八

（三）瓢蟲的行進公比 $r = 1.01$，我們取轉向次數為 $k = 300$ 次時，瓢蟲於各 θ 角所對應的終點坐標所形成的軌跡圖形（如圖十九）。當遠看時，是否覺得其圖形很像蜘蛛呢？



▲圖十九

伍、研究結果架構圖

研究結果 三

假設瓢蟲轉向角為θ時，其收斂點為P，已知U為x軸上任意點且坐標為$(u,0)$。當θ改變時，$\overline{P_1P_2}$與$\overline{UP}$之交點S的軌跡為圓錐曲線（點、直線、拋物線、橢圓、雙曲線）。

研究結果 二

當機器瓢蟲的轉向角θ為任意角時，各收斂點P恰形成圓C。

研究結果 一

當機器瓢蟲的轉向角為$\dfrac{360°}{N}(N\in\mathbb{N},N\geq 3)$時，各收斂點均位於圓$C$上。

$k\to\infty$

高中數學 一
機器瓢蟲等比繞行之原題

呈現當機器瓢蟲轉向的次數為k次$(k\in\mathbb{N})$且轉向角θ為任意角時之終點軌跡圖形。

研究結果 五

當機器瓢蟲轉向次數$k=2$時，圖形為一蚶線且其經平移後的極座標方程式為$R=r+2r^2\cos\theta$

研究結果 四

當瓢蟲以行進公比r、轉向角為α行進時，各轉向點$P_n(n\in\mathbb{N})$位於一個極坐標方程式為

$R=m\cdot r^{\frac{\theta-\pi}{\alpha}}$

$m=\overline{OP}=\sqrt{\dfrac{1}{1-2r\cos\alpha+r^2}}$

，定角為$\cot^{-1}(\dfrac{\ln r}{\alpha})$之等角螺線上。

研究結果 六

展示行進公比$r\to1^-$時其圖形像似黑洞、$r=1$時其圖形像似磁力線、$r\to1^+$其圖形像似蜘蛛的圖形並猜測其圖形與自然界應有深切關係。

陸、研究結果

一、當機器瓢蟲的轉向角為$\dfrac{360^\circ}{N}\left(N\in\mathbb{N},N\geq 3\right)$時，各收斂點均位於圓$C$上。(圓 C : $\left|\dfrac{1}{1-z}-\dfrac{1}{1-r^2}\right|=\dfrac{r}{1-r^2}$)

二、當機器瓢蟲的轉向角θ為任意角時，各收斂點P恰形成圓C。

三、已知U為x軸上任意點且坐標為$\left(u,0\right)$，假設瓢蟲轉向角為θ時，其收斂點為P。當θ改變時，$\overleftrightarrow{P_1P_2}$與$\overleftrightarrow{UP}$之交點$S$的軌跡為圓錐曲線，且$u$點在各範圍時所對應的圖形如下表：

u 值	離心率 e	S 的軌跡	S 的軌跡方程式
$u=\dfrac{1}{1+r^2}$	不存在	一直線	$R=-\dfrac{1}{2\cos\theta}$
$u=1$	0	一點	$R=0$ (極點)
$u=\dfrac{1-2r}{\left(1-r\right)^2}$ or $\dfrac{1+2r}{\left(1+r\right)^2}$	1	拋物線	$R=\dfrac{M\times\frac{1}{2}}{1-M\cos\theta}$ ($M=\dfrac{2r\left(u-1\right)}{r^2u+u-1}$)
$u<\dfrac{1-2r}{\left(1-r\right)^2}$ or $u>\dfrac{1+2r}{\left(1+r\right)^2},u\neq 1$	$0<e<1$	橢圓	
$\dfrac{1-2r}{\left(1-r\right)^2}<u<\dfrac{1+2r}{\left(1+r\right)^2}$	$e>1$	雙曲線	

四、當瓢蟲以行進公比r、轉向角為α行進時，各轉向點$P_n\left(n\in\mathbb{N}\right)$位於一個極坐標方程式為$R=m\cdot r^{\frac{\theta-\pi}{\alpha}},m=\overline{OP}=\sqrt{\dfrac{1}{1-2r\cos\alpha+r^2}}$，定角為$\cot^{-1}(\dfrac{\ln r}{\alpha})$之等角螺線上。

五、當機器瓢蟲轉向次數$k=2$時，圖形為一蚶線且其經平移後的極坐標方程式為$R=r+2r^2\cos\theta$，當行進公比$0<r\le\frac{1}{4}$時其對應的圖形為凸蚶線、$\frac{1}{4}<r<\frac{1}{2}$為凹蚶線、$r=\frac{1}{2}$為心臟線、$\frac{1}{2}<r<1$為蚶線。

六、展示行進公比$r\to1^-$時其圖形像似黑洞、$r=1$時其圖形像似磁力線、$r\to1^+$其圖形像似蜘蛛的圖形並猜測其圖形與自然界應有深切關係。

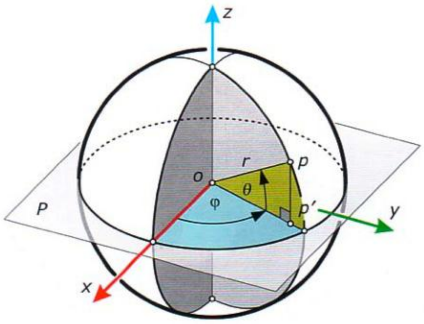
柒、討論與推廣

除了以上所述之討論，亦可繼續做更深入的延伸及推廣如下：

一、由討論結果可知k為不同值時有其對應圖形。當$k=2$時，其圖形為「蚶線」，而當k=2,3,4,5,6⋯時所形成圖形間是否有關聯？

二、當行進公比$r\to1^-$、$r=1$、$r\to1^+$時其圖形像似黑洞、磁力線、蜘蛛等圖形，而其圖形與自然界的關聯性，則有待未來研究。

三、本研究原為二維平面之討論，倘若將瓢蟲之行進路線推廣至三維空間，我們定義：

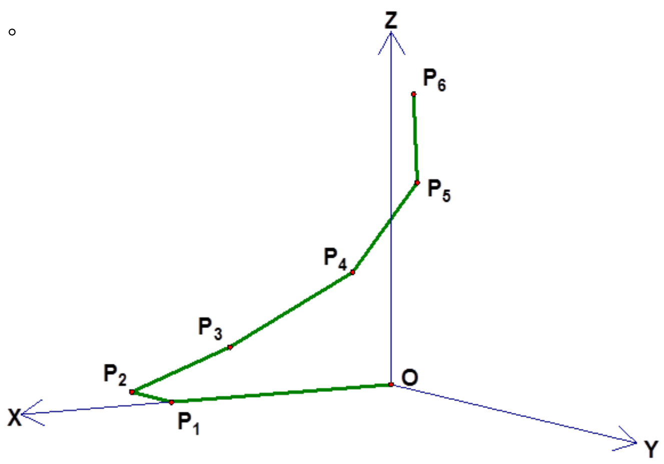


φ：方位角度座標值（$0\le\varphi\le2\pi$）；從x軸逆時針旋轉角

θ：仰角角度座標值$\left(-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}\right)$；從$xy$平面向上旋轉角

則數對(φ,θ)表示瓢蟲沿著方位角φ、仰角θ的方向行走，如：瓢蟲從原點O向$(0°,0°)$前進1單位後來到點P_1；再以(φ,θ)方向前進r單位來到點P_2；再以$(2\varphi,2\theta)$方向前進r^2單位來到點P_3；再以$(3\varphi,3\theta)$方向前進r^3單位來到點P_4；......，繼續不斷，如下圖所示，每次移動距離都是前一次的r倍，故這隻機器瓢蟲最後幾乎停止於一點P，則點P的坐標及其形成的軌跡為何？若轉向次數k為有限次時，瓢蟲的終點P_{k+1}所形成的軌跡為何？

相關問題則有待未來研究。



捌、參考資料及其他

一、趙文敏（民 77）。大學微積分。台北市：書銘。

二、趙文敏（民 79）。蚶線，科學月刊第 21 卷第 7 期。(557-560 頁)。

三、國立台灣師範大學－陳創義教授個人網頁 http://math.ntnu.edu.tw/~cyc/。

四、余文卿（民 98）。高級普通中學數學第一冊（4 版）。台南市：翰林。

五、余文卿（民 88）。高級普通中學數學第二冊（1 版）。新北市：龍騰。

六、李虎雄（民 95）。高級普通中學數學第四冊（1 版）。新北市：康熹。

七、常庚哲(民 90)。神奇的複數－如何利用複數解中學數學難題。台北市：九章。

八、趙文敏。等角螺線及其他，科學月刊第 20 卷第 9、10 期。

九、毛爾（民 89）。毛起來說 e。台北市：天下遠見。

十、嚴鎮軍（民 82）。高中數學競賽教程。台北市：九章。

十一、 圖十六：節錄自網路－對撞機引發恐慌 掉進黑洞將會發生什麼事。(2008）
http://tech.china.com/zh_cn/science/universe/1030/20080914/15086873.html

十二、 圖十八：節錄自網路－8.022 / ESG.8022 物理學 II：電磁學。(2006）
http://www.core.org.cn/OcwWeb/Physics/8-022Fall-2006/CourseHome/index.htm

十三、 第 29 頁討論與推廣之三－球坐標圖形：節錄自網路－空間球坐標系統。（2010）http://moodle.ncku.edu.tw/mod/resource/view.php?id=20050

Designing Linkage with Decomposition

Abstract

Invented in 1864, the Peaucellier Cell was the first planar linkage capable of transforming a rotary motion into a perfect straight line motion, and vice-versa. If a camera moves along the “rotor” then the output of the linkage is observed to trace out an interesting curve satisfying $r = \cos(t) + a \cdot \sec(t)$ in polar coordinates, the equation of Conchoids of de Sluze. Wishing to construct the curve-tracing animation of the linkage, three methods of decomposition are developed. Each method takes care of one family of curve: Conchoids of de Sluze, Trochoids and Lemniscates.

Application 1: The formation with the accompanying linkages and animations of Conchoids, Trochoids and Lemniscates are shown. Application 2: The new linkage is invented by hooking up two linkages; for instance, the ellipsograph. Application 3: Double generation property of trochoids. Two pairs of circles are shown to trace out the same curve, with each consisting of a fixed circle and a rolling circle with different ratios in radius. Application 4: The tangent of three families of curves is formed by the systematic steps of using linkages instead of Calculus. Application 5: The osculating circle of Trochoids is formed by the systematic steps of using decomposition, double generation property and tangent instead of Calculus.

All animations make use of Cabri II Plus for animation, ruler and compass constructions, the Maple for black-box curve constructions.

Contents

Chapter 1 Introduction

The Peaucellier Cell was the first (1864) planar linkage capable of transforming the rotary motion into a perfect straight line motion and vice versa (Fig. 1).

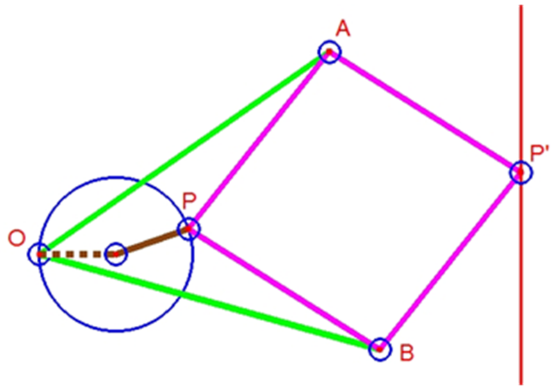


Fig.1

The principle behind the Peaucellier Cell (Fig. 2) is based on the identity

$$\overline{OP} \cdot \overline{OP'} = \overline{OA}^2 - \overline{PA}^2 \text{……(1)}$$

which follows by adding the identities

$$\overline{OM}^2 + \overline{AM}^2 = \overline{OA}^2$$

$$\overline{PA}^2 = \overline{PM}^2 + \overline{AM}^2$$

$$\overline{OM}^2 \text{-} \overline{PM}^2 = \overline{OP} \cdot \overline{OP'}$$

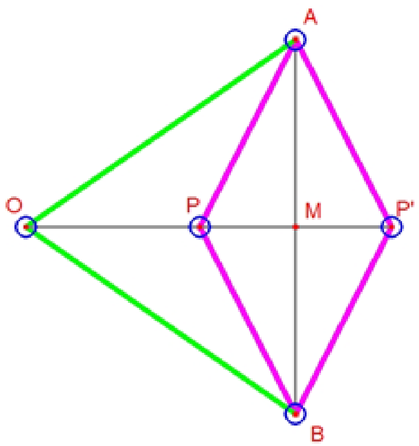


Fig. 2

Since both $\overline{OA}^2$and $\overline{PA}^2$ are constant, so is$\overline{OP} \cdot \overline{OP'}$ Therefore, the linkage transforms any circle passing through O into a straight line.

Interchanging the role of the rotor and the fixed rod, the output is observed to trace an interesting curve (Fig. 3) satisfying the equation

$$r = cos\,\theta + a \cdot sec\,\theta ......(2)$$

in polar coordinates.

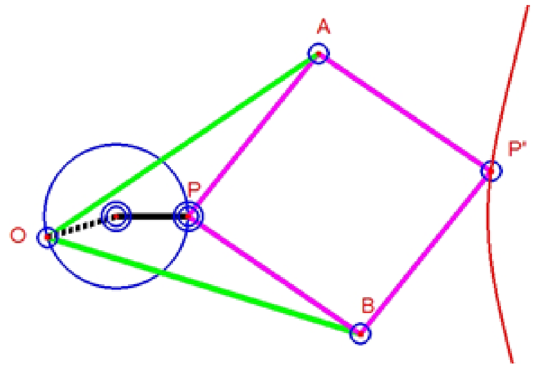


Fig. 3

Checking in the handbook, the equation matches that of the Conchoid of de Sluze. The dual role of the Peaucellier Cell hence offers a novel perspective to study the duality between the circle-conchoid pair in the same way as the telescope- microscope pair.

Designing linkage took up mathematical minds before the 1900s. Most of the work appears isolated from one another. Borrowing from integer factoring into a product of primes, the first step in a systematic method in designing of the linkage is to identify the most important component, to be called the decomposition, and the rest follows recursively. Just as the case of integers, there is no universal method of decomposition.

This project focuses on the decomposition for three families of curves: the Conchoids of de Sluze, the Trochoids and the Lemniscates.

Chapter 2 Definition

I. Conchoids: The Conchoids of de Sluze are given by $r = \cos\theta + a\ \sec\theta$ (a real constants) in polar coordinates (Fig. 4).

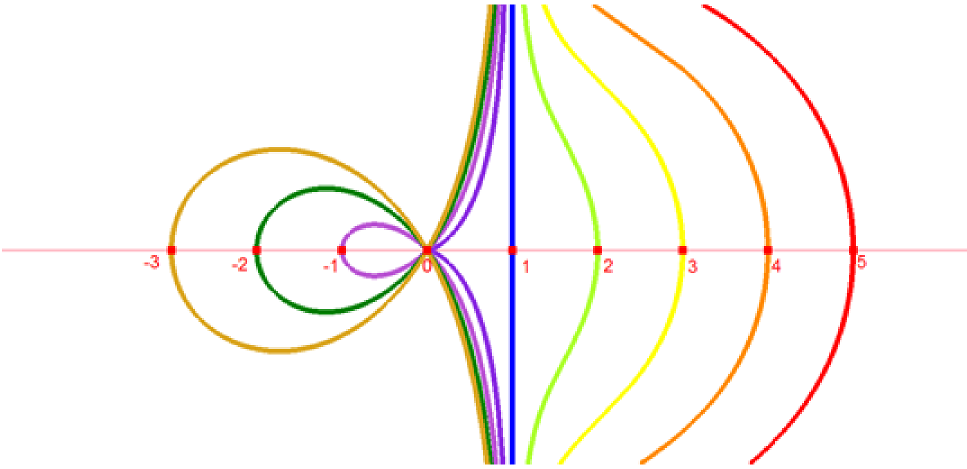
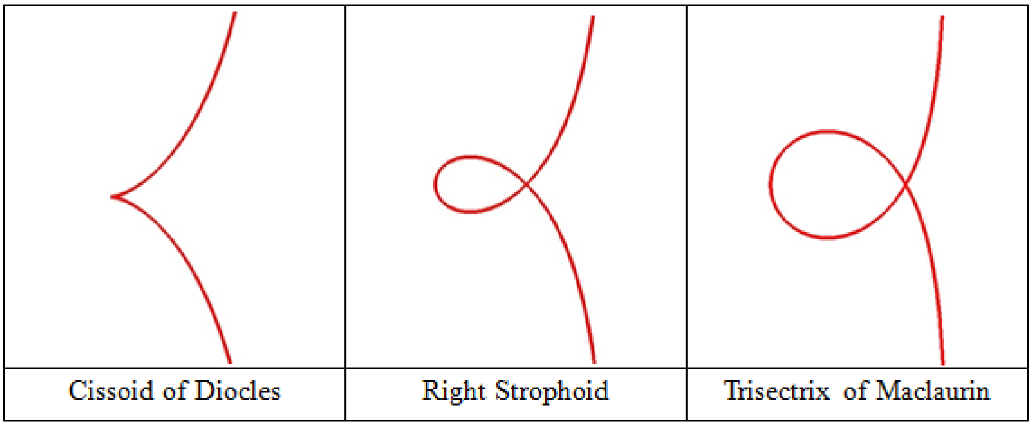


Fig. 4



Important conchoids of de Sluze include the Cissoid of Diocles (*a*=-1), the Right Strophoid (*a*=-2) and the Trisectrix of Maclaurin (*a*=-4).

II. Trochoids: The trochoids, in this project, refer to the family of curve which was generated by the rolling circle and fixed circle. It includes the epitrochoid and hypotrochoid.

A. Epitrochoid: Epitrochoid is a roulette traced by a point *P* attached to a circle rolling around the outside of a fixed circle. (Fig. 5)

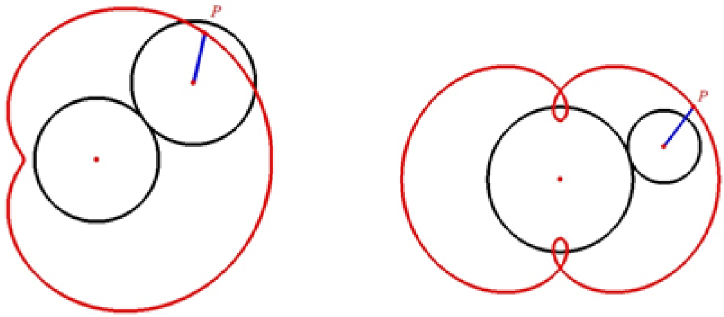


Fig. 5

The epitrochoid has epicycloid as special case, including Cardioid and Nephroid (Fig. 6).

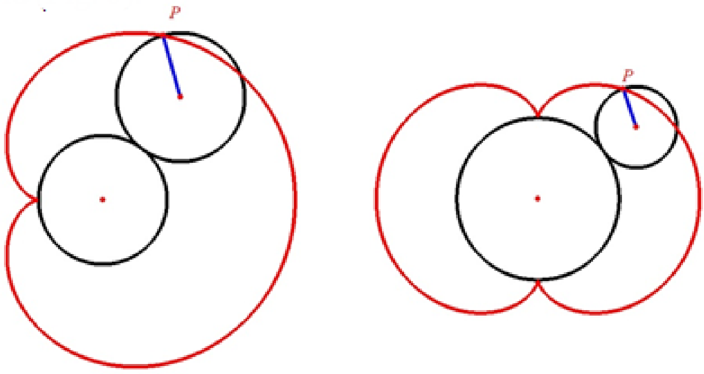


Fig. 6

B. Hypotrochoid: A hypotrochoid is a roulette traced by a point P attached to a circle rolling around the inside of a fixed circle (Fig. 7).

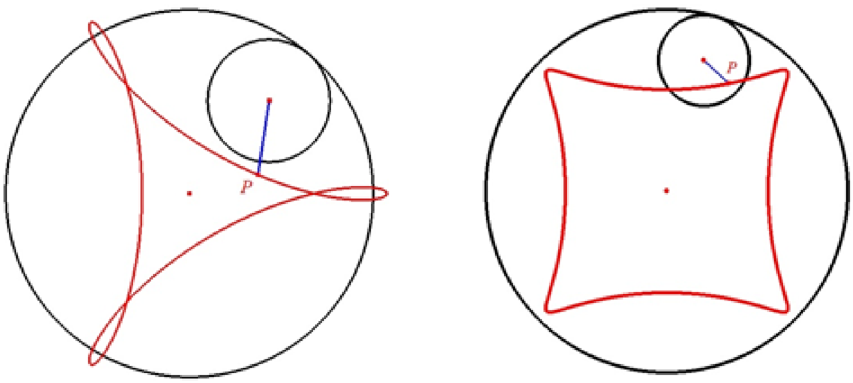


Fig. 7

The hypotrochoid has hypocycloid as special case, including Deltoid and Astroid (Fig. 8).

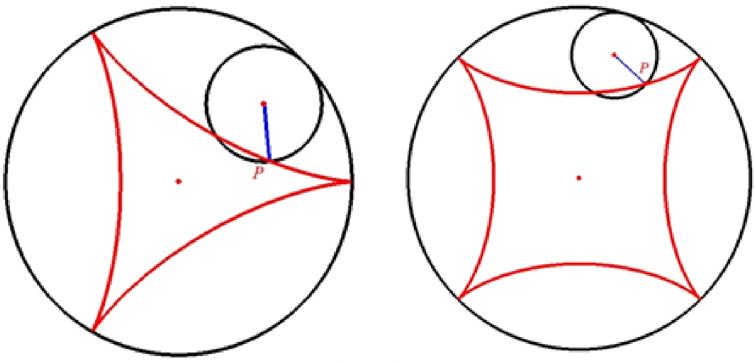


Fig. 8

III. Lemniscate: In algebraic geometry, a lemniscate (from Latin lēmniscātus meaning "decorated with ribbons") may refer to any of several figure-eight or ∞ shaped curves, including the Lemniscate of Bernoulli as special case (Fig. 9).

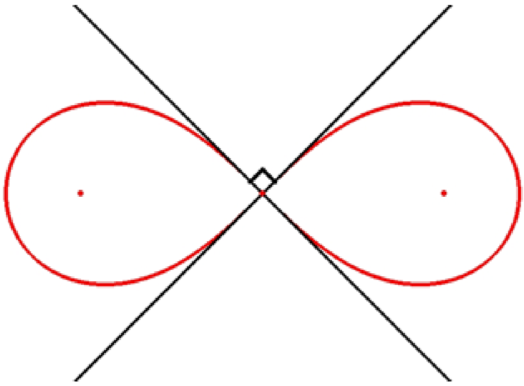


Fig. 9

In this project, The Lemniscates are given in polar coordinates by

$$r^2 = 4 - 4a^2 \, sin^2 \theta, \;\; a \text{ constant}......(4)$$

(Fig. 10).

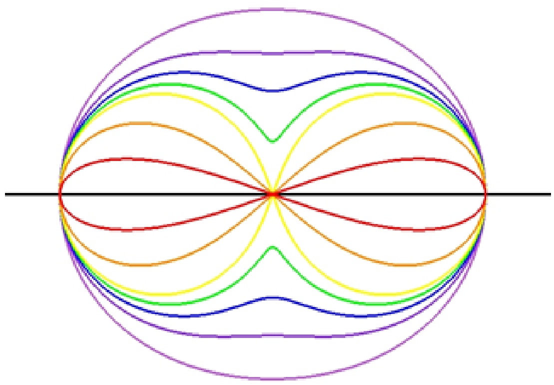


Fig. 10

Chapter 3 Linkages

I. Peaucellier Inversor: Invented in 1864, it was the first planar lnkage capable of transforming a rotary motion into a perfect straight line motion, and vice-versa (Fig. 11).

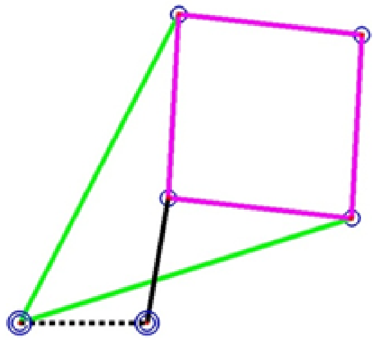


Fig. 11

II. Hart's Inversor: It is also known for transforming a rotary motion into a straight line motion.

<Proof>: (Fig. 12) Assume the ratio of $\overline{AO}$ to $\overline{OB}$ is $m{:}n$. Take perpendicular bisector through A, C to $\overline{AC}$ and obtain the intersections E, F the extension of $\overline{BD}$ and the bisectors, so

$$\overline{AC}\cdot\overline{BD} = \overline{EF}\cdot\overline{BD} = (\overline{ED}+\overline{EB})(\overline{ED}-\overline{EB}) = \overline{ED}^2-\overline{EB}^2,$$

then $\overline{AD}^2 = \overline{AE}^2+\overline{ED}^2$ and $\overline{AB}^2 = \overline{AE}^2+\overline{EB}^2$

Now $\overline{AD}^2-\overline{AB}^2 = \overline{ED}^2-\overline{EB}^2$

$$\frac{\overline{OP}}{\overline{BD}} = \frac{m}{m+n}\ ,\ \frac{\overline{OP'}}{\overline{AC}} = \frac{n}{m+n}$$

$$\overline{OP}\cdot\overline{OP'} = \frac{mn}{(m+n)^2}\overline{AC}\cdot\overline{BD}$$

Therefore $\overline{OP}\cdot\overline{OP'}$ is constant.

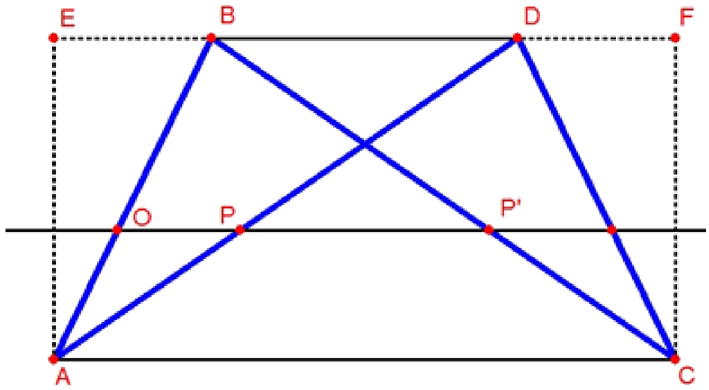


Fig. 12

III. Kite and Dart: The kite and dart are the famous four-bar linkage with two links and joins (Fig. 13).

There are two rotors make the whole linkage moving. It is a special case of Watt's Linkage.

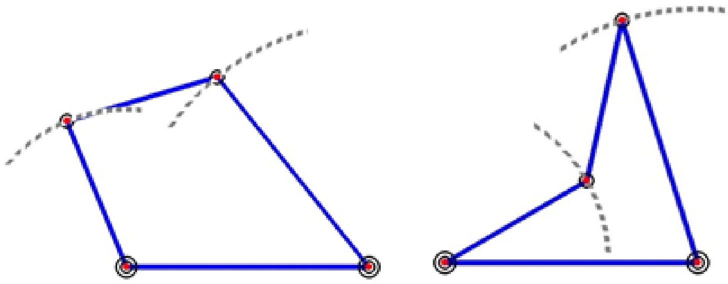


Fig. 13

IV. Watt's Linkage: Watt's linkage consists of a chain of three rods, two longer and equal length ones on the outside ends of the chain, connected by a short rod in the middle (Fig. 14). The outer endpoints of the long rods are fixed in place relative to each other, and otherwise the three rods are free to pivot around the joints where they meet. Thus, counting the fixed-length connection between the outer endpoints as another bar, Watt's linkage is an example of a four-bar linkage.

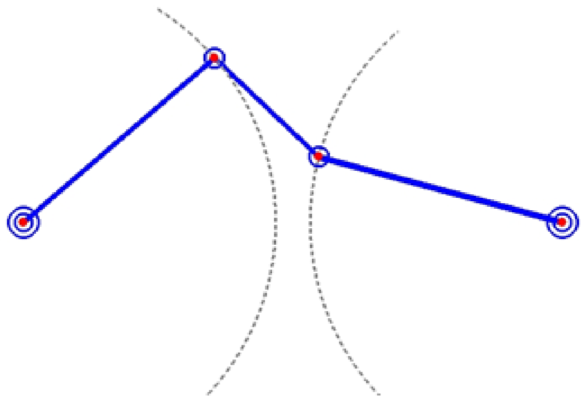


Fig. 14

Chapter 4 Results

Designing linkage took up a handful of mathematical minds before the 1900's [3]. Most of the work appears isolated from one another. Borrowing from the prime factorization of an integer, the first step in a systematic method in designing linkage is to identify the most important component, to be called the decomposition throughout the project, and the rest follows recursively. Just as the case of integers, there is no universal method of decomposition. This project focuses on the decomposition for three families of curves:

A. Decomposition for Conchoids of de Sluze

B. Decomposition for Trochoids

C. Decomposition for Lemniscates

I. Decomposition for Conchoids of de Sluze

A. Formation (Fig. 15)

Conchoids of de Sluze are given by the equation $r=\cos\theta+a\sec\theta$.

1. Draw a circle having the center at the origin O and (1,0) as diameter.

2. Draw a variable line through O and a variable point P' on the circle.

3. Fix a constant a on the x-axis, then draw the perpendicular bisector to the x-axis through (a,0) to meet the variable line at Q.

4. Take the reflection of O w.r.t. the perpendicular bisector of P' and Q at P.

5. The locus of P as P' moving is the required curve, conchoids of de Sluze.

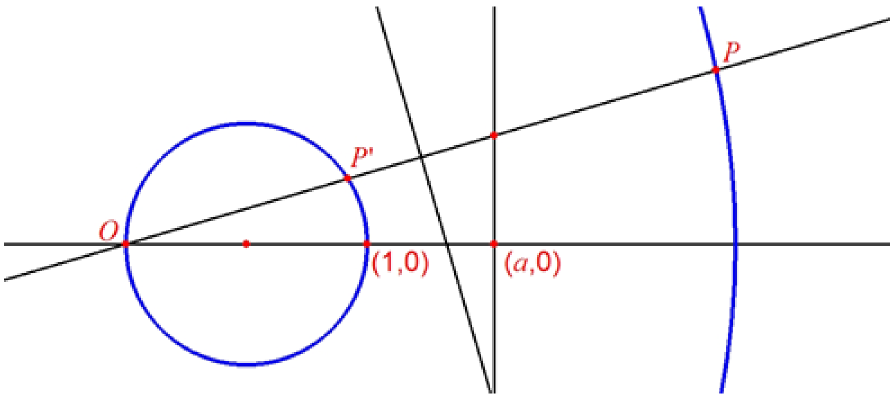


Fig. 15

B. Decomposition

1. Fix a constant L>2.

2. A unique rhombus having O, P as opposite vertices and side L can be constructed.

   Since $\overline{\mathrm{OP}'}=\cos\ \theta$, it follows that $\overline{OP'} \cdot \overline{PP'}$ is constant a.

   The decomposition holds for all points P on the curve with $\overline{OP'} < 2L$.

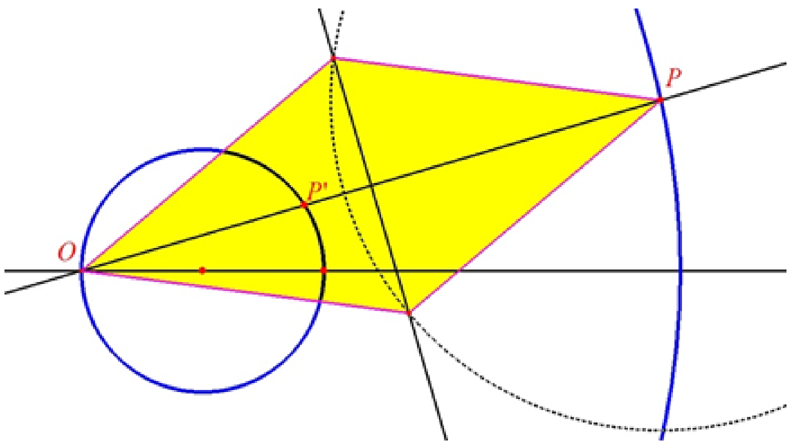


Fig. 16

C. The Linkage to Draw the Conchoids

The complete linkage is formed by attaching the four green bars to the rhombus as indicated (the dotted bar fixed, Fig. 17) .

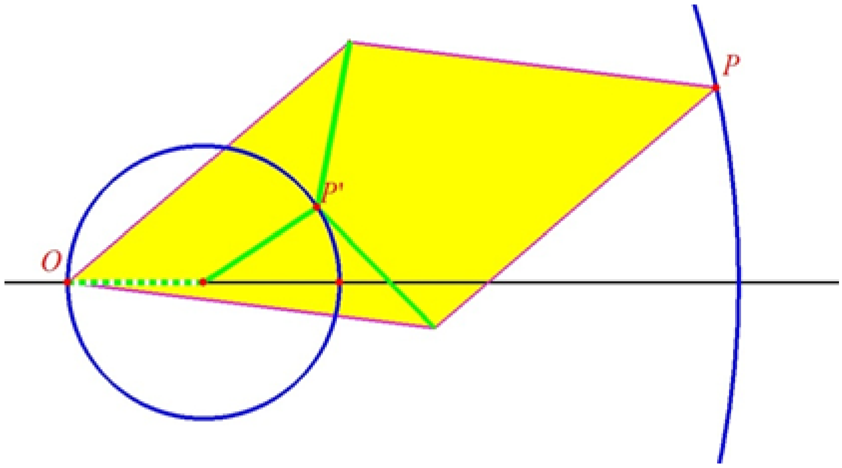


Fig. 17

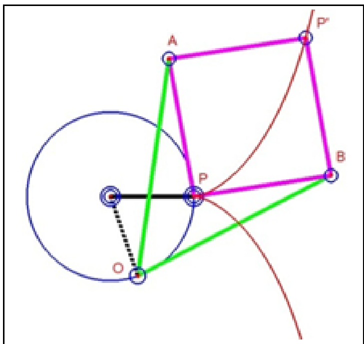
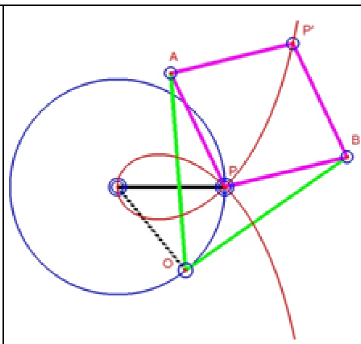
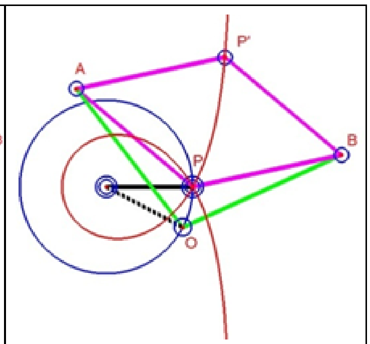
D. Special Cases

By the decomposition, the linkage is shown to draw the conchoids of de sluze and treats all of them, including the Cissoid of Diocles, the Right Strophoid and the Trisectrix of Maclaurin as special cases

1. The Cissoid of Deocles

2. The Right Strophoid

3. The Trisectrix of Macaulrin

		
Cissoid of Diocles	Right Strophoid	Trisectrix of Macluarin

II. Decomposition for Trochoids

A. Formation

Trochoids are given by the complex equation $a \cdot ne^{i\theta} + b \cdot e^{ni\theta}$.

1. Epicycloid (For example: the Cardioid, Fig. 18)

a. Let O be a circle.

b. Take two points P, Q having polar angle θ and 2θ.

c. Take dilation of Q w.r.t. P with the factor 1/3 at R.

d. The required typical epitrochoid, the Cardioid, is the locus of R as θ varying.

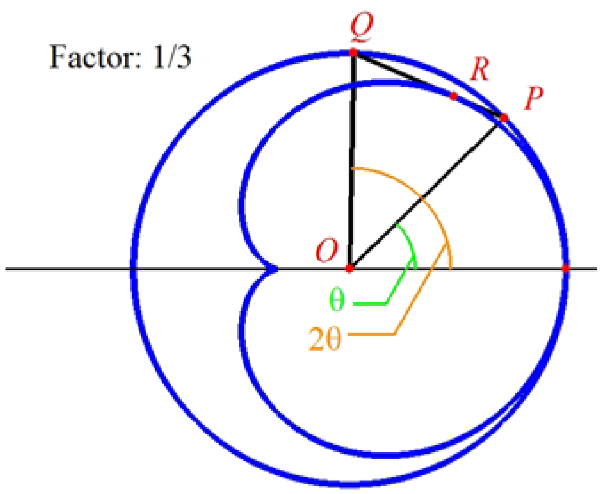
2. Epitrochoid (Fig. 19)

a. Let O be a circle

b. Draw a line through two points P, Q having polar angle θ and 2θ.

c. Fix a constant ratio between P, Q at R.

d. The required epitrochoid is the locus of R as θ varying.



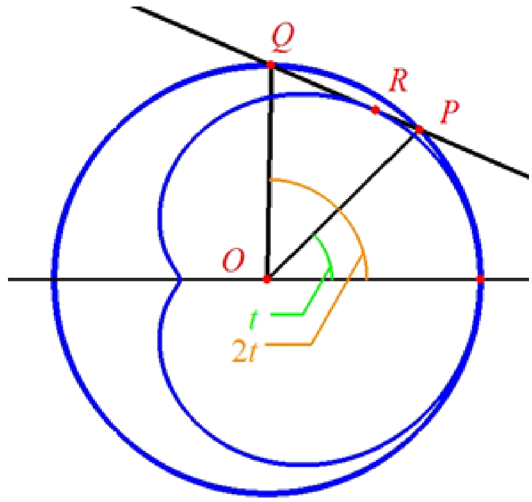


Fig. 18 Fig. 19

3. Hypocycloid (For example: the Deltoid, Fig. 20)

a. Let O be a circle.

b. Take two points P, Q having polar angle θ and -2θ.

c. Take dilation of Q w.r.t. P with the factor 1/3 at R.

d. The required typical hypotrochoid, the Deltoid, is the locus of R as θ varying.

4. Hypotrochoid (Fig. 21)

a. Let O be a circle

b. Draw a line through two points P, Q having polar angle θ and -2θ.

c. Fix a constant ratio between P, Q at R.

d. The required epitrochoid is the locus of R as θ varying.

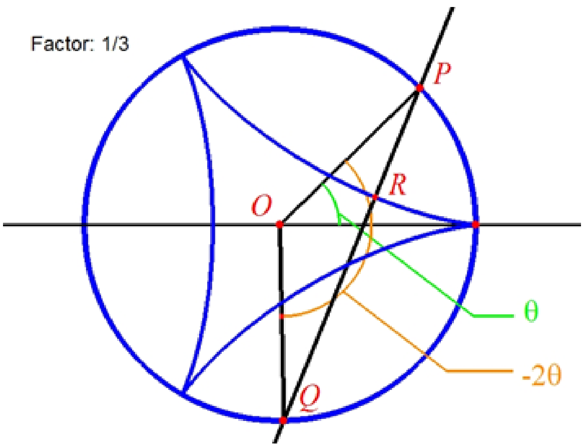


Fig. 20

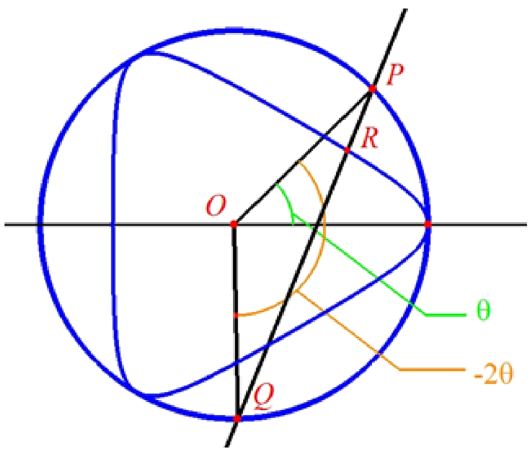


Fig. 21

B. Decomposition

1. Take projection P' through R parallel to $\overline{OQ}$.

2. Take projection Q' through R parallel to $\overline{OP}$.

3. The required decomposition is the parallelogram $OP'RQ'$.

The parallelogram OP' RQ' with $P'=a\cdot ne^{i\theta}$, $R=a\cdot ne^{i\theta}+b\cdot e^{ni\theta}$ and $Q'=b\cdot e^{ni\theta}$ forms the required decomposition. Note that the sides of the parallelogram remain constant for all θ (Fig. 22).

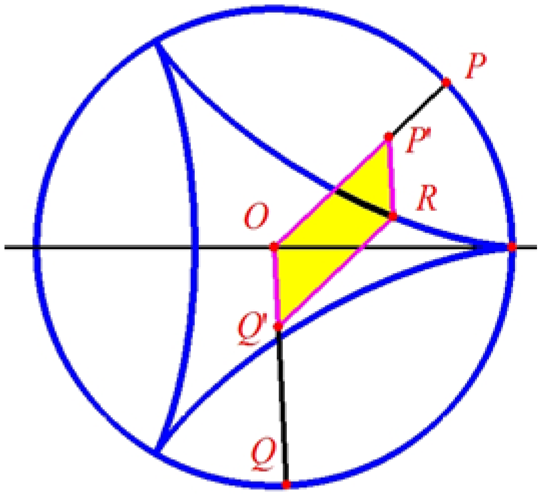


Fig. 22

C. The Linkage to Draw the Trochoids

The remaining part of the linkage is formed by attaching a series of rhombi to maintain the relative angular velocity as indicated by the green (Fig. 23) .

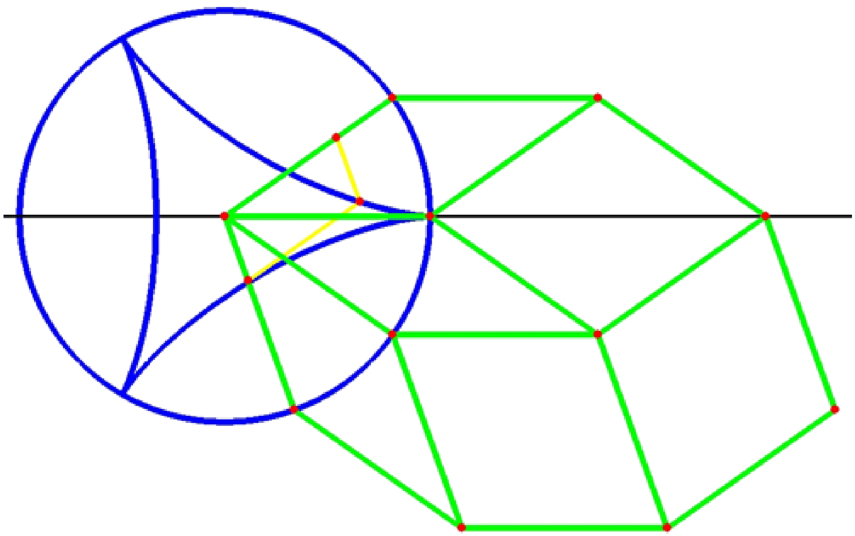


Fig. 23

III. Decomposition for Lemniscates

A. Formation

The lemniscates are given in polar coordinates by $r^2=4\text{-}4a^2 \sin^2\theta$, a constant.

1. Draw a unit circle, then choose a variable point A having polar angle θ.

2. Fix a constant a on the x-axis at D.

3. Take the reflection of O w.r.t. $\overline{AD}$ at C.

4. Take the reflection of C w.r.t. A at P.

5. The required Lemniscates is the locus of P as A varying.

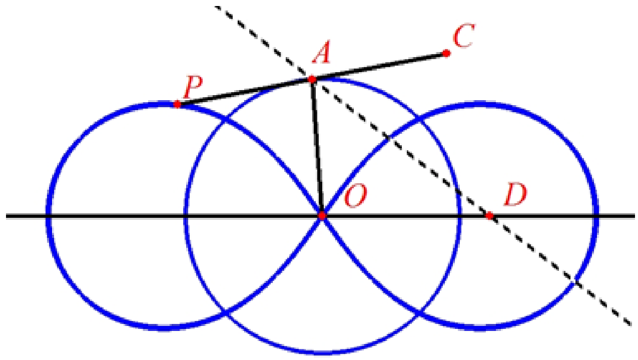


Fig. 24

When a>1, the curve was called Lemniscate (Fig. 24); when a<1, the curve was called Hippopede (Fig. 25).

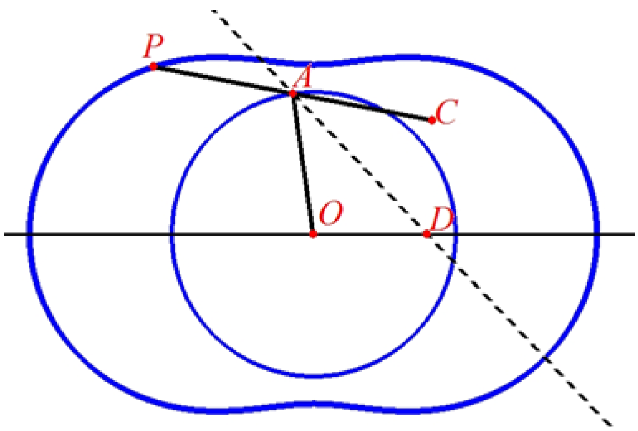


Fig. 25

B. Decomposition (Fig. 26)

1. Take reflection of A w.r.t. $\overline{OP}$ at B.

2. The required decomposition is the rhombus $AOBP$.

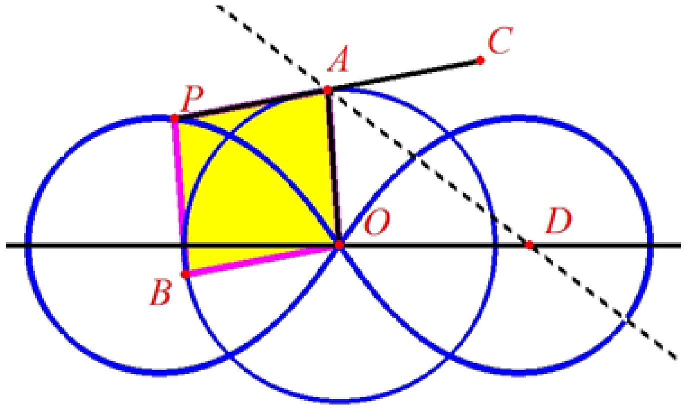


Fig. 26

C. The Linkage to Draw the Lemniscates

There are two linkage designed by the decomposition shown to draw the Lemniscates.

1. The first linkage consists of the kite $ACDO$ and the additional bar $\overline{PA}$ (Fig. 27).

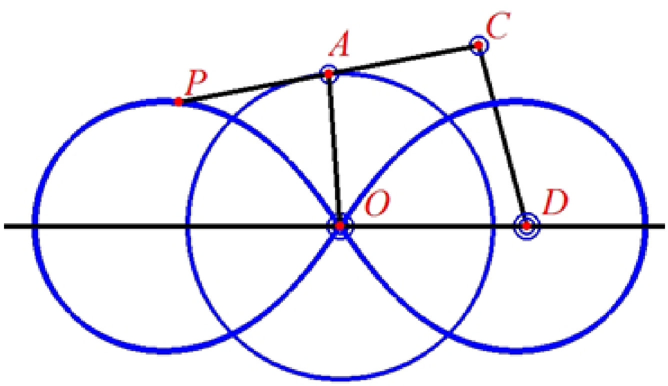


Fig. 27

2. The second linkage is formed by the anti-parallelogram DED' E' where E is the reflection of D w.r.t. O and $\overline{D'E'}$ is the reflection of $\overline{DE}$ w.r.t. $\overline{AB}$ (Fig. 28).

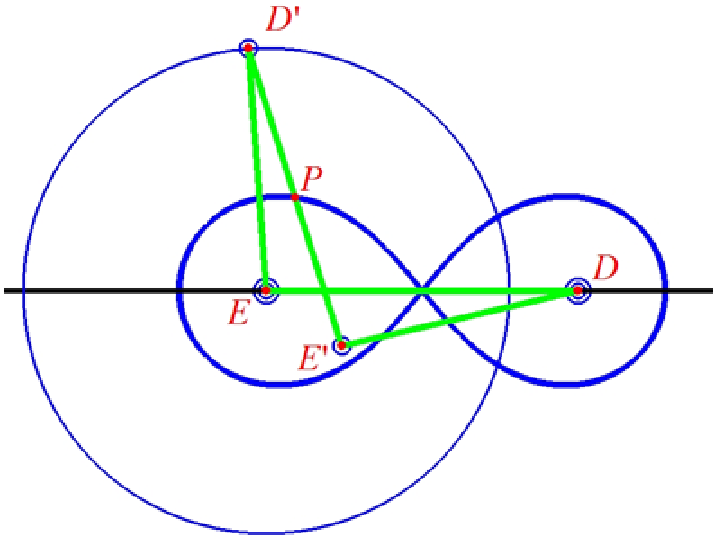


Fig. 28

Chapter 5 Applications

I. New Linkages Formed by hooking up Linkages

The new combination can be invented by hooking up the output of the first linkage with the input of the second linkage.

A. Ellipsograph

Each of the linkages to draw the lemniscates has two kinds of bars, either short or long. By readjusting the relative lengths, the modified linkages can be used to draw the hippopede, the inverse of the ellipse of the same center. Therefore the ellipsograph may be formed by the output of any inversor (such as the Peaucellier Cell or Hart's Inversor) whose input is hooked up with the output of the modified linkage (Fig. 29, 30, 31, 32).

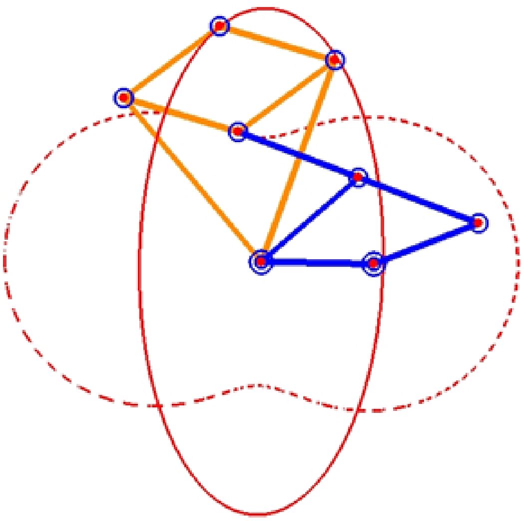


Fig. 29

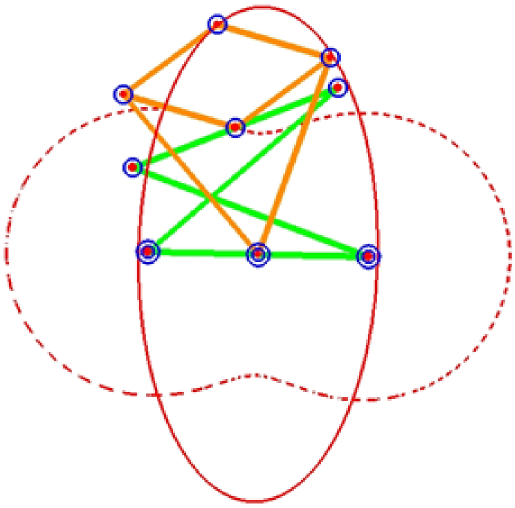


Fig. 30

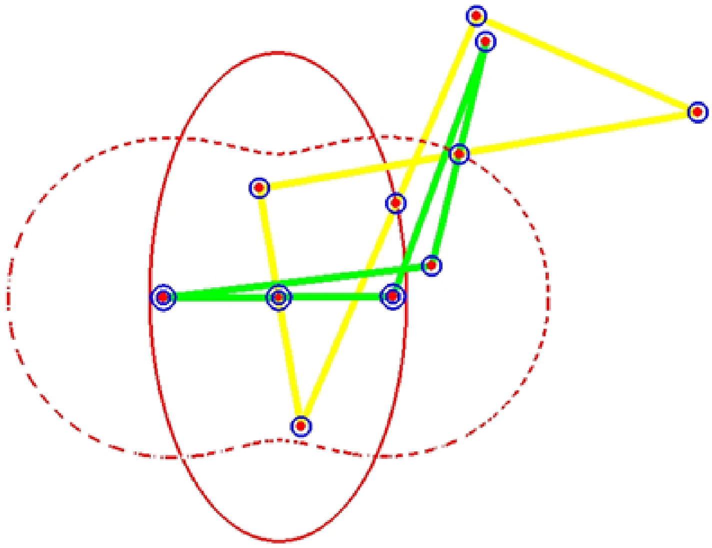


Fig. 31

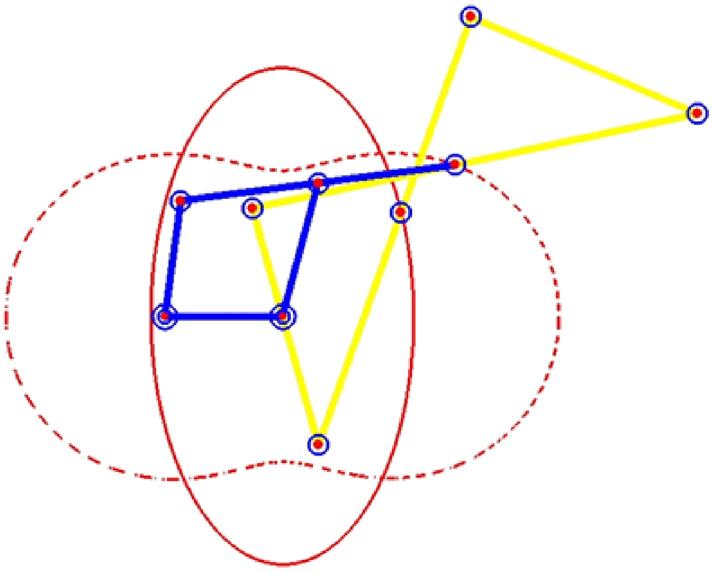


Fig. 32

B. Special Interesting Curve

In the same sence of the design, the linkage to draw the special curve is invented.

The output of the linkage of Trochoids hooks up with the input of Peaucellier Cell, then the locus of the output of the Peaucellier Cell is observed to trace an interesting curve (Fig. 33).

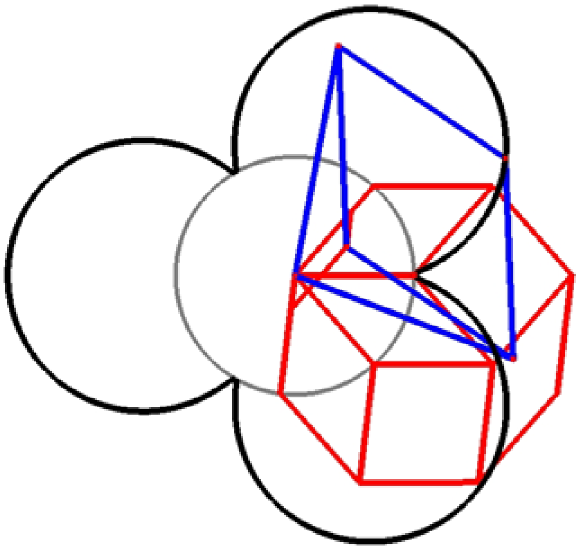


Fig. 33

II. Double Generation Property of Trochoids

Every trochoid may be generated in two ways: two pairs of circles are shown to trace out the same curve, with each pair consisting of a fixed circle and a disk with different ratios in radius and rolling in opposite directions. The visual demonstration can be constructed from decomposition for trochoids (Fig. 34):

a. The rolling circles are centered at the opposite vertices of the parallelogram.

b. The points of contact are located at $P''=a\cdot(n+1)\cdot e^{i\theta}$, $Q''=b\cdot(n+1)\cdot e^{ni\theta}$.

Note that the three points P'', Q'' and R all lie on the normal of the curve.

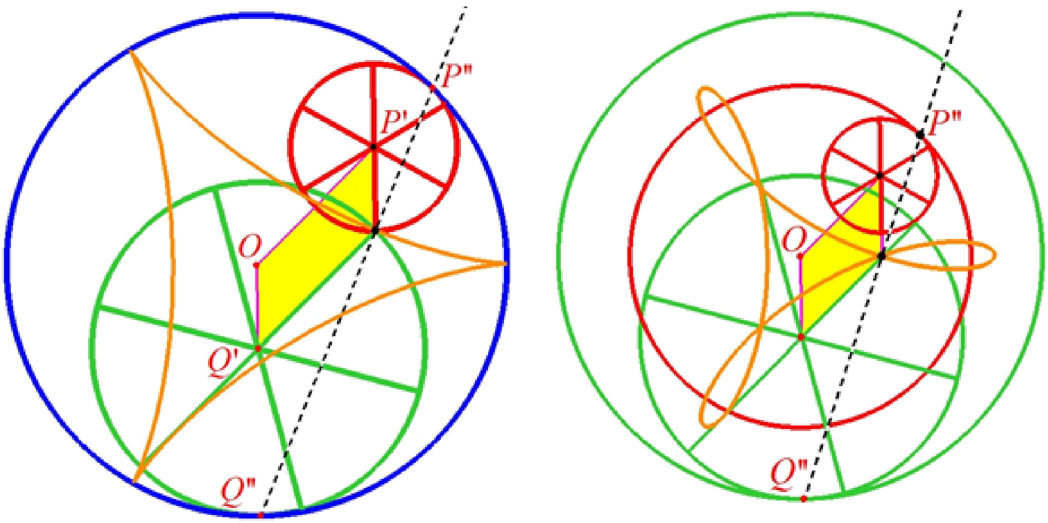


Fig. 34

The Other Case

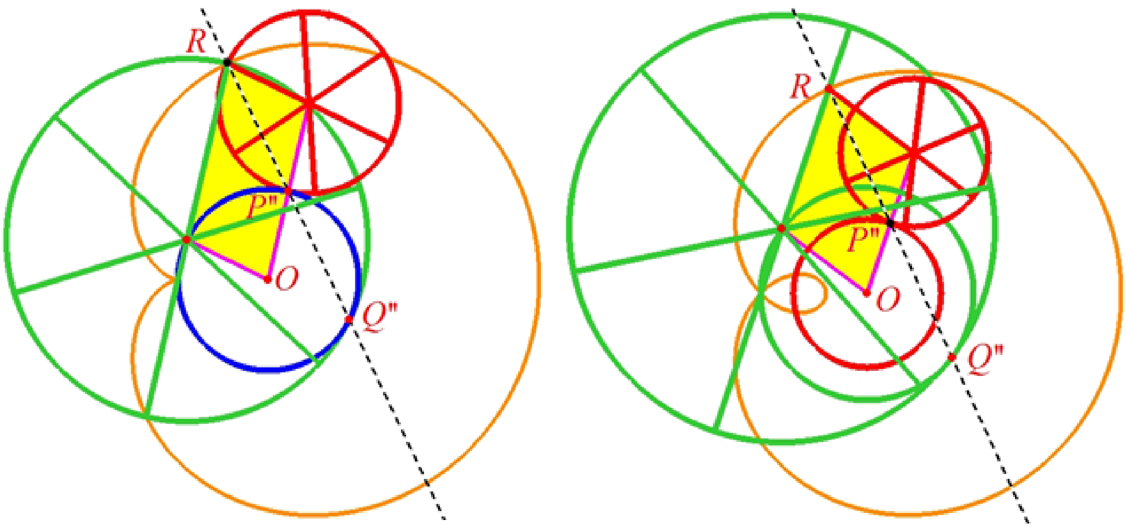


Fig. 35

III. Tangent Construction, No Calculus Needed

In Differential Geometry, the tangent is determined by the velocity which is obtained by taking the first derivative. For each construction below, the tangent is drawn perpendicular to the normal direction determined by the linkage.

A. Tangent of Conchoids of de Sluze

1. Draw the diagonal of the decomposition

2. Draw a line through O parallel to x-axis to meet the diagonal at S

3. The line (SP')¯ is the normal direction.

4. The line through P perpendicular to the normal is the tangent.

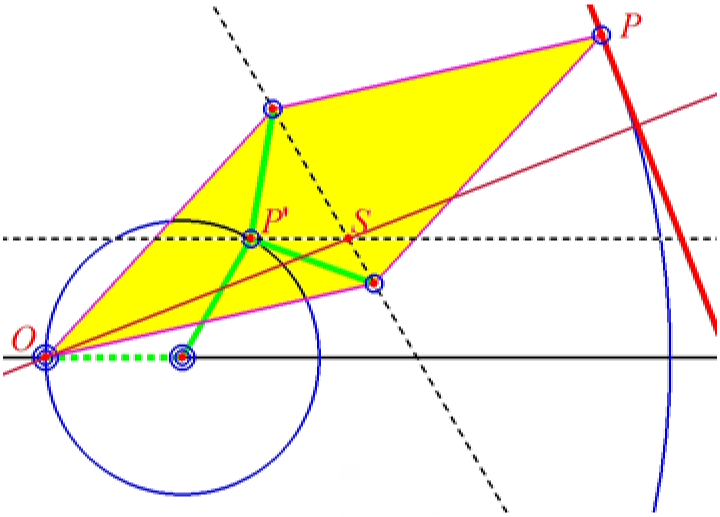


Fig. 36

B. Tangent of Trochoids

1. Extend the decomposition according to the relative angular velocity.

2. Draw the diagonal of the bigger parallelogram. It is the normal.

3. The line through R perpendicular to the normal is the tangent.

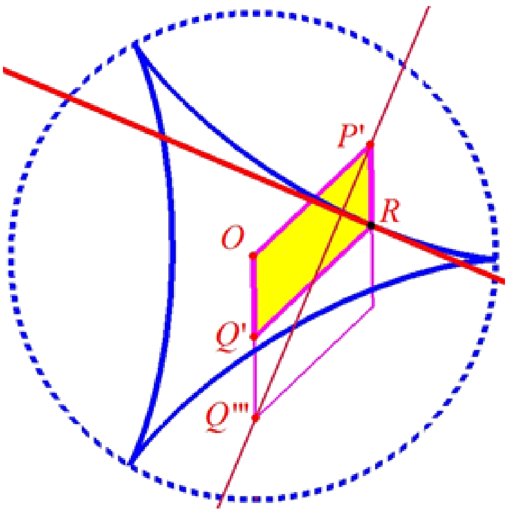


Fig. 37

C. Tangent of Lemniscates

1. Draw the diagonal of the decomposition.

2. Extend the rotor to meet the diagonal at *S*.

3. $\overline{SR}$ is the normal to the curve.

4. The line through *R* perpendicular to the normal is the tangent.

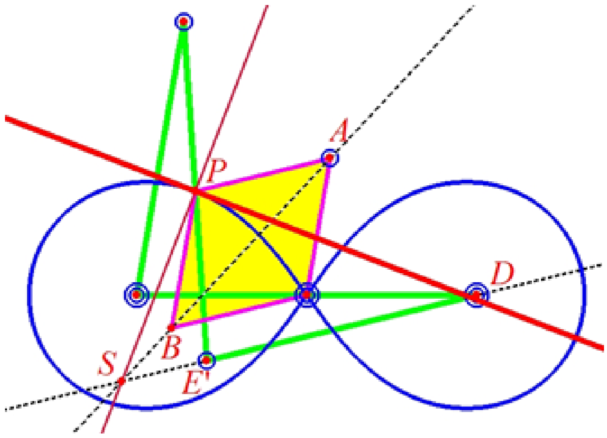


Fig. 38

IV. Osculating Circle Construction, No Calculus Needed

In Differential Geometry, the center of curvature is given in terms of the second derivative. Based on the decomposition, the construction of the osculating circle without going through the limiting process is possible.

A. Mapping

1. Draw the normal to the curve as above.

2. Through the point of contact as in Application 2, draw the parallel to the tangent to meet the extension of $\overline{P'R}$ at *S*.

3. The required osculating circle has the intersection of $\overline{OS}$ with the normal to the curve as the center, the center of curvature.

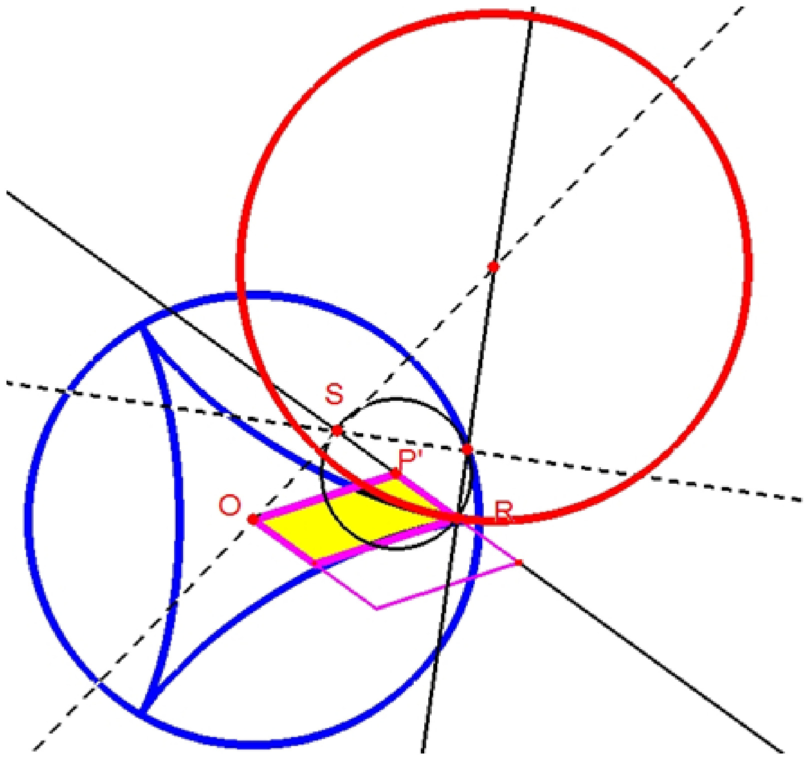


Fig. 39

To check the validity of the construction, we apply the built-in command "envelope" in software Cabri II to the normal. The center of curvature is observed to lie on the evolute of the trochoid.

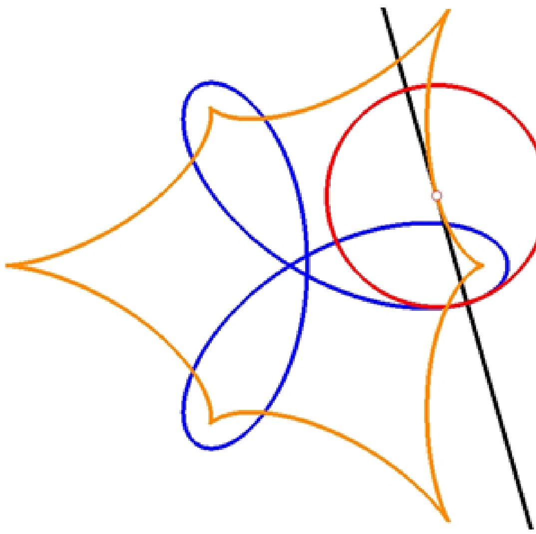


Fig. 40

B. Definision of κ (Kappa)

For a plane curve $(x(t), y(t))$ its curvature is given by

$$\kappa = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

The quantity

$$r = \frac{1}{\kappa} = \frac{((x')^2 + (y')^2)^{\frac{3}{2}}}{|x'y'' - x''y'|}$$

is called the **radius of curvature.**

The point (x_c, y_c) with coordinates given by

$$\mathrm{c}x_c = x - \frac{(x')^2+(y')^2}{x'y''-x''y'}y'$$

$$y_\mathrm{c} = y + \frac{(x')^2 + (y')^2}{x'y'' - x''y'}x'$$

is called the **center of curvature.**

The circle with center (x_c, y_c) radius r is called the osculating circle or the circle of curvature. The center of curvature (x_c, y_c) lies on the normal of the curve at the point (x, y).The line segment joining (x, y) with (x_c, y_c) is perpendicular to the tangent at (x, y).

C. Proof in Maple

```
> with(plots);
```

[animate, animated3d, animatecurve, arrow, chanecoords, complexplot3d, conformal, conformal3d, contowplot, contowplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, graphplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listconplot, listconplot3d, listdensityplot, listplot, listplot3d,loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot,

polyhedral_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setopions3d, spacecurve, sparsematrixplot, surfdata, texplot, textplot3d, tubeplot]

```
> x:=n*cos(t)-cos(n*t);y:=n*sin(t)-sin(n*t);
```

$$x := n\cos(t) - \cos(n\,t)$$

$$y := n\sin(t) - \sin(n\,t)$$

```
> n:=-2;
```

$$n := -2$$

```
> x1:=diff(x,t);y1:=diff(y,t);
```

$$x1 := 2\sin(t) + 2\sin(2\,t)$$

$$y1 := -2\cos(t) + 2\cos(2\,t)$$

```
> x2:=diff(x1,t);y2:=diff(y1,t);
```

$$x2 := 2\cos(t) + 4\cos(2\,t)$$

$$y2 := 2\sin(t) - 4\sin(2\,t)$$

```
> m:=(x1^2+y1^2)/(x1*y2-x2*y1);
```

$$xc := -2\cos(t) - \cos(2\,t) - ((2\sin(t) + 2\sin(2\,t))^2 + (-2\cos(t) + 2\cos(2\,t))^2)\,(-2\cos(t) + 2\cos(2\,t))/((2\sin(t) + 2\sin(2\,t))\,(2\sin(t) - 4\sin(2\,t)) - (2\cos(t) + 4\cos(2\,t))\,(-2\cos(t) + 2\cos(2\,t)))$$

```
> xc:=x-m*y1; yc:=y+m*x1;
```

$$xc := -2\cos(t) - \cos(2\,t) - ((2\sin(t) + 2\sin(2\,t))^2 + (-2\cos(t) + 2\cos(2\,t))^2)\,(-2\cos(t) + 2\cos(2\,t))/((2\sin(t) + 2\sin(2\,t))\,(2\sin(t) - 4\sin(2\,t)) - (2\cos(t) + 4\cos(2\,t))\,(-2\cos(t) + 2\cos(2\,t)))$$

$$yc := -2\sin(t) + \sin(2\,t) + ((2\sin(t) + 2\sin(2\,t))^2 + (-2\cos(t) + 2\cos(2\,t))^2)\,(2\sin(t) + 2\sin(2\,t))/((2\sin(t) + 2\sin(2\,t))\,(2\sin(t) - 4\sin(2\,t)) - (2\cos(t) + 4\cos(2\,t))\,(-2\cos(t) + 2\cos(2\,t)))$$

```
> r:=(x1^2+y1^2)^(3/2)/abs(x1*y2-x2*y1);
```

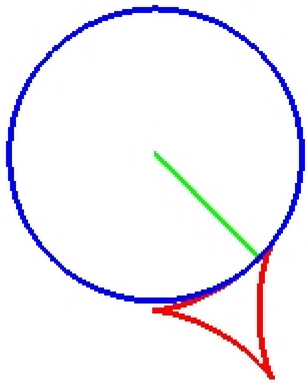
$$r := ((2\sin(t) + 2\sin(2t))^2 + (-2\cos(t) + 2\cos(2t))^2)^{(3/2)} / |(2\sin(t) + 2\sin(2t))(2\sin(t) - 4\sin(2t)) - (2\cos(t) + 4\cos(2t))(-2\cos(t) + 2\cos(2t))|$$

```
>
aa:=animate([xc+r*cos(s),yc+r*sin(s),s=0..2*Pi],t=0
..2*Pi,frames=200,color=blue,thickness=3):

> bb:=plot([x,y,t=0..2*Pi],thickness=3,color=red):

>
cc:=animate([(1-s)*xc+s*x,(1-s)*yc+s*y,s=0..1],t=0.
.2*Pi,frames=200,color=green,thickness=3):

> display(aa,bb,cc,axes=none,scaling=constrained);
```



V. Clock Face Designed by Linkages

Each linkage studied here can be applied directly to design the clock face.

Fig. 41 is a clock face inspired by the Peaucellier Cell.

Fig. 42 is a clock face inspired by the anti-parallelogram.

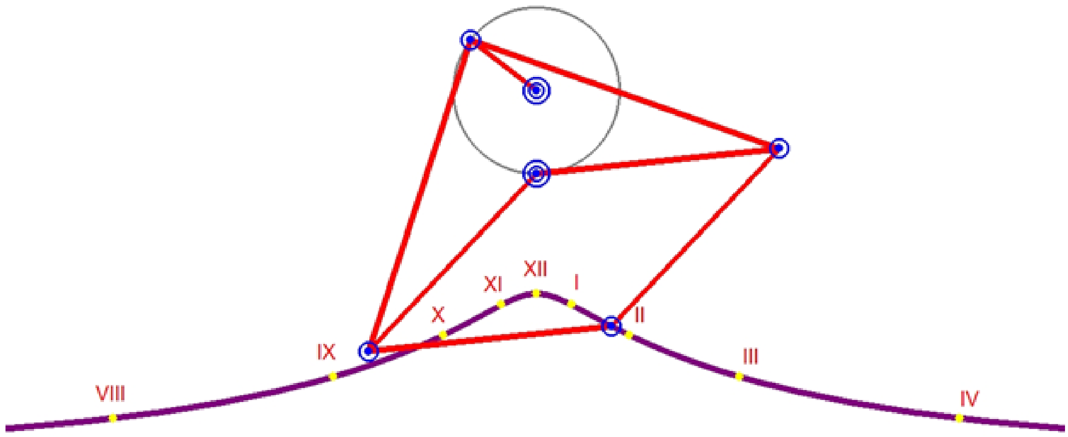


Fig. 41

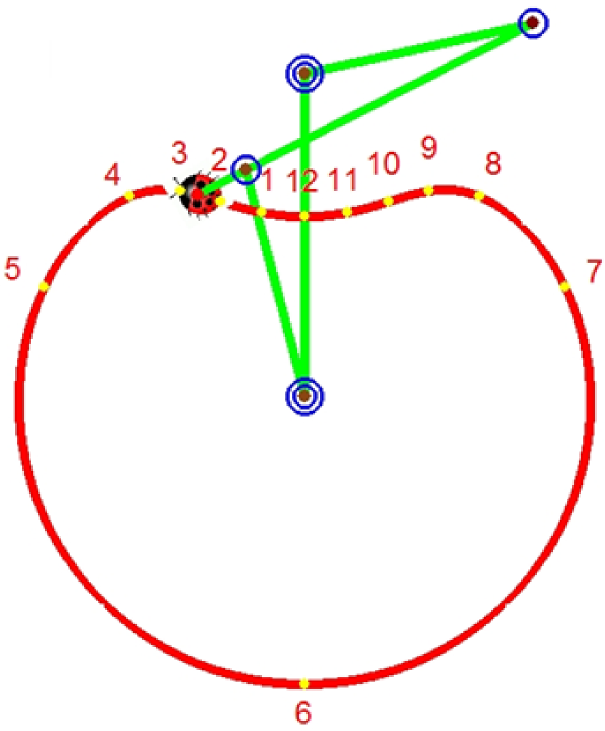


Fig. 42

Chapter 6 Conclusions

1. The decomposition identifies the most important component of the linkage.

2. Complicated linkages such as the ellipsograph can be built by hooking up the basic ones such as the Peaucellier Cell and the anti-parallelogram.

3. It is shown visually that every trochoid may be generated in two ways.

4. No Calculus is needed to construct the tangent and the osculating circle of trochoids.

5. Linkages can be regarded as a natural setting to study the relative motions, an important step behind the Theory of Relativity.

Chapter 7 References

[1] A.B. Kempe, How to draw a straight line; a lecture on linkages, Macmillan, London, 1877.

[2] E.H. Lockwood, A Book of Curves, Cambridge University Press, 1961.

[3] MacTutor History of Mathematics Archive. http://www-groups.dcs.st-and.ac.uk/~history/Curves/Curves.html.

[4] R. C. Yates, Curves and Their Properties, NCTM, 1974.

[5] Richard Courant/H. E. Robbins, What is mathematic, Oxford University Press, 1941. p.157-p.158

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Mr. Dan-Wei Lin, National Hsinchu Commercial Vocational High School

評語

本作品考慮有限形態的等比螺旋線的收斂點的問題，不同角度的收斂點軌跡形成一個圓，作者也給出一個證明，除此之外，作者對於有限點的軌跡，以及收斂點之間的比值也有進一步的探討。從這個作品中可以感受到作者對於數學的熱忱。再者作者也得到有限模擬的圖形與許多自然界現象的連結，這些連結雖然並不見得可以建立在紮實的理論上，但是光是作為呈現數學成果，即已經是可喜的嘗試。