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作品名稱

傑克船長的心機

The Trick of Treasures

得獎獎項

數學科大會獎一等獎

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我喜歡閱讀、打球、摺紙,最喜歡的則是數學。

回顧第一次接觸數學競賽後,開啟了我數學美麗殿堂的大門。也因此有機會參加各式各樣不同的競賽,不但增廣見聞,我也結識了許多同好。更從此深深著迷於變化多端的數學。除了上學的時間外,假期中,我參加了一些數學活動及俱樂部,如:九章數學愛好者聯誼、數學夏令營等,最有趣的還是加拿大 Alberta大學辦的「加拿大青少年數學夏令營」。

Abstract

Captain Jack and his pirates had acquired some treasure chests as loot. Each treasure chest was known to contain some gold and some diamonds. The crew agreed to let Captain Jack take half of the chests for himself.

Captain Jack would like to have at least half the gold and at least half the diamonds. Could he be guaranteed to do so, for any distribution of gold and diamonds among the chests?

The problem has been solved by means of higher mathematics. This project uses only elementary mathematics, and deals with some related problems.

The approach may applied to the distribution of relief goods among disaster areas, or the sharing of intangible resources like professional expertise or versatile machines.

摘要

傑克船長和他的海盜們掠奪到許多箱珍寶，每箱含有數量不等的金幣及鑽石。船員們深怕傑克船長又出什麼陰謀，一致同意讓船長任選一半的箱子拿走。

當然，傑克不知道金幣和鑽石的價格比，為了保證可以得到一半的利益，傑克希望他拿到的金幣和鑽石都各占一半。傑克的願望會實現嗎？

這個問題已用高等數學證明其解，而本研究利用初等數學的方法，除證明傑克需取的最少箱子數外，同時也能更快速的算出取法。

類似的結果可應用至分配災區物資等情形，或任何無法轉移、獨立的資源，如各式專長的人才、多功能的機械等，期望可對更有效的分配做出貢獻。

壹、前言

(一)、研究動機

前陣子和教授吃飯時，教授講了一個小故事：

海盜們搶了一些裝有金幣和鑽石的箱子，分贓時，船長希望可以拿到至少一半的黃金及一半的鑽石，請問他至少要拿多少箱子？

其實很容易就可以發現，在大部分的情況下，只要拿的比一半還多一點點，就可以達到題目的要求。我們的解釋是：即使在拿取箱子時，傑克會同時拿到金幣及鑽石，但無論傑克是如何「一魚兩吃」的，拿一半箱子、吃一半魚，聽起來合情合理。

但是另一方面，由於有價物品是可以隨意分配的，不僅要明確地找出有效的拿法，並且要構造出符合解答的模型，最重要的，是找出這兩者的規律以便推廣。於是我開始著手這個有趣的題目及其推廣。

(二)、 研究目的

針對任意的：

1.r，比例，代表傑克想要得到多少比例的東西。

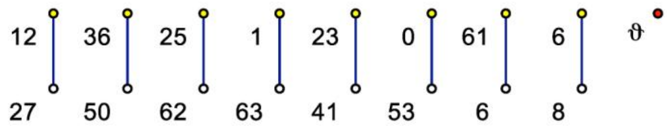
2.k，有價物品的種類，代表箱子裡共有多少種東西。

3.n，箱子的數量，代表共有多少個箱子待選。

找出無論有價物品如何分布（代表船員可以隨意配置這些東西），傑克至少需要拿幾個箱子，使得對任意的有價物品，他拿到的部分至少占了全部的r。

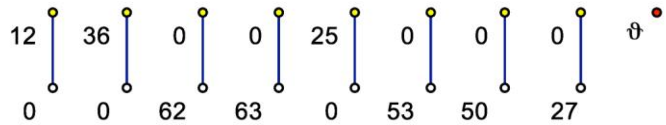
更精確地說，這並不是為「每一種物品分布」找出最少箱子解，而是傑克在得知（或者說他想要的）比例、種類、數量這三者之後，他就必須決定出一個數字，使得無論這些物品是如何分布的，他都可以只取這麼多個箱子，就達到他的

目標。下面用1/2、兩種物品、八個箱子為例。一條線段代表一個箱子；黃點代表金幣、白點代表鑽石；左下角的數字是重量，單位是盎司（黃金）、克拉（鑽石）；右上角的紅點則是傑克（圖片定位用）。如圖：

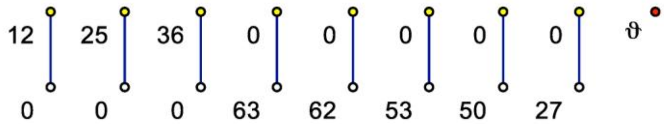


大豐收阿！這時傑克只要拿這四個箱子即可。如圖：

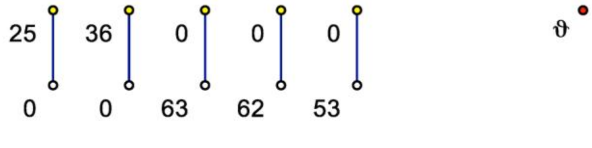
當然事情不會總是那麼順利，船員可能偷偷動過一些手腳，至於怎麼做到的，就是另一個故事了。如圖：



適當的排序總是必要的。如圖：



雖然賣鑽石賺的錢幾乎可以填補黃金的差額，不過為了保險起見，傑克還是得拿五個箱子。如圖：



但不會因為「五」這個數很大或「5/8」很大就代表他是答案（雖然的確是沒錯）。充其量只能說明「至少要五個」而不是「五個就夠了」。本研究將致力於這兩者（分別是數字的上、下限）的證明。

貳、研究方法

（一）定義：≡ (r, k, n)

這個函數代表在某種情況下，傑克至少需要拿多少個箱子才能滿足心中的慾望，其中r是比例、k是有價物品的種類、n是箱子的總數。這樣子的順序是為了讓數字小的在前、也較好記。

（二）引理1 - 均分金幣

這個引理是說，我們可以將箱子平均分成兩堆，使得兩堆的金幣數量差不超過單一箱子最多金幣的量。以下是較正式的表述：

將箱子編號為$A_1, A_2, A_3, \ldots, A_{2n}$，再重複令$A$（沒有足碼）是裝最多金幣的箱子、而$a$（也沒有足碼）是其數量，首先將箱子大略的分成兩堆，我們希望這兩堆的金幣差不超過a。

如圖：

$$A_1, A_2, A_3, \ldots, A_n$$

$$A_{n+1}, A_{n+2}, A_{n+3}, \ldots, A_{n+n}$$

如果上下兩堆的總量差不超過a，那麼均分就結束了。若否，則在多的那一堆（假設是上面）中，必存在某個箱子的金幣比少的那一堆（假設是下面）的某個箱子還多。這並不是說最多金幣的箱子會在總量多的那一堆，而是點出我們可以藉由交換其中兩個箱子使得雙方的差距縮小。

由於金幣的數量：$a_1, a_2, a_3, \ldots, a_{2n}$是有限的，因此他們的「組合」（指的是形如$a_2+a_3-a_5+a_8$這種係數為-1, 0, 1的線性組合）也是有限的，換句話說，每交換一次箱子，雙方的差距至少會縮小某個固定的量（這是因為良序性）。於是較少（下

面）的一堆的金幣數遲早追過較多（上面）的一堆。

就在追過的這一次交換中，我們知道差距的變化量取決於被交換的兩個箱子的金幣差，顯然無論哪兩個箱子的金幣差都不會超過最多的金幣，即$|a_i-a_j|\leq a_i\leq a$，故此時雙方差距不超過a、符合條件。

這個引理隱含的意思是：如果我們把一堆箱子中最多金幣的箱子挑出來，再把剩下的分成兩堆，此時，無論把挑出來的箱子加入哪一堆，那一堆的總量就會比另一堆多，也就是會超過一半。

於是，這就造成一種「自由」，若無論選擇加入哪一堆都符合條件，我們就可以把選擇權「讓」給其他條件。後面就會提到，我們可以選擇鑽石較多的那一堆，使得鑽石的部分也可以符合條件。

更甚者，我們也可以將箱子依照金幣數量均分成任意堆，只要把不符合條件的兩堆抓來「調整」（更乾脆一點：先混合再重分），由良序性保證有限步之內可換完。

（三）引理2 - 餘數處理

引理1強歸強、遇到箱子數不是偶數（或某個整數的倍數）時一樣沒轍，所以特別用一個引理的篇幅，說明當箱子數不被整除時應該如何處理。

我們用分兩堆、箱子個數為奇數為例。在這樣的情況下，我們可以完全抄襲引理1的證法，唯一癥結在於：箱子少的那一堆在「交換」之後，究竟有沒有辦法「追過」箱子最多的那一堆。

如圖：

$$A_1, A_2, A_3, \ldots, A_n$$

$$A_{n+1}, A_{n+2}, A_{n+3}, \ldots, A_{n+n-1}$$

答案雖然是「不行」，但如果我們硬做的話，換到最後就會發現少的那一堆全部都是金幣最多的箱子。於是有：

$$(a_1+a_2+a_3+\ldots+a_n)-(a_{n+1}+a_{n+2}+a_{n+3}+\ldots+a_{n+n-1})$$
$$=(a_1-a_{n+1})+(a_2-a_{n+2})+(a_3-a_{n+3})+\ldots+(a_{n-1}-a_{n+n-1})+a_n$$
$$\le a_n \le a$$

顯然最後不是變成負的（那麼就會在中途符合條件）、就是變成比a小的正數，無論如何都符合條件。

同樣的道理也適用於「調整」多堆之間差異，在此便不贅述。

（四） 引理3 - 加權平均

加權平均是一個很普遍的觀念，在這篇報告中，並沒有用到很多加權平均的性質，以下是幾個需要特別注意的面向：

第一：兩個數的加權平均一定介於兩數之間。也就是說，如果我們將箱子分成兩堆，在其中一堆拿到某個比例的金幣，在另一堆拿到另一個比例的金幣，那麼總合起來，我們拿到的金幣佔全部的比一定介於這兩個比例之間，等號成立若且唯若其中一堆加權是零，其實就是沒有東西。

第二，若我們反過來看，我們也可以主動把箱子分開，並且分別證明分開後都可以拿到某個固定比例，則合在一起之後仍保有該比例的物品。特別地，如果分開之後某一邊特別容易證明，那麼這樣的動作就可視為增加（或化簡）另一邊的條件，對證明有莫大的助益。

第三，這個引裡可視為「真分數上下同加定數」等命題的延伸，特別是某些比例很極端（0或1）的地方，其實都可以用加權平均解釋。

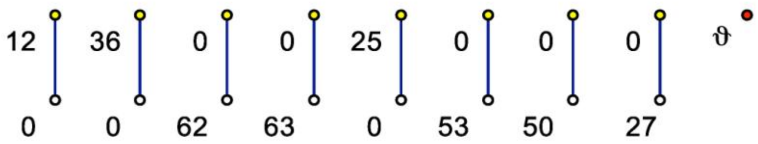
叁、研究過程

在這篇報告中，不乏因奇偶性、四捨五入等需要深入討論的細節，省略它們不代表證明沒有完成，為的是把重要的部分更清楚地展現，而不是把時間浪費在此等小事上。

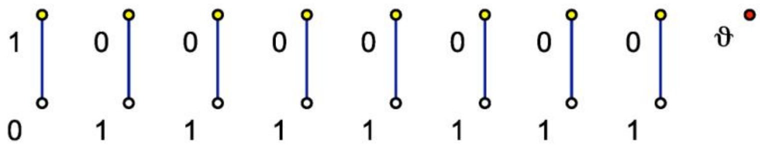
（一）≡ **(1/2, 2, *n*)**

首先我們討論最原始、最簡單的情況。

一開始，我們需要構造幾個例子，來幫助我們了解究竟答案落在哪個範圍內，例如一開始的例子。如圖：



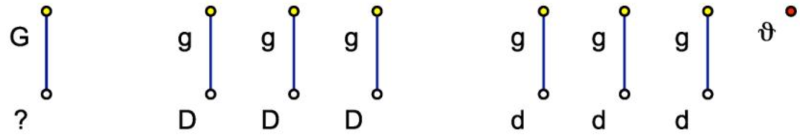
或者更簡單的例子。如圖：



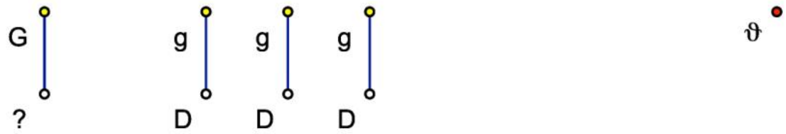
顯然只拿一半的箱子不能滿足條件。

同時，這告訴了我們兩件事：一方面，只要多付出一個箱子的代價，就可以使黃金的部分也達成條件。另一方面，這一個多出來的箱子可能跟「第」兩種有價物品有關，稍微用心一下即可「構造」出的拿法。以下將重述一次證明。

首先我們將金幣最多的箱子(以大G表示)挑出來,並將剩下的箱子分成兩堆,由引理1可知，我們可以將剩下的箱子依照金幣「均分」。如圖：



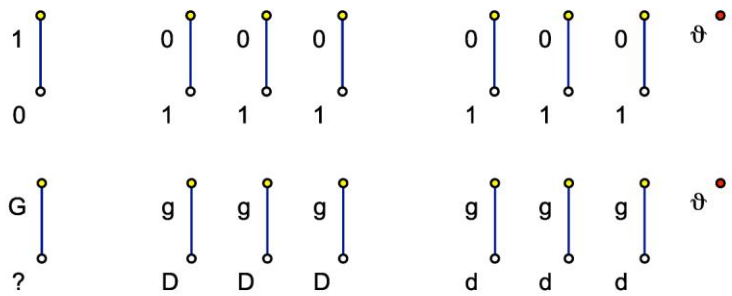
在這兩堆中，必有鑽石多（以大D表示）的一堆、也有鑽石少（以小d表示）的一堆，拿取鑽石多的那堆、再加上金幣最多的箱子。如圖：



現在來計算這次的收穫。從金幣的角度來看：由引理1可以保證傑克拿的超過一半。而鑽石方面，傑克拿到的「D」已經比「d」多，再加上第一個箱子，就已經夠了。

於是有 $\equiv(1/2, 2, n)=[(n-1)/2]+1$，其中「[]」是天花板函數。

有趣的是，如果我們將解法的圖形和構造出來的例子放在一起看，會發現其中還有更多意想不到的關聯、似乎在暗示甚麼。如圖：



其中一個方向，很容易就讓人聯想到如下的子題。

（二）$\equiv$ (*r*, *1*, *n*)

理論上，這一個子題時在無趣到了極點，要拿超過某個比例的金幣，拿超過那個比例的箱子就好了，實在沒有任何理由需要提醒別人還有這一點。

容易有 $\equiv$ (*r*, *1*, *n*)=[*nr*]，其中「[]」是天花板函數。

但是這牽涉到另一個問題，每當傑克提出一個數字時，首先就是得說服自己「舉不出反例」，因此有效率地舉出例子便非常重要。（「有效率地」不只是指構造出例子不需要花太多時間，更多是希望這個例子可以很快地被理解。）

一個方向是，假設一個箱子只能裝一種物品，這麼做的話：

1.即使是窮舉所有的物品分布也不難。

2.計算要拿幾個箱子時變得非常方便。

原因是，一旦我們提出這樣的假設，每一個問題都會化簡成多個k=1的問題。（端看原本的k值為何。）徹徹底底的將每種有價物品分出來討論，是謂「獨立運作」。就如同本節的開頭所說，接下去的研究要難也難。

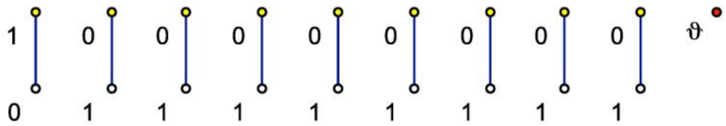
事實上，在接下來的研究中，「異物異箱」（注意不是「一物一箱」喔！）的確扮演著重要的角色，忠實的驗證所有的解答。我們原本擔心會由某種「神奇的分布」造成等號成立的情況，但是「異物異箱」構造出的例子—如 $\equiv$ (1/2, 2, *n*)—都非常契合證明，讓人不禁懷疑「神奇分布」的存在性。

甚至這個角色還要更極端一些，與其說「異物異箱」驗證了我們的解答，不如說我們依照他給出的「模型」構造出解答。如果可以證明：遵守「異物異箱」即可達成等號成立的條件。問題或許會變得簡單許多。

（三）≡ (1/3, 2, *n*)

假設副船長也想獨立於船員、參與分贓的話，傑克就只能要求其中的三分之一了。

同樣地，我們也需要一些模型。如圖：

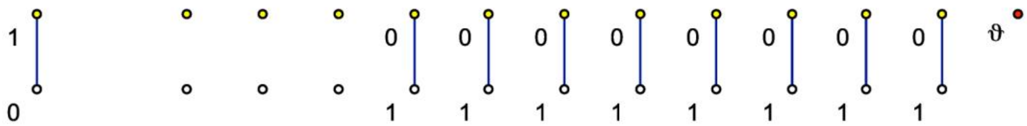


這個地方沒甚麼訣竅，可以完全仿造≡ (1/2, 2, *n*)的證法，唯一的差別在於我們要把箱子分成三堆。而分成三堆之後、挑出來的那一個箱子加入任何一堆也都會超過1/3，從而保證這樣拿是正確的。

於是有≡ (1/3, 2, *n*)=[(*n*-1)/3]+1，其中「[]」是天花板函數。

（四）≡ (1/*p*, 2, *n*)

這樣的證法可以歸推廣至所有的單位分數，下限的部分，我們則有萬用的例子，要幾個有幾個。如圖：

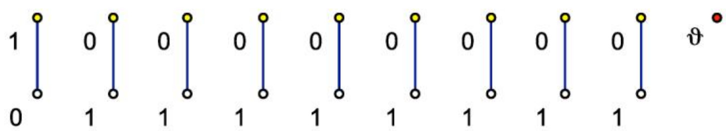


於是有≡ (1/*p*, 2, *n*)=[(*n*-1)/*p*]+1，其中「[]」是天花板函數。

（五）≡ (2/3, 2, *n*)

讓我們繼續考慮下一個分數。

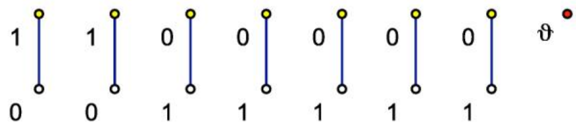
首先還是要看模型。如圖：



同樣可仿造≡ (1/2, 2, *n*)的證法。

於是有≡ (2/3, 2, *n*)=[(*n*-1)2/3]+1，其中「[]」是天花板函數。

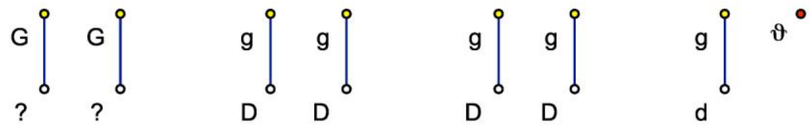
當然事情不會總是那麼順利，讓我們仔細地看這個例子。如圖：



再回來看之前的敘述：「那一堆的總量就會比另一堆多，也就是會超過一半。」可改成：「挑出來的那一個箱子加入任何一堆也都會超過1/3。」卻不能改成：「挑出來的那一個箱子加入任何一堆也都會超過2/3。」。這就是問題所在。

（六）≡ (2/3, 2, *n*) - 續

真正的做法是，挑出金幣最多的兩個箱子，才將剩下的箱子分堆。如圖：



這樣才能保證金幣的部分有⅔。

於是有≡(2/3, 2, *n*)=[(*n*-2)2/3]+2，其中「[]」是天花板函數。

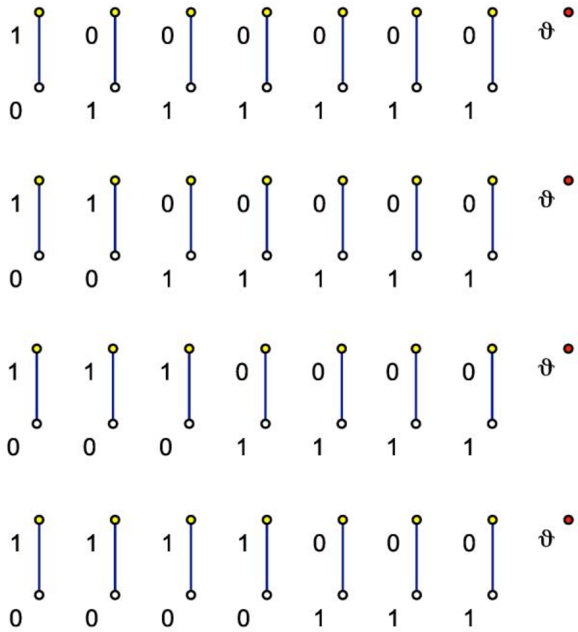
究竟挑兩個和挑一個有甚麼差別呢？從這裡我們還看不出真正的原因。唯一可以確定的是，挑兩個比較能「浪費箱子」（比前者多「浪費」一個）。但是對於其他的比例r呢？

這時我們就需要用到「異物異箱」的概念了。

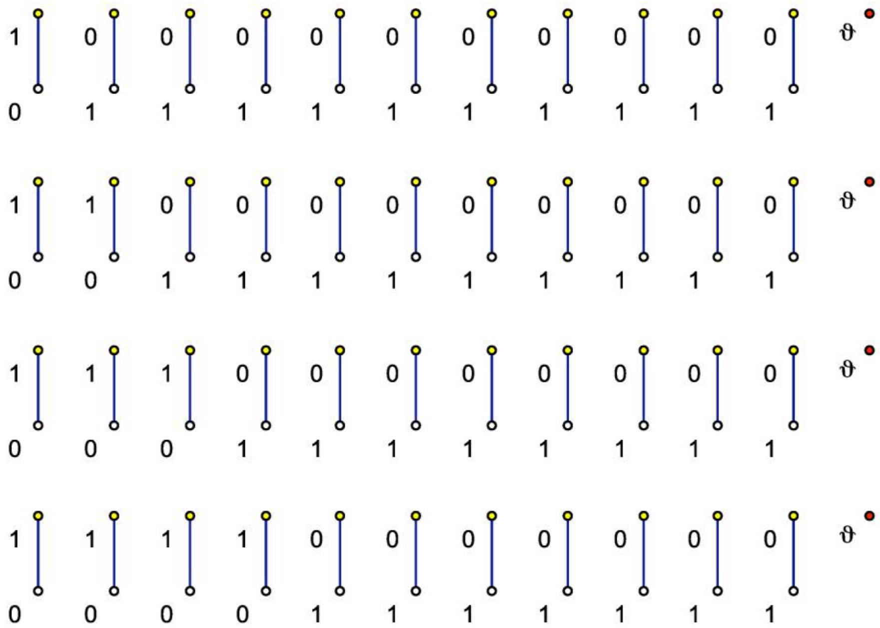
就如同在該節所說，我們想知道究竟挑幾個比較浪費，只要將其帶入≡(*r*, 1, *n*)再加總即可。換句話說，我們有≡ (*r*, 2, *n*)≥Max: {≡(*r*, 1, *a*)+≡(*r*, 1, *b*) | *a*+*b*=*n*}。

（七）≡ (2/5, 2, *n*)

讓我們看下一個需要討論的比例，首先「找」模型。如圖：



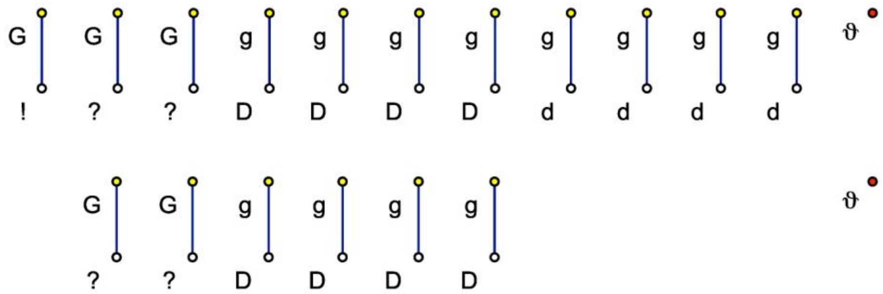
分別需要4、3、4、4個箱子，還不夠明顯，再換一組模型。如圖：



分別需要5、5、6、5個箱子，夠明顯了，看來挑三個是正確的模型。請注意：我們自始至終都沒有認定所謂的「模型」就是「解答」，嚴格說來，「模型」只

能幫助我們尋找解答，並且在某些特殊的情況下可以提供上、下限讓我們參考。就好像模型車不能動，卻能夠幫助我們了解車的結構。

但是問題沒有解決，在剛剛的模型裡，我們只須從三個裝金幣的箱子裡選兩個拿。也就是說，在解答裡，我們有只能選兩個箱子拿。如圖：



這會產生兩個問題：首先，對金幣來說，後半段幾乎可以說是任選的，我們捨棄了第三多的箱子，會不會不夠2/5？更重要地，對鑽石來說，我們沒有考慮到「驚嘆號」究竟有多驚人，如果該箱匯集了大部份的鑽石，後半段再怎麼心機也是徒勞。

讓我們引用之前的敘述：「而鑽石方面，傑克拿到的「D」已經比「d」多，再加上第一個箱子，就已經夠了。」這就是所謂的「真分數上下同加定數」，由引裡3易知其值上升，可是在這裡分母分子增加的量大不同。我們得換個想法。

（八）≡ (2/5, 2, *n*) - 續

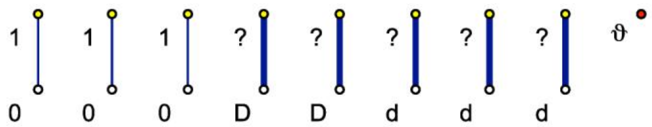
如果稍微動一下腦筋的話，會發現驚嘆號對應到模型時，其實是鑽石最少的箱子。

（雖然在這樣畸形的模型裡說「最」沒甚麼太大的意思，可是我們還是要發揮一下聯想的精神。）如果我們挑出來的三個箱子分別是：金幣最多、金幣次多、鑽石最少，會發生甚麼事？

對鑽石來說，只要在後半段拿到超過2/5的鑽石，傑克拿到的鑽石占的部分就是2/3和2/5的加權平均，由引理3，必達成條件。（特別是等號成立的條件是前三個箱子均無鑽石，例如上面給出的模型。）

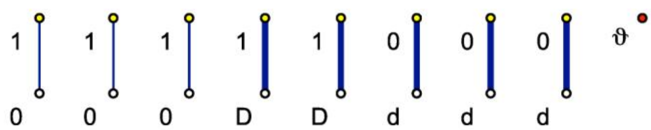
金幣方面，不妨假設前三個箱子的金幣都一樣，原因是我們可以把金幣最多的兩箱的金幣扣到跟第三多的一樣多，這麼做只會減少傑克獲得的金幣量，（這次變成真分數上下同減一數，）不影響結果。

我們用粗線代表（多個）分好堆的箱子，由左至右依鑽石多寡排序，現在來計算金幣的獲取量。如圖：

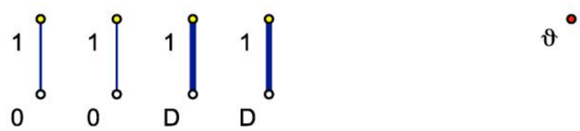


重述一次：我們要做的事情是，在後半段選出兩堆箱子，使得這兩堆的鑽石數量超過2/5，並且在加上前兩個箱子之後，金幣的部分也達成2/5。接下來，我們將用調整法將盤面調往不利傑克的方向，再一舉證明傑克可以達成目標。

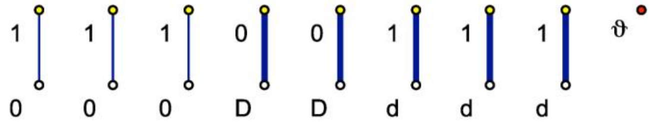
首先由引理1，我們知道後半段這五堆的金幣的差不超過最大者（其實就是1），另由引理3，我們可以將重複的部分（指這五堆的基本盤）和其他部分分開來算，所以可以講他們減掉，於是假設它們的金幣量皆介於0和1之間。如圖：



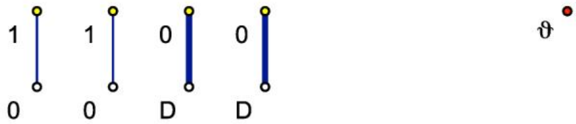
上面的例子中，只要拿中偏左四個（堆）箱子即可。如圖：



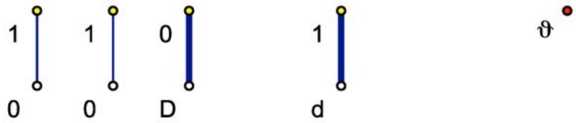
但如果稍微調換一下金幣的位置，如此拿法就無用了。如圖：



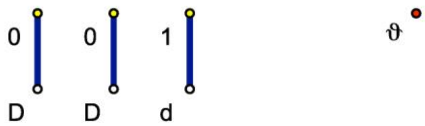
注意到取得的金幣比是2/6，此數和2/5做加權平均之後仍小於2/5，由引理3，傑克無法如願。如圖：



天無絕人之路，這裡還是有緩頰空間的，特別注意到五個數裡的第一大及第三大加起來超過全部的2/5。如圖：



現在真的來計算金幣的獲取量，由於中間這三堆是二選一，因此必可得到超過1/2的金幣。（有趣的是：現在角色完全反過來，變成鑽石無論如何都會符合條件，因此把選擇權讓給金幣。）如圖：



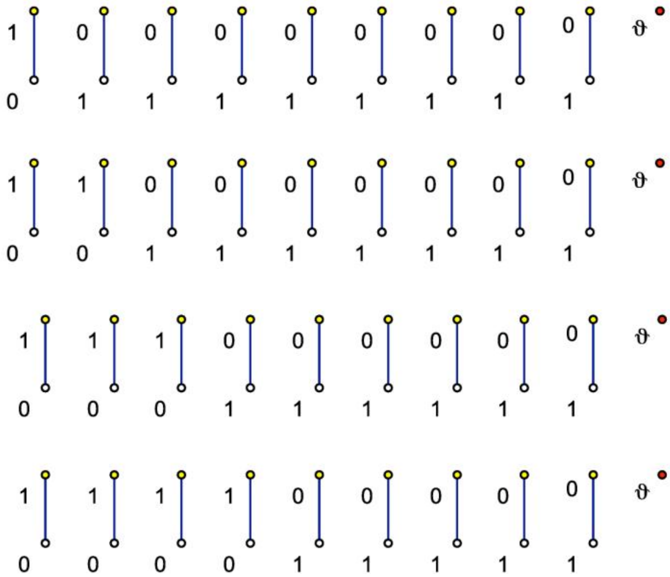
至於其他五個，我們則保證會拿到金幣最多的兩個。（由引理1保證。）如圖：

由加權平均，傑克保證可以達成目標。特別地，所有的等號成立條件都指向上一節提出的模型，又一次地，「異物異箱」展現了極強的配合度。

於是有 $\equiv (2/5, 2, n)=[(n\text{-}3)2/5]+2$，其中「[]」是天花板函數。

（九）≡ (3/5, 2, *n*)

首先要找模型。如圖：

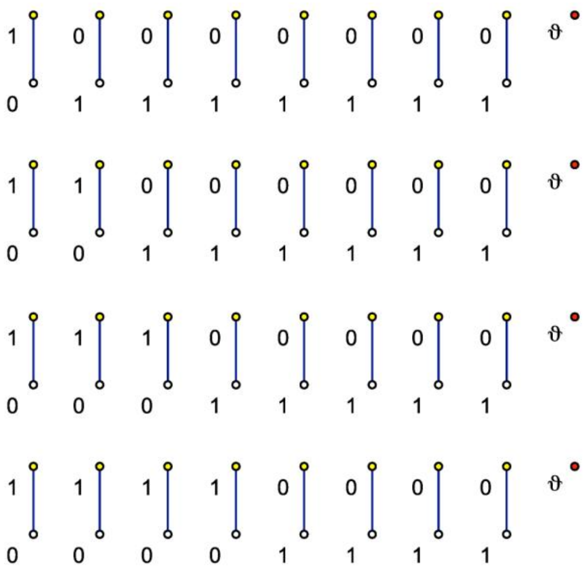


分別需要6、7、6、6個箱子，因此選擇挑兩個的模型，特別地，挑出來的這兩個可以都拿，故可仿造≡ (2/3, 2, *n*)的證法。

於是有≡ (3/5, 2, *n*)=[(*n*-2)3/5]+2，其中「[]」是天花板函數

（十）≡ (4/5, 2, *n*)

首先要找模型。如圖：

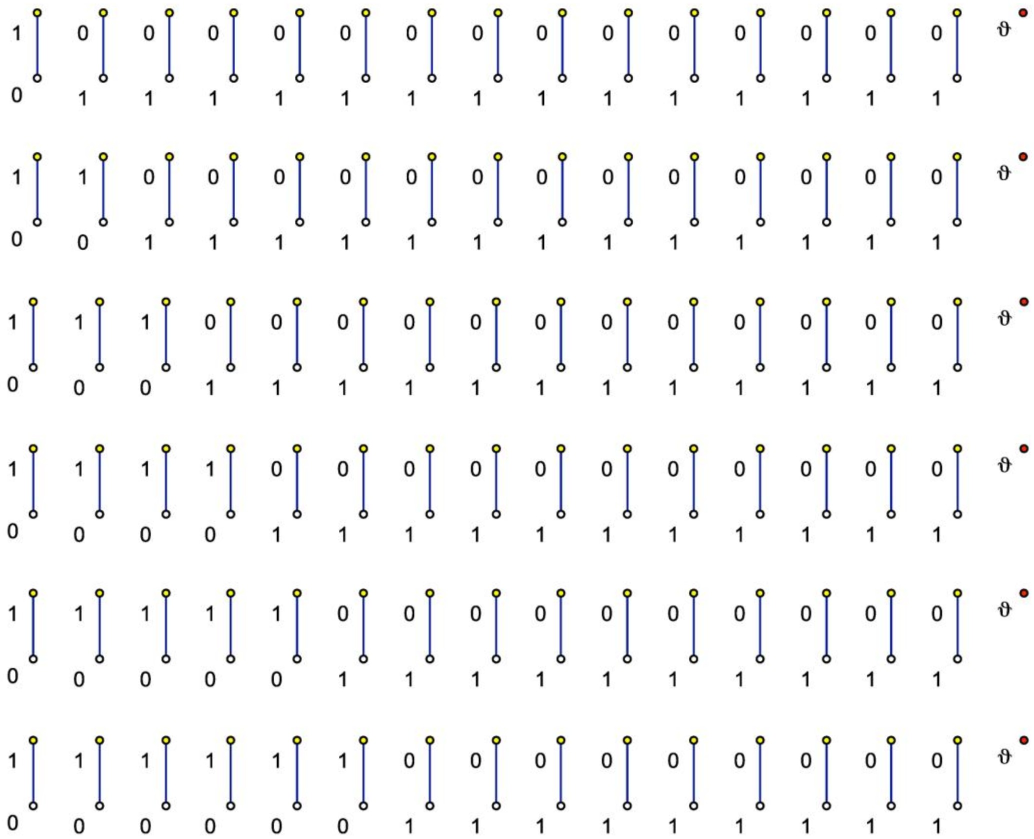


分別需要7、7、7、8個箱子，因此選擇挑四個的模型，特別地，挑出來的這四個可以都拿，故可仿造≡ (2/3, 2, *n*)的證法。

於是有≡ (4/5, 2, *n*)=[(*n*-4)4/5]+4，其中「[]」是天花板函數

（十一）≡ (2/7, 2, *n*)

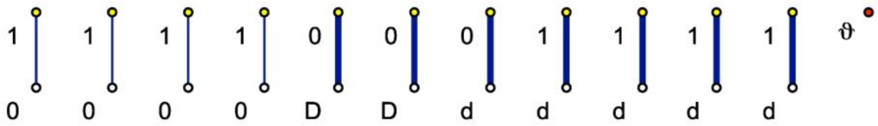
讓我們看下一個需要討論的比例，首先找模型。如圖：



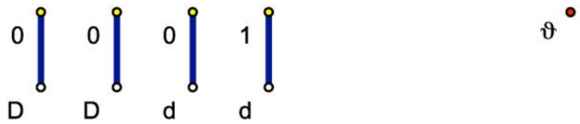
分別需要5、5、5、6、5、5個箱子，因此選擇挑四個的模型，和(2/5,2,*n*)一樣，這四個也不能全拿，只能選兩個。

類似地，我們假設前兩個是黃金最多，再接下來兩個是鑽石最少的。接下來再變成，四個箱子的金幣都一樣多、沒有鑽石。

我們用粗線代表（多個）分好堆的箱子，由左至右依鑽石多寡排序，再把重複的金幣刪去。如圖：



再分成兩個部分，中間這四堆箱子中，取第一個和任一個均可使鑽石超過2/7，並且經適當的選擇可使金幣超過1/3。如圖：



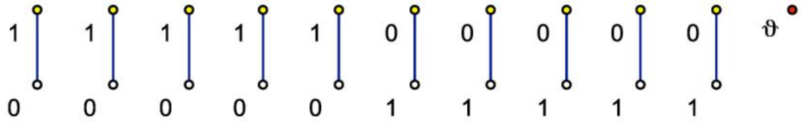
至於其他七個，我們則保證會拿到金幣最多的兩個。如圖：

所有的等號成立條件都指向上一節提出的模型，又一次地，「異物異箱」展現了極強的配合度。

於是有 $\equiv (2/7, 2, n)=[(n\text{-}4)2/7]+2$，其中「[]」是天花板函數。

（十二）$\equiv$ (3/7, 2, *n*)

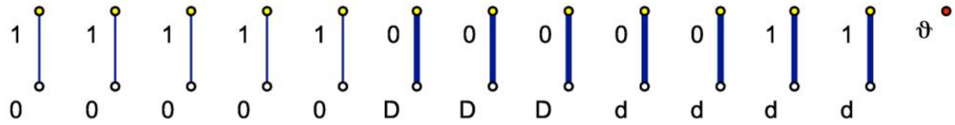
首先跳過找模型的步驟，我們已經悄悄地找到了。如圖：



需要6個箱子，選擇挑五個的模型，和 $\equiv(2/5, 2, n)$一樣，這五個也不能全拿，只能選三個。

類似地，我們假設前兩個是黃金最多，再接下來兩個是鑽石最少的。接下來再變成，四個箱子的金幣都一樣多、沒有鑽石。

我們用粗線代表（多個）分好堆的箱子，由左至右依鑽石多寡排序，再把重複的金幣刪去。如圖：



接下來就比較複雜了，這一次我們可以拿三個箱子，而這三個箱子究竟該怎麼選呢？

以下我們將從鑽石的觀點，討論究竟有多大的自由。全部的結果均可由簡易的不等式得出，故不詳列。

首先，我們知道第一個箱子一定得拿。如圖：

再來，下一個可以選第二、三個箱子。如圖：

最後，最多選到第五個箱子，再下去鑽石就不夠多了。如圖：

值得注意的是，儘管不能往下選，往上選倒是沒有被禁止。

現在我們把後半段的箱的堆也分類了，對每一類來說，都可以在該類裡取到至少1/2的金幣，由引理3我們把它忽略掉。

至於其他七個，我們則保證會拿到金幣最多的三個。如圖：

於是有≡ (3/7, 2, *n*)=[(*n*-5)3/7]+3，其中「[]」是天花板函數。

肆、研究結果與討論

（一）結果

經由以下的過程，我們可以為≡ (*q*/*p*, 2, *n*)找出值及其取法。

1.找出q對模p的倒數r。

2.扣除前r個箱子（分別是金幣最多及鑽石最少）。

3.為剩下的箱子分堆並均分金幣。

4.將分堆後的箱子分類。

5.分別在每一類中選擇要拿的堆。

6.再加上前r個箱子中要拿的，大功告成

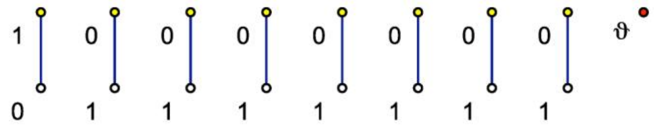
（二）討論

其實值得討論的地方還有很多，這裡選擇提一個比較有意思的部分：

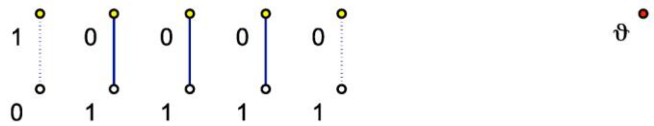
從之前的式子可以看到，平均而言，傑克幾乎多拿一個箱子。（畢竟[(*n*-1)*q*/*p*]

和[(*n*)*q*/*p*]大部分的時候是一樣的。）但是在解答中，我們並沒有看到任何一個多拿的箱子，每一個箱子都是必須的，這究竟是怎麼一回事？

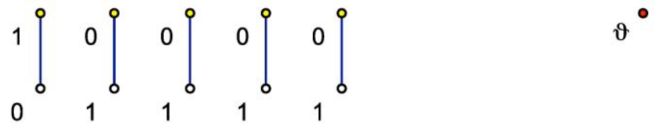
我們認為，這是因為嚴格來說，傑克多拿的不是「一個箱子」，而是兩個「半箱子」（Two of half a box）。請再仔細地看一次≡ (1/2, 2, *n*)給出的例子。如圖：



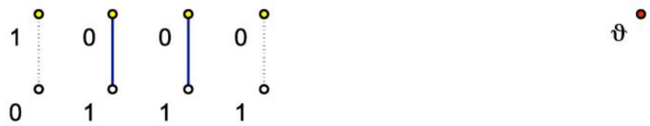
注意到黃金全都集中在某個箱子，因此最經濟實惠的拿法應是只拿半個箱子，（容我用虛線表示半個箱子。）同理，鑽石也應只拿3.5個箱子。如圖：



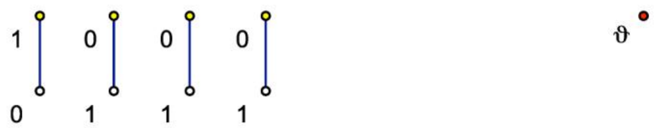
不爭地，虛線畢竟還是線的一種，就像傑克不能只拿「半個箱子」一樣，這就是所謂的「多出兩個半箱子」。如圖：



同樣的例子換成≡ (1/3, 2, *n*)，這回虛線代表1/3個箱子。如圖：



總共得多拿2/3+2/3個箱子，還不是普通的多。如圖：



（三）補充

這裡我們再稍微提一下有關「異物異箱」的事，我們希望可以證明：「『異物異箱』可以構造出所有最糟的情況」。所謂「最糟的情況」是指，同一組參數（也就是比例、物品種類、箱子數）下，需要拿最多箱子（時的物品分佈）。

也就是說，這個引理其實是在「度量」「異物異箱」究竟有多糟。當然，這不代表「只有」異物異箱是最糟的，（反例很容易構造），而是凡有最糟的情況出現，「異物異箱」一定有份。

好處是，儘管非「異物異箱」的分佈可以是「最糟的」，但是我們卻可以不用管，只要專心研究「異物異箱」中的最糟即可，因為我們已經知道這樣的最糟是真的最糟。而「異物異箱」裡的最遭要比普通的最糟還來得好刻劃。

舉個例子，我們想知道一個班上成績最差的是誰，我們可以去問每個人的成績。或者，我們可以「證明」小明每一次都是最後一名，這樣只要問小明的成績就好了，而剛好小明這個人比其他人都願意公佈自己的成績。

壞處是，這樣並沒有真正解決問題，當我們要「證明」小明是最後一名的時候，我們也需要其他人的成績（問題又回到原點），但是論如何，這都是一個階段性的看法。

伍、結論與應用

（一）結論

經由類似分堆、分類等步驟，我們將證明推廣至所有的分數。換言之，我們求出了所有$\equiv(r, 2, n)$的解並且證明它。$\equiv(q/p, 2, n)=[(n-r)q/p]+[rq/p]$，其中「[]」是天花板函數、$r$是$q$對模$p$的倒數。

過去曾有人用高等數學證明$\equiv (r, k, n)$大部分的解（詳見參考文獻1），但礙於其引理的限制，該結論只適用於n足夠大的情況（相較於r及k）。特別是r為無理數時，雖然可以由有理數無限逼近，但逼近時會使n不夠大，成為另一個證明缺乏的部分。

本文從最基礎的情形出發，利用適當的餘數處理，不僅能處理各式的n值，連帶也使r為無理數的情形獲得解釋。

此外，異物異箱也是重要概念之一，若能善加利用，必能更進一步地解釋更多問題。

（二）應用

這項研究的結果可應用至分配災區物資時，所有的物資（包括水、食物等）皆已裝箱不易重新分裝，因此適當的分配救援包使其盡量平均便相當重要。同時也適用於任何一種無法重組的資源，例如開新的分公司時，也必須考慮如何分配能力無法轉移的人才。以及如何用最實惠的價錢買齊所有功能的家具，或是各種營養的套餐等。

期望在現實上可以加強此類分配的效率。

陸、參考文獻

1.Robbers Sharing Boxes Loot,for Summer Seminar of the International Mathematics Tournament of the Towns,in Mir Town, Belarus, from August 1 to August 9, 2005.

Maximizing the Lion's Share

Hsin-Po Wang, student, Taipei Municipal Jianguo High School

Captain Jack and his pirates had acquired some treasure boxes as loot. Each treasure box was known to contain some gold and some diamonds. The crew agreed to let Captain Jack take half of the boxes for himself.

Captain Jack would like to have at least half the gold and at least half the diamonds. What is the least number he be guaranteed to do so for any distribution of gold and diamonds among the boxes?

General Formulation

There are n boxes each contains a number of gold and a number of diamonds. Jack would like to get at least q/p of the gold and q/p of the diamonds by selecting the minimum number of boxes. Is there a systematic way to do so?

Let k is number of categories of treasure. Let $f(q/p, k, n)$ be that unique number f, such that:

1. There exist f boxes fulfilling the requirements for all possible DFDB distributions of the fruits;
2. There exists a particular DFDB distribution of the fruits that any $f-1$ boxes fail to fulfill the requirements.

Lemma 1A

It is possible to divide the boxes into two parts equally, so that the sum of gold of one pile is almost same as the other. Where "almost same" means: the difference between these two sums is not more than the maximum number of gold in all boxes.

Proof:

By swapping the "heaviest box" (which contains the most gold) of the "heavier pile" with the "lightest box" of the "lighter pile", the difference between the sums of piles is strictly reduced. The desires result reached when this operation is repeated to the piles.

Lemma 1B

Lemma 1A can be also applied to the case of more than two piles, even if the number of piles is not a factor of the number of boxes.

Proof:

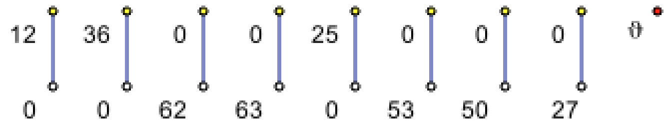
Alternately "Adjust" any two piles by Lemma 1A, and add empty boxes into some pile if necessary. (Adding empty boxes only re-enforce the argument psychologically. There is no harm to the logic.)

In this report, we will not explain such details as parity or rounding too specifically. The omission of these details does not invalidate our proof; the primary goal is to present the more important parts clearly.

Case 1 $\quad$ $\boldsymbol{f(1/2, 2, n)}$

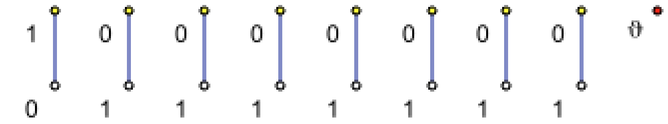
We will start with the simplest and most basic case.

To start, we should construct several examples to give us an idea of where the solution is. See the diagram:



Where a single line represents a box, with two numbers represents number of gold and diamonds.

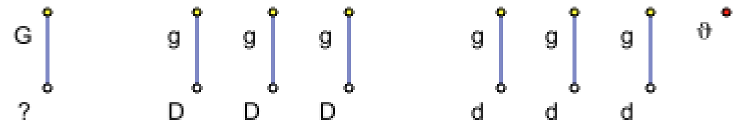
Or consider an even simpler case, namely:



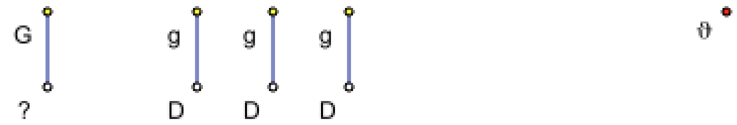
Clearly, taking just half of the boxes is not sufficient.

At the same time, these examples tell us two things: firstly, by just sacrificing one extra box, we can bring the amount of obtained gold to the needed level. Secondly, the selection of this extra box may depend on the existence of the second object. It is not hard to construct a selection based on this idea. Below is the formalized proof.

First, let us find the box with the most gold (call it G), and separate the remaining boxes into two piles. By Lemma 1, we can separate the boxes in a balanced way.



In these two piles, there must be one pile with more diamonds and one pile with less. Pick the one with more diamonds and add the box G.



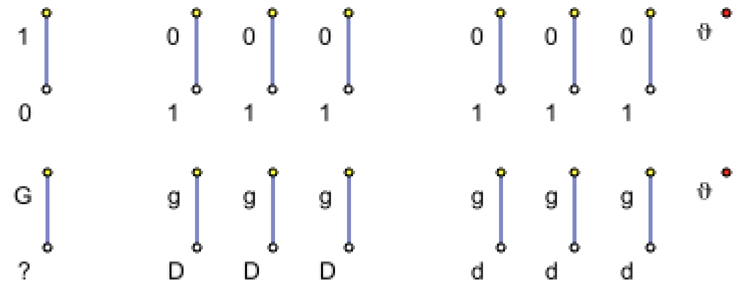
Now, let's look at how much treasure that we have got. By Lemma 1 we know that Jack has gotten more than half of the gold; considering the diamonds, since Jack has taken the pile with more diamonds, we can also guarantee that Jack has more than half of all the diamonds.

therefore,

Theorem 1

$f(1/2, 2, n)=\lceil (n-1)/2 \rceil +1$, Where $\lceil \ \rceil$ is the ceiling function.

Interestingly, if we look at our example and at the diagram used in our solution, we note the two have a curious correspondence:



One of the resultant ideas quickly leads to the following case:

Case 2 f **(*r*, 1, *n*)**

This case is absolutely trivial. To take more than a certain ratio of the coins, simply take more than that ratio of the boxes. It should not be necessary to explain this case to anybody.

Therefore,

Theorem 2

$f(r, 1, n) = \lceil nr \rceil$

However this simple case leads us to another problem. Whenever Jack comes up with a certain ratio, the first thing he must do is to convince himself that no counterexample exists. Therefore the efficient construction of examples is important as well. (Here "efficient" does not just mean that the construction should not take too much time, but also that it should be understood quickly.)

One idea is to assume that each box can only contain one type of treasure. If we make this assumption:

1. Listing all distributions is much easier.
2. It is convenient when calculating the number of boxes to take.

The reason is that once this assumption is made, and each problem reduces to a set of cases where k=1. (Note the k-value of the case we are discussing.) If we can cleanly sort out all the treasures, the problem will be very trivial.

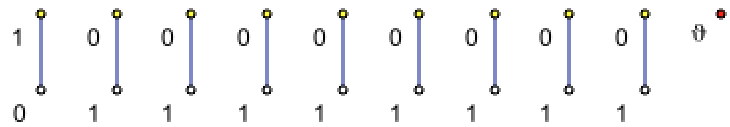
In fact, in the remaining part of this report, this concept of "different treasures in different boxes" plays quite an important role, verifying each and every case we test. At first, we worried that some "magical distribution" would cause the worst number of boxes, but all the examples produced by this idea, as in the $f(1/2, 2, n)$ above, work perfectly with the proof, and make us suspicious of the existence of such a "magical distribution".

In the other hand, if we could prove that the worst case in any evaluation of f could be provided with an example with different treasures in different boxes, this problem would become much simpler.

Case 3 f **(1/3, 2, *n*)**

If the vice captain also wanted his/her share of the treasure, Jack would only be able to require one third of the treasure.

Once more we need a model:



Here no special trick is needed: we can prove it just as we proved $f(1/2, 2, n)$: the only difference being that we split the boxes into three piles instead of two. Once this split has been accomplished, the box we took out, plus any pile, will be at least 1/3. Thus this method also works.

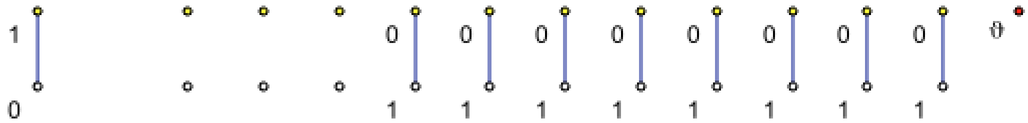
Therefore,

Theorem 3

$f(1/3, 2, n) = \lceil (n-1)/3 \rceil + 1$

Case 4 f **(1/*p*, 2, *n*)**

The same proof can be extended to cover all reciprocals of integers. To prove the lower bound, we have a perfect example for every case:



Therefore,

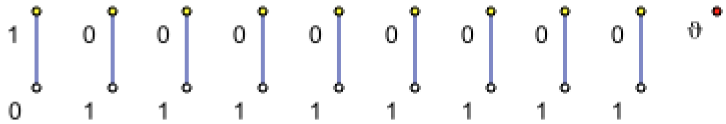
Theorem 4

$f(1/p, 2, n)=\lceil (n-1)/p \rceil +1$

Case 5 $\quad$ ***f*** **(2/3, 2,** ***n*****)**

Let's consider the next fraction.

The first step is still to look at our model:



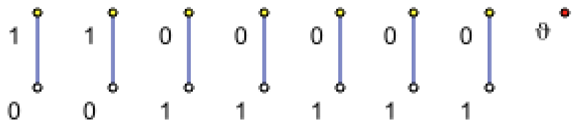
The proof is just the same as in $f(1/2, 2, n)$.

Therefore,

Theorem 5

$f(2/3, 2, n)=\lceil (n-1)2/3 \rceil +1$

Unfortunately, things don't always go that smoothly. Let's observe this example carefully:



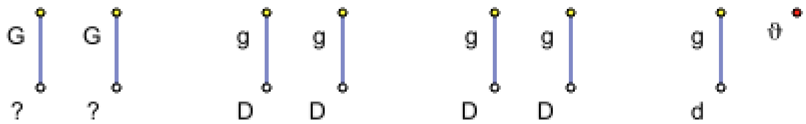
Let's go back to our Lemma:

"the difference between these two sums is not more than the maximum number of gold in all boxes."

Here lies the problem.

Case 5 $\quad$ $f(2/3, 2, n)$ **-con't**

The real solution is to pick out the two boxes with the most gold, and then split the boxes into three piles:



Then we can ensure that we get at least 2/3 of the gold.

Therefore,

Theorem 5 -con't

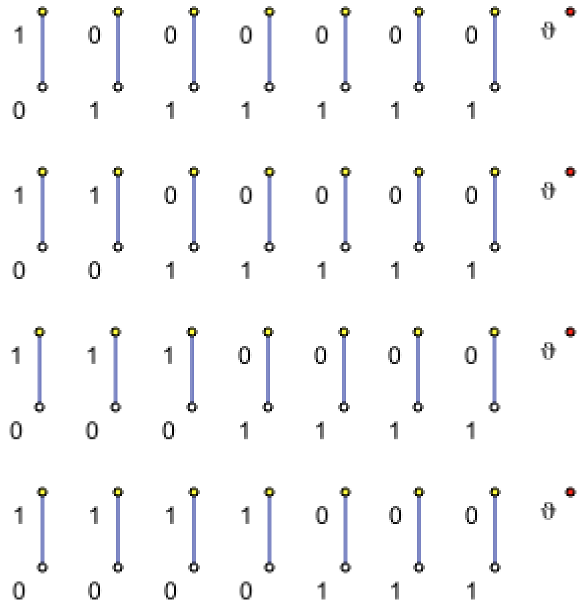
$f(2/3, 2, n)=\lceil (n-2)2/3 \rceil +2$

What is the difference between picking out one and two boxes? From this example we can't see the real reason. The only thing we can be certain of is, by picking out two boxes we are able to "waste" one more box when taking the piles. But what about other ratios for r? Now we will need to use our concept of "different treasures in different boxes".

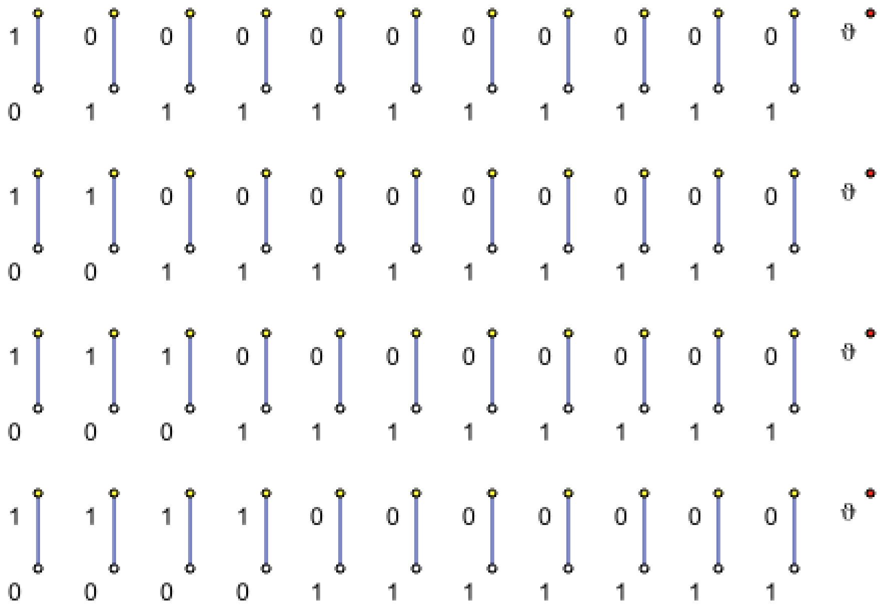
As explained in this section, we want to know how many boxes we should pick out to be the most wasteful. All we have to do is plug it into *f* (r, 1, *n*) and sum the results. In other words, *f* (*r*, 2, *n*)≥Max: {f (*r*, 1, *a*)+*f* (*r*, 1, *b*) | *a*+*b*=*n*} .

Case 6 *f* **(2/5, 2,** ***n*****)**

Now let's move on to the next ratio. To give us an idea, let's look for examples:

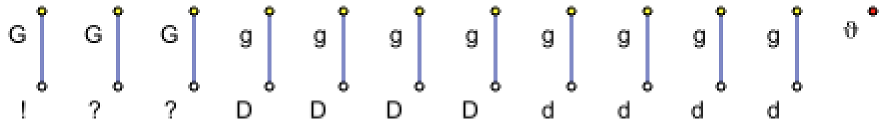


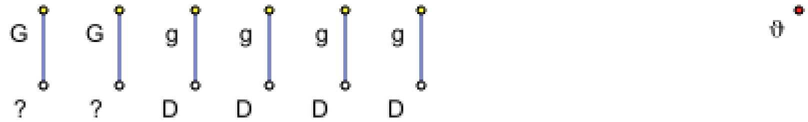
We must take 4, 3, 4, 4 boxes in each of these cases, respectively. The lower bound is not obvious yet, so let's take a new example:



Here we need 5, 5, 6, 5 boxes respectively. Now our lower bound is more obvious: we should pick out three of the boxes with most of the gold. Note that this example-searching is not a solution in itself, and never will be; it is merely an aid in helping us find the answer, and sometimes can give us upper and lower bounds. Its function is like that of a model car: even though it can't move, it can help us understand how a car works.

But the problem is not over. In the previous example, we only had to take two boxes from the three with the most coins. In other words, in this solution, we can only take two of the three we picked.





Here we have two problem: First, we chose the piles without considering the gold, Second and more important, we even chose the box without considering the "!", and this time Lemma cannot promise us.

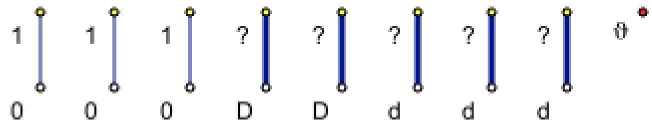
Case 6 *f* **(2/5, 2,** ***n*****) -con't**

In fact, we discover that in the model, "!" is the box with the least diamond. Which means we can try to pick up two boxes with the most gold and one with the least diamond.

In view of diamond, if we can get at least 2/5 of diamond from the piles, then we approach the goal because we can get at least 2/3 in the first three (especially, the equation made if these three boxes has no diamond.)

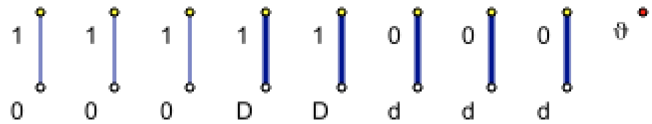
In view of gold, assume that these three are equal in gold, or we can adjust them into the same.

The thick line represent a pile, and ranking them in increase order in diamond. As the diagram.

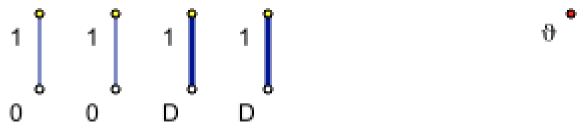


Again: what we want o do now is, to choose two piles containing 2/5 of diamond, and containing 2/5 of gold with the other two boxes.

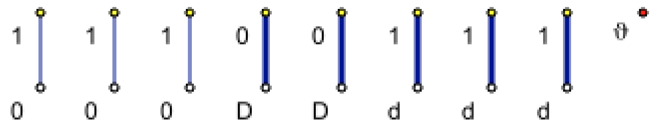
By the Lemma, we can eliminate the five piles as five boxes, where the number of gold is not larger than the three.



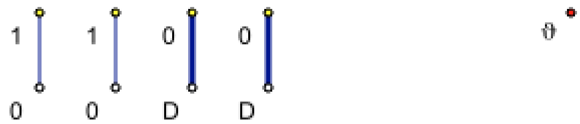
For the example, we take these four boxes, as the diagram:



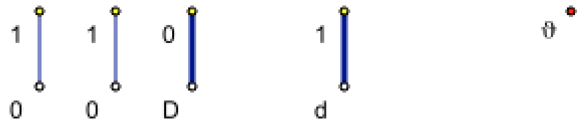
But the method fail if we exchange some gold:



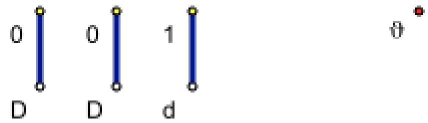
In this case, we got 2/6 of gold, less than 2/5.



Fortunately, we can choose the piles ranking 1 and 3, and still get 2/5 of gold.



Because we can choose from ranking 2 and 3, we get 1/2 of diamond.



Within the other five unit, we get two of them.

Jack approaches his goal again. And also, “different treasures in different boxes” help to find the answer.

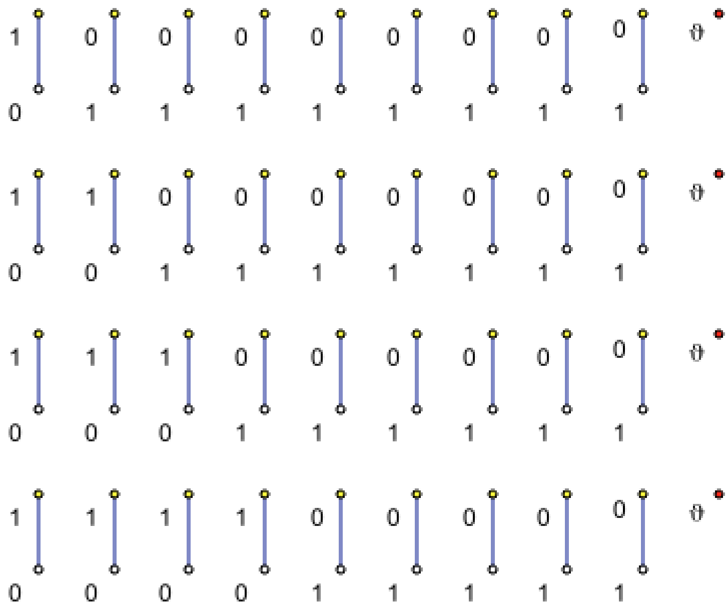
Therefore,

Theorem 6 -con’t

$f(2/5, 2, n) = \lceil (n-3)2/5 \rceil + 2$

Case 7 $\quad$ ***f*** **(3/5, 2,** ***n*****)**

Let's look for examples:



Here we need 6, 7, 6, 6 boxes in each of these cases, we should pick out two of the boxes with most of the gold. The same proof of $f(2/3, 2, n)$ can be extended to this case.

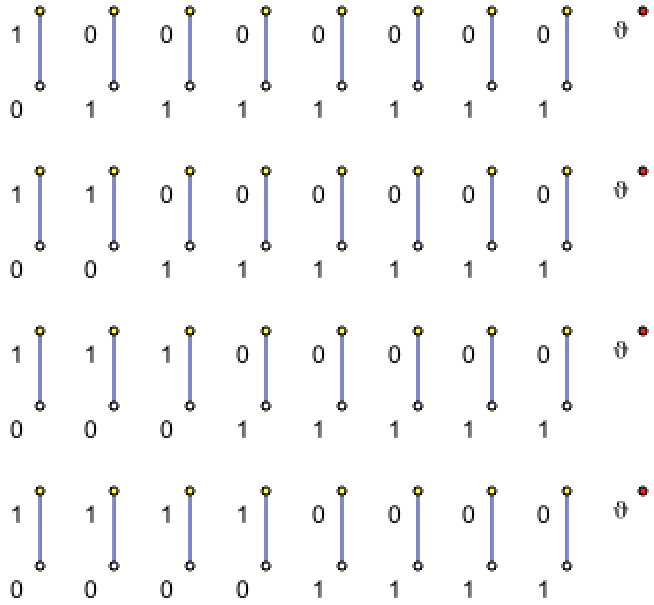
Therefore,

Theorem 7

$f(3/5, 2, n) = \lceil (n-2)3/5 \rceil + 2$

Case 8 $\quad$ ***f*** **(4/5, 2,** ***n*****)**

Let's look for examples:



We must take 7, 7, 7, 8 boxes in each of these cases, so we should pick out four of the boxes with most of the gold. The same proof of *f* (2/3, 2, *n*) can be extended to this case.

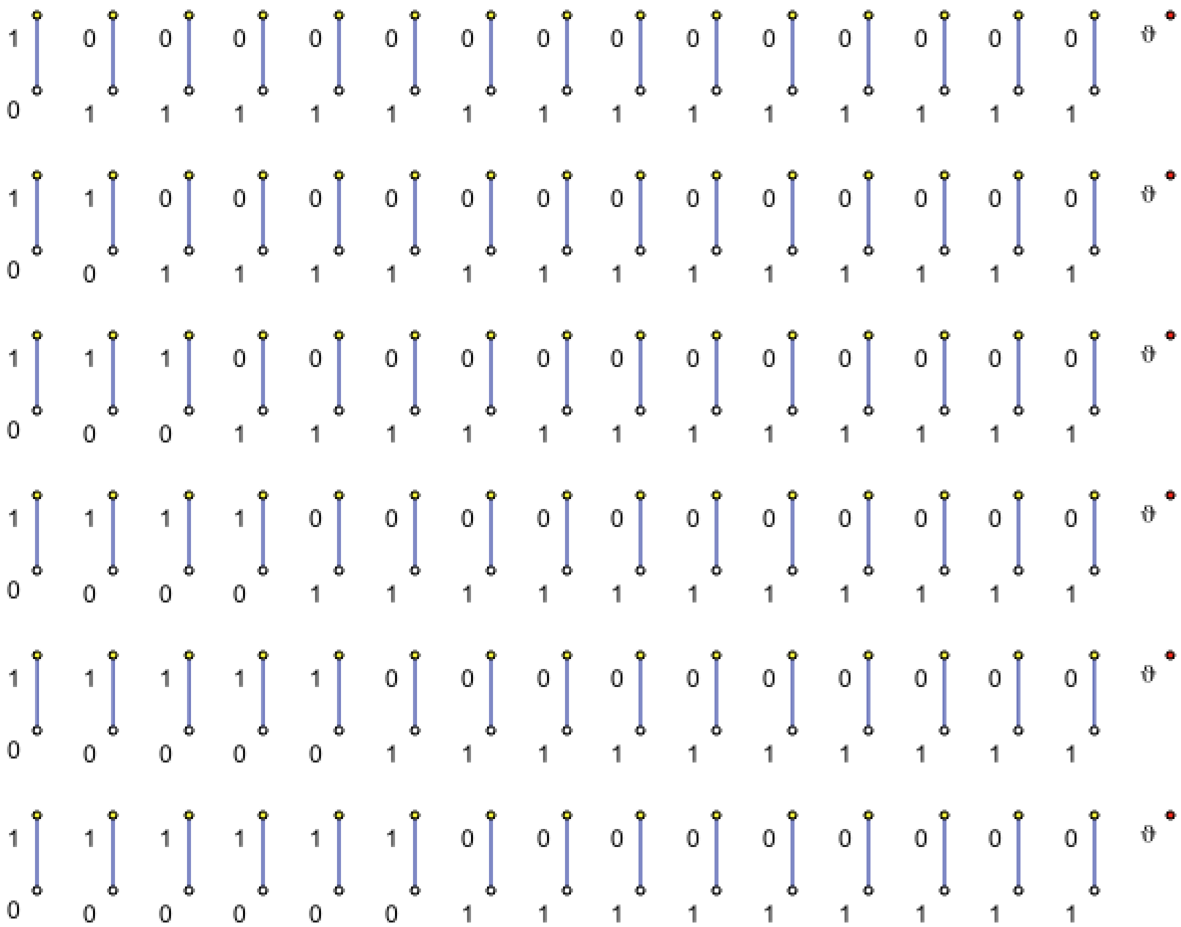
Therefore,

Theorem 8

f (4/5, 2, *n*)=⌈ (*n*-4)4/5⌉ +4

Case 9 *f* **(2/7, 2,** ***n*****)**

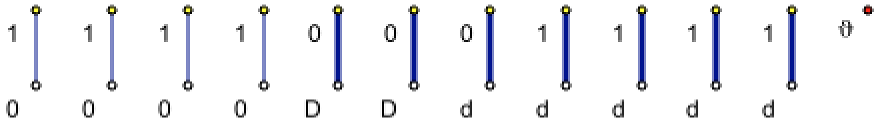
Let's look for examples:



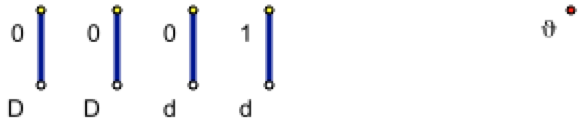
Here we need 5, 5, 5, 6, 5, 5 boxes respectively, so we should pick out four of the boxes with most of the gold. But as the case of $f\,(2/5, 2, n)$, we can't take all of these four but two.

Similarly, assume that the first two boxes contains the most gold, and the third and forth contains the least diamond. And then we adjust them into four boxes within the same gold but no more diamond.

Use, a thick line to represent a pile, and rank them in increase order in diamond. As the diagram.



In the middle four piles, the pile with the most diamond and either one contains more than 2/7 of diamond, and one of them contains more than 1/3 of gold. As the diagram.



In the other seven units, we can take the two boxes withe the most gold. As the diagram.

Again, the equation holds when "different treasures in different boxes".

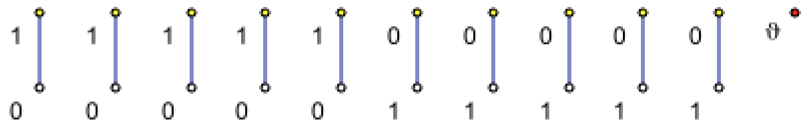
Therefore,

Theorem 9

$f\,(2/7, 2, n) = \lceil (n-4)2/7 \rceil + 2$

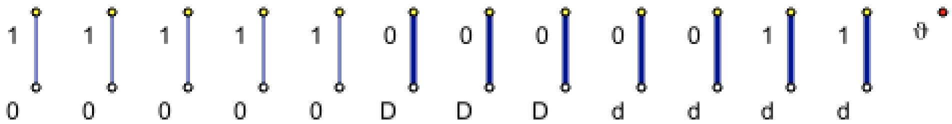
Case 10 $f\,(3/7, 2, n)$

Assume that we have already found out the example:



Here we need 6 boxes and we should pick out five of the boxes with most of the gold. But as the case of $f\,(2/5, 2, n)$, we can't take all of these four but three.

Use a thick line to represent a pile, and rank them in increase order in diamond. As the diagram.

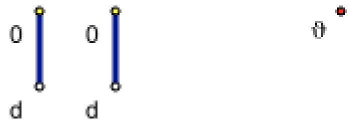


There is a little bit trouble here. This time we can choose three piles, but how to?

First, we all know that the first pile must be taken.

Second, we can choose one from the second or third.

Third, we can choose one from the forth or fifth.



In the other seven units, we choose the three boxes with the most gold.

Therefore,

Theorem 10

$f(2/7, 2, n)=\lceil (n\text{-}4)2/7 \rceil +2$

So far we discuss about a lot of cases, now we will introduce the general method.

General Cases

Let r to be the unique number satisfying $rq \equiv 1 \pmod p$. We claim that:

$$f\left(\frac{q}{p}, 2, n\right) = \left\lceil (n-r)\cdot\frac{q}{p} \right\rceil + \left\lceil r\cdot\frac{q}{p} \right\rceil$$

To substantiate this claim, it suffices to show that:

1. There is a systematic way to select $\lceil (n-r)q/p \rceil + \lceil rq/p \rceil$ boxes to achieve the goal.
2. There exists a particular distribution of the fruits that any $\lceil (n-r)q/p \rceil + \lceil rq/p \rceil - 1$ boxed will never reach the goal.

Proof

To show condition (1) is satisfied, we take these steps:

Step 1: Mark r boxes with the most apples. Divide the remaining into p piles each containing $\lceil (n-r)q/p \rceil$ or $\lfloor (n-r)q/p \rfloor$ boxes such that the differences among the total numbers of apples in each pile is no larger than the number of apples in each boxes of the r boxes. This is possible in view of Lemma 1.

Step 2: Arrange the piles in the increasing order of bananas. Pick the pile B that contains the most bananas, and by applying the one-dimensional greedy algorithm to the remaining $p-1$ piles to form $q-1$ superpiles. The boxes from B and the one single pile containing the most apples selected from each of $q-1$ superpiles comprise the required selection.

Step 3: Of the r boxes marked in Step 1, pick the box with the most apples. Repeat the same selection as in Step 2 by exchanging bananas with apples to obtain the boxes to be selected in Step 4.

Step 4: $\lceil rq/p \rceil$ boxes selected in Step 3 together with the $\lceil (n-r)q/p \rceil$ boxes taken from the q piles selected in Step 2 make the desired selection.

Condition (1) is now fulfilled. To see condition (2), it suffices to consider the distribution of the fruits with r boxes each containing one apple and $(n-r)$ boxes each containing one banana.

Result

From the above process, we can find a value and a strategy for $f(q/p, 2, n)$:

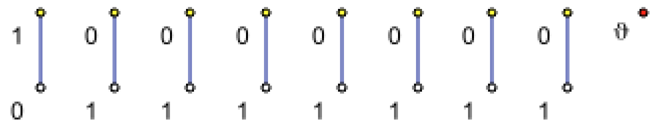
1. Find the inverse of q mod p, and call it r.
2. Take out the first r boxes (with the most gold and least diamonds).
3. Split the remaining boxes into p piles.
4. Sort the piles.
5. Determine which piles should be taken.
6. Finally add the boxes from the first r boxes that should be taken.

Discussion

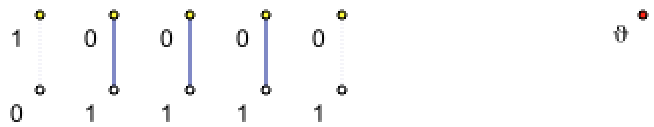
In fact, there are a lot more places where we can discuss. Here we pick out one of the more interesting parts:

From the previous formulas, we discover that on average Jack is forced to take one extra box. (After all, $\lceil (n\text{-}1)q/p \rceil$ and $\lceil (n)q/p \rceil$ are usually the same.) But in our solutions, we never see Jack take an extra box. Every box is necessary to make the lower bound. Why is this?

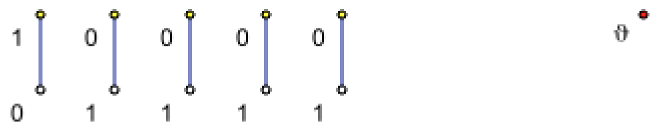
We think this is because, strictly speaking, Jack is not taking one "extra box", but two "extra half-boxes". Let's look again at the example given in $f(1/2, 2, n)$:



Note how all the gold is put in a single box. Therefore, in Theorem, the best method should be to just take half of that box. (Here, a dashed line will represent half a box.) Similarly, only 3.5 of the diamond boxes should be taken:

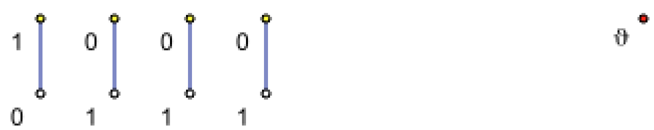


The dashed line is still a line, so since Jack can't take half a box, he ends up taking two "extra half-boxes":



Swapping the example to be $f(1/3, 2, n)$, where a dashed line is one third of a box:

Here we must take 2/3+2/3 extra boxes, which isn't actually a lot:



Matrix

It is possible to identify a distribution of fruits by a single permutation matrix in which all entries are 0 except the (i, j)-entry is 1 if a box has i^{th} ranking in apple and j^{th} ranking in banana.

By swapping appropriate rows and columns successively, taking extra care that the determinant of the associated two-by-two submatrix decreases, we may convert the permutation matrix into an "anti-diagonal" matrix.

Now with the Greedy algorithm, it is possible to select corresponding boxes fulfilling the required condition. Tracing back, we obtain the required selection.

Extension

Here we will discuss the concept of "different treasures in different boxes" (DFDB) in a little more detail. We want to prove that "all of the worst cases can be constructed with 'different treasures in different boxes'". Here "worst case" means a configuration with a fixed set of parameters (the ratio, types of treasure, and box number) and the maximum required number of boxes.

In other words, this would-be lemma is measuring how bad "different treasures in different boxes"-configurations are. Obviously, this doesn't mean that only "different treasures in different boxes"-configurations can be the worst. Just that, when we list the worst configurations for a set of parameters, the list must include configurations with this property.

The advantages of this is that, even if other distributions could be the worst, we can still safely ignore them and focus on determining the worst of "different treasures in different boxes"-configurations, since we know that the worst value here would be the worst value of the whole problem.

As an analogy, if we want to know what the lowest grade in a class is, we can ask every person's grade and find the minimum. Or, if we can "prove" that John always has the lowest grade in the class, all we have to do is ask for John's grade. At the same time, John is more likely to be willing to tell us his grade.

The disadvantage is, this doesn't really solve the problem: when we want to prove that John always has the lowest grade, and we need the grades of the other people (effectively right where we started). But no matter what, this is a progressive idea.

However, in the case of DFDB, we already have its own result. Let k be the number of categories of fruits. It is less complicated to substantiate the claims that:

1. There exist $\lceil (n-(k-1)r)q/p \rceil + (k-1)\lceil rq/p \rceil$ boxes fulfilling the requirements for all possible DFDB distributions of the fruits;
2. There exists a particular DFDB distribution of the fruits that any $\lceil (n-(k-1)r)q/p \rceil + (k-1)\lceil rq/p \rceil - 1$ boxes fail to fulfill the requirements.

Conclusion

Using various techniques such as splitting into piles and sorting by type, we have extended our proof to all fractions. In other words, we have determined and proved the value of $f(r, 2, n)$, $f(q/p, 2, n) = \lceil (n-r)q/p \rceil + \lceil rq/p \rceil$, where $\lceil\ \rceil$ is the ceiling function and r is the inverse of q mod p.

In the past, values of $f(r, k, n)$ for most sets of parameters have been proved with highe r mathematics (see reference [1]), but due to limitations in its lemmas, the result only applies to when n is relatively large, as compared to r and k. Particularly when r is irrational, it can be infinitely closely approximated with rational numbers, but as the approximation is tightened n will no longer big enough. This is another unfortunate shortcoming of the previous paper.

This paper starts from the most basic cases, and with manipulation of moduli, and we can not only process all values of n, but also the case where r is irrational.

Furthermore, the idea of "different treasures in different boxes" is also important. If it can be used well, more problems will certainly be solved.

Application

The result of this paper can be applied to the distribution of resources in disaster areas, when the resources (water, food, and so on) are prepackaged and not easy to repackage, so that adequate distribution to ensure the equality of resources is necessary. This applies also to any resource which can't be recombined. For example, when starting a new branch of a company, one problem to consider is the assignment of workers with untransferable abilities. Other problems include frugally buying all furniture with a set of required capabilities, or combos of various nutritious meals.

We hope that the efficiency of this type of distributions can be improved.

Reference

[1] 16th Summer Conference International Mathematical Tournament of Towns, http://www.turgor.ru/lktg/2004/12.en/index.htm

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[3] Jack Robertson and William Webb, Cake-Cutting Algorithms: Be Fair if You Can, A K Peters, 1998.

[4] Anne Smelker, Sharing Fairly, http://www.colorado.edu/education/DMP/activities/fair_division/amsact02.html

[5] Martin Gardner, The Last Recreations: Hydras, Eggs, and Other Mathematical Mystifications, Springer, 2007.