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作品名稱

單淘汰制賽程分析

Analysis of knockout tournament

得獎獎項

數學科大會獎二等獎

美國正選代表:美國第 **61** 屆國際科技展覽會

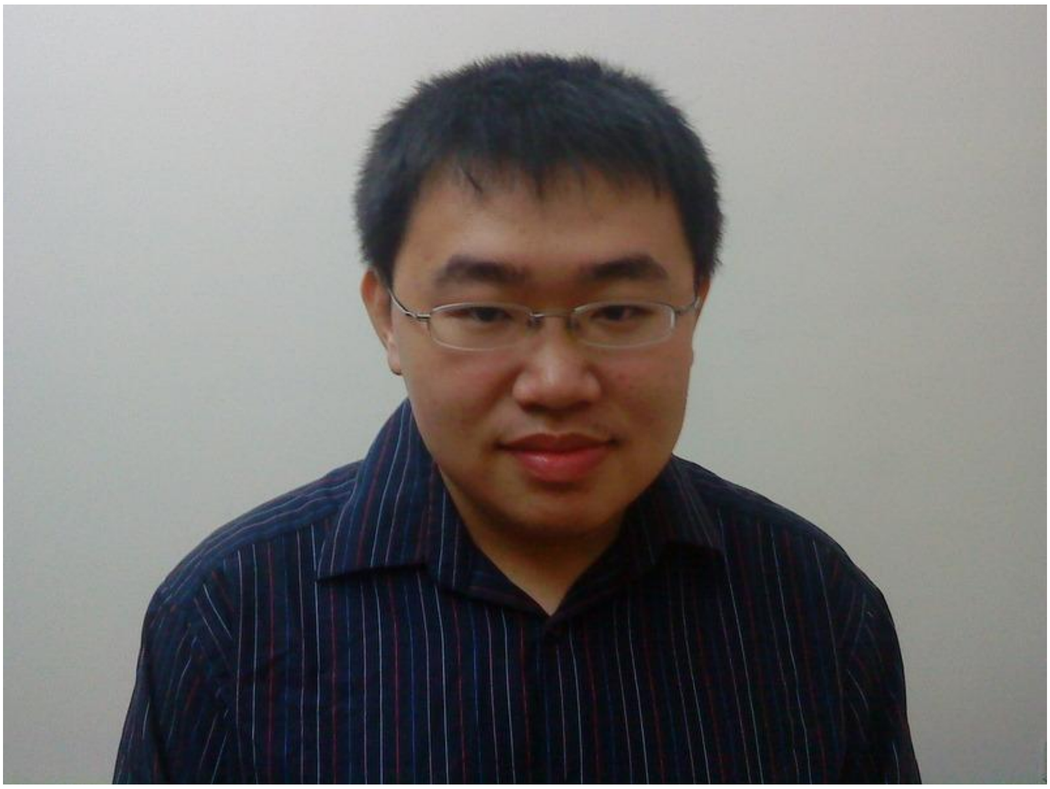
學校名稱：國立屏東高級中學

作者姓名：王啟樺

指導老師：張宮明

關鍵詞：威脅門檻、勝率一般式、賽程表現率

作者簡介



我是王啟樺，就讀屏東高中數理資優班三年級。平時喜愛閱讀寫作，國中時期勤於創作新詩，完成了個人第一部詩集——戀羽。高中始嘗試散文寫作，並將作品匯集成第一部散文集——瀲瑩。品味當代文學之餘，亦喜愛閱讀英文報章雜誌，並自習日語。人類創造許多珍貴的知識，等待我們親身閱讀、理解；掌握語言，我們便能獲得全人類努力與知識的結晶。

除此之外，我對數學更有興趣。數學是科學的語言，從小便鍾情數學的我，在高中有了機會，藉由數學，描述這份熱愛發現的心情。幸運地，於區域與全國科展時，得到了評審教授的肯定；並在青少年人才培育計畫期中報告營時，邂逅了改變我價值觀的人生導師。這段過程所學到的一切，是我踏入成年的勇氣自信，是我生命歷程中，最閃亮的一頁。因此，我更加努力地投入研究，希望在這個領域，有這個機會，能夠有所發揮。

握在手中完成的報告書，不止的顫抖與心跳同步。證明的喜悅、無解的怨懟，每當輕輕地翻開閱讀，曾經的狂喜或氣餒，於此，一一地，滿足心靈地收穫。

摘要

本研究報告針對單淘汰制賽程中存在的迷思，提出方法並加以討論。單淘汰制賽程規則為每場比賽皆有勝負(即沒有和局)，負方即失去奪得冠軍的機會，且不得出現於另一場賽程。全文分別對影響選手勝率的因素——選手實力與賽程安排，提出方法與概念討論。

第一部分討論選手實力變化對勝率的影響。利用「假想選手」的概念，討論選手於各場賽程中，所有可能遇上的對手所造成的威脅。並透過「威脅門檻」判斷對手實力的變化對自己勝率的利弊。藉由「勝率一般式」計算選手於賽程中奪冠的機率，並以各種角度觀看賽程，判斷個體與群體的實力。

第二部分討論位置安排對勝率造成的影響。由「勝率實力比」討論賽程安排對選手的公平性，並定義「賽程表現率」討論選手因賽程安排對勝率所造成的影響。

文中並以實際數據範例，希望閱讀以後的你(妳)，能認識並喜歡上淘汰賽的世界。

Abstract

This paper discusses some myths in the knockout tournament. A knockout tournament satisfies the property that each game has a winner and a loser ; a loser of a game is not involved in any further game.

In the first part, we discusses the influence of the variation in contestant's strength on winning probability. By using the conception of“Imaginary Contestant,”we can judge the threat which caused by the contestant who we may encounter in the game. Through “Threshold of Threat,”we can learn the advantages and disadvantages of the variation in contestant's strength. By applying“Formula of Winning Probability,”we can calculate the winning probability of contestant. Furthermore, by observing the tournament from various angles, we can know the strength of the individual and the group.

In the second part, we discusses the influence of the scheme of contestants on winning probability. By evaluating“Ratio of Winning Probability to Contestant's Strength,”we can know whether the scheme is fair to every contestant or not. By calculating“Rate of Winning Probability at Normal Scheme to Optimum Scheme,” we can realize the influence of the scheme of contestants on winning probability.

At the lemma, we make some examples to expect you, after reading this paper, to understand and enjoy the astonishment of knockout tournament.

壹、 前言

一、研究動機與目的

在我們的生活中，淘汰賽是一種相當普遍的比賽方式。小從校內比試，大至國際賽事，關心競賽的我們總會想知道自己支持的選手隊伍，於賽程中的勝率如何。

在數學傳播中，閱讀了黃光明教授所發表的淘汰賽一文，初步認識了淘汰賽於數學上的表達方式。另外，在于小涵同學於 2006 台灣國際科學展覽會的作品輸贏一線間-淘汰賽的相關探討，得知包含 8 人的賽程表中，次強選手勝率大於最強選手勝率之可能性。

對於淘汰賽，我們總有著許多強烈的直觀，而這些直觀，是否都無誤呢?

以古典機率切入分析賽程，我們更深入探討下列 8 個存在於淘汰賽的迷思：

(1)未進行任何賽程之前，各選手對我的威脅？

(2)任何選手的實力增加，將使我的勝率下降？

(3)種子選手有絕對的優勢？

(4)自己的實力的增加是否也會增加自己的勝率？

(5)有勝率一般式嗎？

(6)選手於賽程中的勝率實力比如何?

(7)賽程安排對選手勝率的影響?

(8)賽制是否對每個選手都公平?

二、研究方法

為了使問題得以討論，在此訂定幾個前提：

一、每位選手皆有其自身的實力，命名選手Ａ、Ｂ、Ｃ、Ｄ……，則各選手對應之實力依序為 a、b、c、d……。

二、令 P_{AB} 為選手 A 選手 B 比賽時，選手 A 勝出的機率，並規定 $P_{AB}=\dfrac{a}{a+b}$。

三、令 $P_n(A)$為選手 A 晉升至上方第 n 節點的機率。

四、令 X_n 為選手將晉升至第 n 節點時所遇到的假想選手。

五、令 $F_n(Q_i,A_j)$為選手 Q_i 於賽程表 A_j 時晉升至第 n 節點的勝率實力比。

六、令 $S_n(Q_i,A_j)$為選手 Q_i 於賽程表 A_j 時的勝率與其於最佳賽程表時的勝率之比值。

貳、研究過程

一、選手實力變化對勝率的影響

(一)包含 4 人的賽程

1.假想選手 X (Imaginary Contestant)

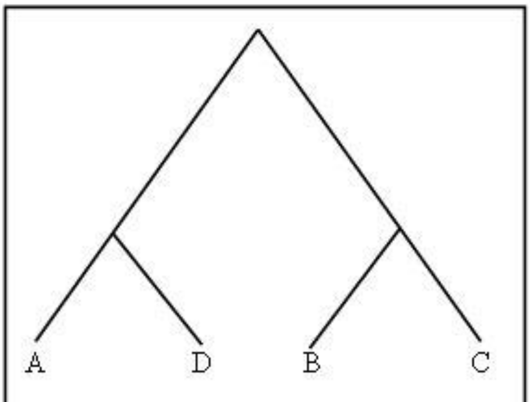


圖 1.1

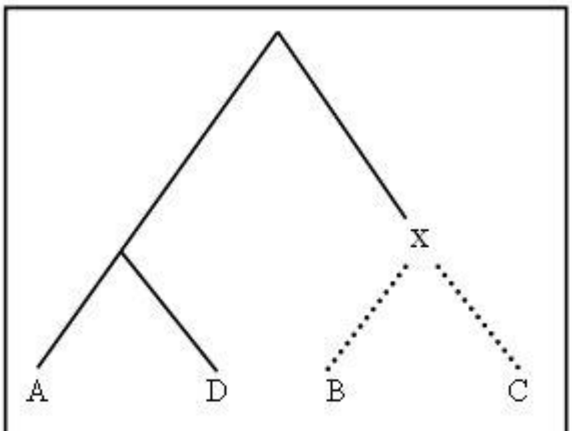


圖 1.2

直觀上我們認為「**只要敵人的實力增強，對我的威脅就會增加**」，而在賽程中，亦即意味著「**任何敵人實力增加，我的勝率就會下降**」。現在，我們想要證明這個直觀。在討論對手實力變化對我的勝率之影響之前，我們必須知道各對手對我所造成的威脅。

如圖 1.1 為一包含 4 位選手的賽程表。就選手 A 而言，第 1 場賽程的對手必然是選手 D，愈晉級所受到的威脅只消比較選手 D 的實力即可判斷。然而，第 2 場賽程可能遇上的對手為選手 B 與選手 C。在未進行任何賽程之前，這兩位選手對我的威脅有多大呢?

我們嘗試建立一假想選手 X，藉由將選手 B 與選手 C 替換為假想選手 X，我們可以得到簡化後選手 A 的梯樹賽程表如圖 1.2 表示。則我們得到一個漂亮的定理：

定理 1　選手 A 於第 2 場賽程所遇上的假想選手 X 之實力 x 為:

$$x=\frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)}$$

證明:

由圖 1.1 與圖 1.2 可分別得到選手 A 的勝率如(1)、(2)：

$$P(A)=\frac{a}{a+d}\left(\frac{b}{b+c}\frac{a}{a+b}+\frac{c}{b+c}\frac{a}{a+c}\right)\ldots\ldots(1)$$

$$P(A)=\frac{a}{a+d}\frac{a}{a+x}\quad\ldots\ldots(2)$$

同一張賽程表中，選手 A 贏得冠軍的實力相同，由(1)、(2)相等可得：

$$P(A)=\frac{a}{a+d}\left(\frac{b}{b+c}\frac{a}{a+b}+\frac{c}{b+c}\frac{a}{a+c}\right)=\frac{a}{a+d}\frac{a}{a+x}$$

$$(a+x)\left(\frac{b}{a+b}+\frac{c}{a+c}\right)=b+c$$

$$a+x=\frac{b+c}{\frac{b}{a+b}+\frac{c}{a+c}}=\frac{(a+b)(a+c)(b+c)}{b(a+c)+c(a+b)}$$

$$x=\frac{(a+b)(a+c)(b+c)}{b(a+c)+c(a+b)}-a$$

$$=\frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)}$$

藉由假想選手 X，我們可以判斷在未進行任何賽程之前，第 2 場賽程可能遇上的選手對我所造成的威脅。

2.包含 4 人賽程之威脅門檻(Threshold of Threat)

由定理 1，我們可以判斷在未進行任何賽程之前，第 2 場賽程可能遇上的選手對我造成的威脅。參照圖 1.2 選手 A 的梯樹賽程表，第二場將遇上的對手為假想選手 X。直觀上，「**選手 B 的實力增加，將造成假想選手 X 的實力增加而使我(即選手 A)的勝率下降。**」因此，我們便**猜測假想選手 X 的實力 x 為選手 B 的實力 b 之嚴格遞增函數**。當我們著手證明此一猜測後，竟發現事實並非如此。我們得到了更令人驚喜的定理 2:

> 定理 2 存在一威脅門檻 b_0，若 B 選手之實力 $b> b_0$，則 b 增加將使 $P_2(A)$下降；若 B 選手之實力 $b< b_0$，則 b 增加將使 $P_2(A)$上升。其中:
>
> $$b_0=\frac{c(\sqrt{2a(a+c)}-a)}{a+2c}$$

證明:

令 x 為函數 f(b)，使 $x=f(\mathrm{b})$。檢驗 $f(\mathrm{b})$是否為 b 之嚴格遞增函數。

$$f(b)=\frac{\mathrm{b}^2(\mathrm{a}+\mathrm{c})+\mathrm{c}^2(\mathrm{a}+\mathrm{b})}{\mathrm{b}(\mathrm{a}+\mathrm{c})+\mathrm{c}(\mathrm{a}+\mathrm{b})}=\frac{(a+c)b^2+c^2b+c^2a}{(a+2c)b+ac}$$

作 $f(b)$之一次導函數：

$$\frac{df(b)}{db}=\frac{(a+c)(a+2c)b^2+2ac(a+c)b-(ac^2(a+c))}{((a+2c)b+ac)^2}$$

其一次導函數的分母恆正，分子則為 b 之二次函數，令其為 g(b)，即:

$$\mathrm{g}(b)=(a+c)(a+2c)\mathrm{b}^2+2ac(a+c)b-(ac^2(a+c))$$

g(b)的領導係數恆正且由判別式大於 0 知其圖形與橫軸交於兩點，開口向上之拋物線。

令 g(b)為 0，解出 b 之 2 根，得一正根一負根：

$$b=\frac{-2ac(a+c)\pm\sqrt{1a^2c^2(a+c)^2+4(a+c)^2(a+2c)ac^2}}{2(a+c)(a+2c)}=\frac{-ac\pm c\sqrt{2a(a+c)}}{a+2c}$$

$$=\frac{c\,(\,-a\pm\sqrt{2a(a+c)}\,)}{a+2c}$$

取其正實根，並命名為 b_0，即:

$$b_0=\frac{c(\sqrt{2a(a+c)}-a)}{a+2c}$$

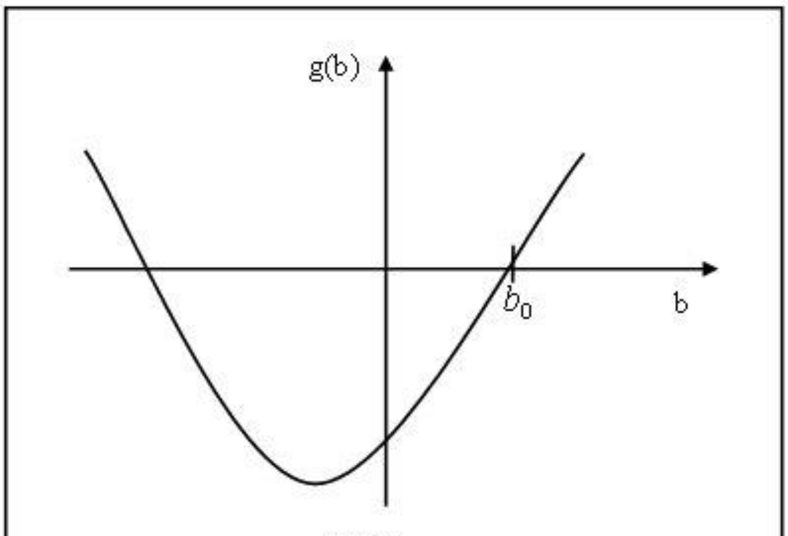


圖 1.3

將 g(b)之函數圖形繪出，如圖 1.3，則：

當 b> b_0，g(b)>0，此時 *df(b)/db>0*，*f(b)*為嚴格遞增函數，此時 b 上升將使 x 上升而使 P(A)下降。

當 0<b< b_0，g(b)<0，此 *df(b)/db>0*，*f(b)*為嚴格遞減函數，此時 b 上升將使 x 下降而使 P(A)上升。

定理 2 是一個十分耐人尋味的結果，代表 B 選手的實力 b 有一個門檻 b_0，當 $b>b_0$ 時，B 選手的實力增加，對 A 選手才會造成威脅(即使 P(A)下降)；而當 $b<b_0$ 時，B 選手實力增加卻對 A 選手之勝率 P(A)有提升的作用。不但**否定了直觀上「只**

要敵人的實力增強，對我的威脅就會增加」的迷思，甚至證明了列寧於統一戰線中所提出的「**連絡次要敵人，打擊主要敵人**」策略的理論基礎。即**培養將來的對手使其實力增加，竟亦能增加自己的勝率**，這真是非常有趣的結果。

現在，讓我們看看一些例子。在圖 1.1 中，我們令選手 C 的實力 c = a，選手 D 的實力 d = a /10，則選手 B 的威脅門檻為 a / 3。我們讓選手 B 的實力 b 由 a / 5 遞增至 a，觀察選手 B 的實力與選手 A 的勝率的關係。結果如下:

選手 B 的實力 b	$\frac{a}{5}$	$\frac{a}{4}$	$\frac{a}{3}$	$\frac{a}{2}$	a
選手 A 的勝率 P(A)	50.51%	50.91%	51.14%	50.51%	45.45%

有趣地，我們可以發現，在選手 B 的實力 b 小於威脅門檻 a/3 時，其實力增加亦會增加選手 A 的勝率；而當選手 B 的實力 b 大於威脅門檻 a/3 時，其實力增加便開始讓選手 A 的勝率下降了。

這個例子告訴我們，若選手 C 的實力與我(選手 A)相當，對我而言，選手 B 的實力最好為 a / 3。若選手 B 的實力小於 a / 3，我甚至能培養選手 B 的實力達到 a / 3，使我能有最高的勝率；反之，若選手 B 的實力大於 a / 3，我則想辦法降低選手 B 的實力至 a / 3，亦可以使我能有最高的勝率。真是有趣的現象。

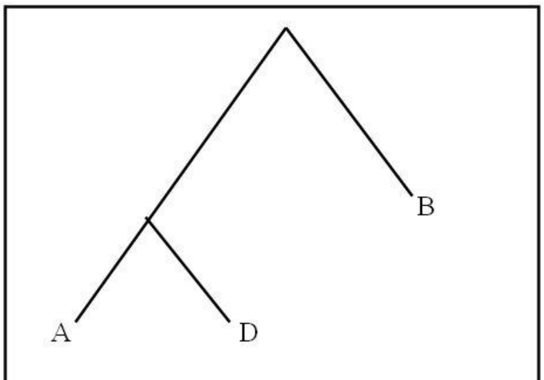


圖 1.4

另外，賽程有時後會安排種子選手，假設一包含種子選手的賽程表如圖 1.4，則:

推論 1 若選手 B 為種子選手，則其威脅門檻 b_0=0

證明:

選手 B 為種子選手，則可視為選手 C 的實力為 0(即選手 B 於第一場賽程必定晉級)。

將 c=0 代入 b_0

$$b_0 = \frac{c(\sqrt{2a(a+c)}-a)}{a+2c}$$

c=0

b_0=0

肯定地，當選手 B 為種子選手，其威脅門檻 b_0=0。

再求出 b_0=0 之其餘可能:

$$b_0 = \frac{c(\sqrt{2a(a+c)}-a)}{a+2c} = 0$$

$$a^2 + 2ac = 0$$

$$a(a+2c) = 0$$

$$a = -2c \text{ 、} 0$$

由於 a>0，c ≥ 0，使選手 B 之威脅門檻 b_0=0 之情況唯 c=0(即選手 B 為種子選手)。

3.自己實力上升對勝率的影響

直觀上，自己的實力增加，自己的勝率亦會跟著增加，努力才有收穫。如圖 1.1 包含 4 位選手的賽程表中，檢驗自己的勝率是否為自己的實力之嚴格遞增函數，則我們有:

引理 1 於包含 4 位選手之完全二元樹賽程表中，自己的實力增加使自己的勝率增加。

證明:

由圖 1.1 可得選手 A 的勝率：

$$P(A)=\frac{a}{a+d}(\frac{b}{b+c}\frac{a}{a+b}+\frac{c}{b+c}\frac{a}{a+c})$$

猜測 P(A)為選手 A 的實力 a 之嚴格遞增函數，檢驗 P(A)對 a 之一次導函數是否恆為正值。令前項為 $\mathrm{G}(a)=\dfrac{a}{a+d}$，$\dfrac{\mathrm{dG}(a)}{\mathrm{da}}>0$，可明顯地知道為變數 a 之嚴格遞增函數，令後項為 $\mathrm{H}(a)$，並求其一次導函數:

$$\mathrm{H}(a)=\frac{b}{b+c}\frac{a}{a+b}+\frac{c}{b+c}\frac{a}{a+c} = \frac{(b+c)a^2+2bca}{(b+c)a^2+(b+c)^2a+bc(b+c)}$$

$$\frac{d\mathrm{H}(a)}{da}=\frac{bc(b+c)^2a+2b^2c^2(b+c)}{((b+c)a^2+(b+c)^2a+bc(b+c))^2}$$

由其一次導函數恆為正值得知，$\mathrm{H}(a)$ 為嚴格遞增函數，自己的實力增加使自己的勝率增加。

(二)包含 2^n 人的賽程

1. 勝率一般式(1) (Formula of Winning Probability)

在賽程進行之前，我們總想先秤秤自己的斤兩，算算自己於這場比賽中得到冠軍的機率。因此，我們想要瞭解選手晉升至第 n 節點的勝率一般式如何表示，則我們有了:

定理 3 選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1)=\prod_{\mathrm{j}=1}^{\mathrm{n}}\sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}}(P_{\mathrm{j}-1}(Q_{\mathrm{i}})P_{\mathrm{Q_1Q_i}})$

證明：

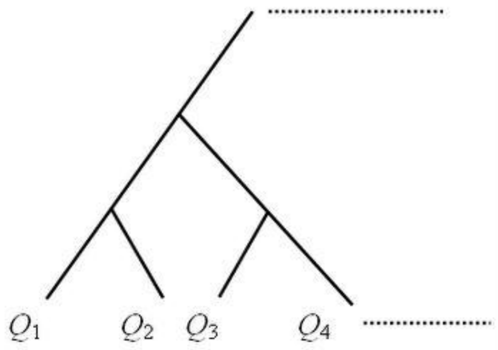


圖 1.5

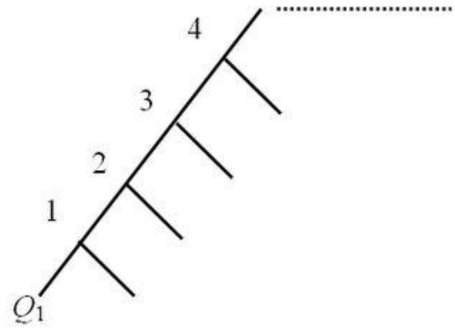


圖 1.6

假設有一包含 2^m 位選手之完全二元樹賽程表，其子節點由左至右依序命名選手為 $Q_1,Q_2,Q_3,\ldots$，如圖 1.5 所示。以 Q_1 為起始點，選手 Q_1 每勝出一場比賽即晉級至下一節點，如圖 1.6 所示。並以 $P_n(Q_1)$表示選手 Q_1 晉升至第 n 節點之機率。則：

$$P_1(Q_1) = P_{Q1Q2}$$

$$P_2(Q_1) = P_1(Q_1)(P_{Q_3Q_4}P_{Q_1Q_3} + P_{Q_4Q_3}P_{Q_1Q_4}) = P_1(Q_1)(P_1(Q_3)P_{Q_1Q_3} + P_1(Q_4)P_{Q_1Q_4})$$

$$P_3(Q_1) = P_2(Q_1)(P_2(Q_5)P_{Q_1Q_5} + P_2(Q_6)P_{Q_1Q_6} + P_2(Q_7)P_{Q_1Q_7} + P_2(Q_8)P_{Q_1Q_8})$$

推得下遞迴式：

$$P_n(Q_1) = P_{n-1}(Q_1)\sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i})$$

由遞迴式求 $P_n(Q_1)$之一般項：

$$P_1(Q_1)=P_0(Q_1)\sum_{i=2^0+1}^{2}(P_0(Q_i)P_{Q_1Q_i})$$

$$P_2(Q_1)=P_1(Q_1)\sum_{i=2^1+1}^{2^2}(P_1(Q_i)P_{Q_1Q_i})$$

$$P_3(Q_1)=P_2(Q_1)\sum_{i=2^2+1}^{2^3}(P_2(Q_i)P_{Q_1Q_i})$$

.
.
.

$$\times)\quad P_n(Q_1)=P_{n-1}(Q_1)\sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i})$$

$$P_n(Q_1)=P_0(Q_1)\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

其中，$P_0(Q_1)$表示選手 Q1 晉升至第 0 節點的機率。因未進行任何比賽之前，選手皆位於第 0 節點，故 $P_0(Q_1)=1$，則選手 Q_1 晉升至第 n 節點的機率為:

$$P_n(Q_1)=\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

由定理 3 與條件機率，我們可以進一步得到一個推論:

推論 2 選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1)=\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

證明:

令 A 事件為選手 Q_1 晉升至第 n 節點的機率；B 事件為選手 Q_1 晉升至第 m 節點的機率: C 事件為選手 Q_1 由第 n 節點晉升至第 m 節點的機率。透過條件機率，且 B 事件完全包含於 A 事件中，我們可以得到:

$$P(\mathrm{C}) = \mathrm{P}(\mathrm{B}|\mathrm{A}) = \frac{\mathrm{P}(\mathrm{A}\cap\mathrm{B})}{\mathrm{P}(\mathrm{A})} = \frac{\mathrm{P}(\mathrm{B})}{\mathrm{P}(\mathrm{A})}$$

$$P_{\mathrm{n,m}}(Q_1) = \frac{P_{\mathrm{m}}(Q_1)}{P_{\mathrm{n}}(Q_1)} = \frac{\prod_{\mathrm{j=1}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j-1}}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j=1}}^{\mathrm{n}} \sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j-1}}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}})} = \prod_{\mathrm{j=n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j-1}}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}})$$

由引理 1 我們可得於包含 4 位選手的賽程表中，自己的實力上升使自己的勝率上升。那麼，在包含 2^{m} 位選手之賽程中，是否亦是如此呢?

定理 4 於包含 2^{m} 位選手之完全二元樹賽程表中，自己的實力增加使自己的勝率增加。

證明：

由定理 3 我們可以知道選手 Q_1 晉升至第 n 節點之機率為 $\mathrm{P_n(Q_1)}$。猜測 $\mathrm{P_n(Q_1)}$ 為選手 Q_1 的實力 q_1 之嚴格遞增函數，檢驗其一次導函數是否恆為正值。

作 $\mathrm{P_n(Q_1)}$之一次導函數:

$$P_n(Q_1) = P_0(Q_1) \prod_{\mathrm{j=1}}^{\mathrm{n}} \sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j-1}}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}})$$

$$\frac{dP_{\mathrm{n}}(Q_1)}{dq_1} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \{ \sum_{\mathrm{i=2^{k-1}+1}}^{2^{\mathrm{k}}} [\frac{\mathrm{q_i}}{(\mathrm{q_i}+\mathrm{q_1})^2} P_{\mathrm{k-1}}(Q_{\mathrm{i}})] \prod_{\mathrm{j=1;j\neq k}}^{\mathrm{n}} \sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j-1}}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}}) \}$$

由其一次導函數恆為正值可得，$\mathrm{P_n(Q_1)}$為選手 Q_1 的實力 q_1 之嚴格遞增函數，證明於包含 2^{m} 位選手之完全二元樹賽程表中，自己的實力增加使自己的勝率增加。

2.假想選手 X_n 之實力

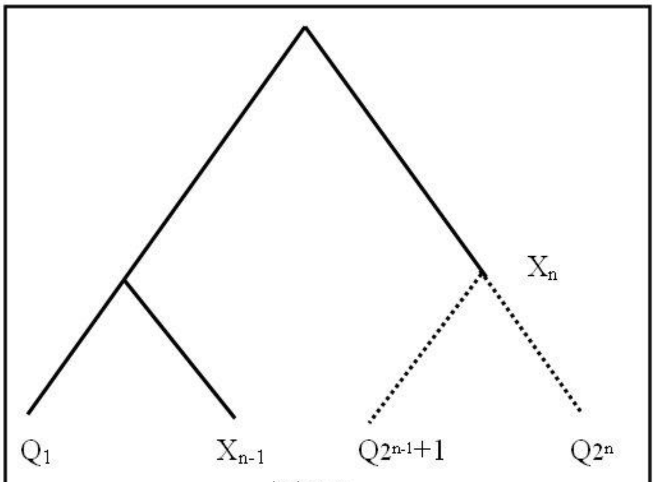


圖 1.7

在定理 1 中，我們得到於包含 4 位選手的賽程之中，第 2 場賽程所遇到的假想選手 X 的實力 x。有了勝率一般式之後，我們進一步地將假想選手 X 廣義化，令選手 Q_1 愈晉升至第 n 節點時所遇上的假想選手為 X_n 為，如圖 1.7 所示。則我們有:

定理 5 選手 Q_1 於第 n 場賽程所遇上的假想選手 X_n 之實力 x_n 為:

$$x_n = \frac{1 - \sum_{i=2^{n-1}+1}^{2^n} (P_{i-1}(Q_i) P_{Q_1 Q_i})}{\sum_{i=2^{n-1}+1}^{2^n} (P_{i-1}(Q_i) P_{Q_1 Q_i})} q_1$$

證明：

利用類似定理 1 的方式，我們將第 n 場賽程所可能遇上的所有選手視為假想選手 X_n，由圖 5.1 我們可以得到:

$$P_n(Q_1) = P_{\mathrm{n}\text{-}1}(Q_1)\frac{\mathrm{q}_1}{\mathrm{q}_1 + x_{\mathrm{n}}}$$

$$x_{\mathrm{n}}P_n(Q_1) + \mathrm{q}_1P_n(Q_1) = \mathrm{q}_1P_{\mathrm{n}\text{-}1}(Q_1)$$

$$x_{\mathrm{n}} = \frac{P_{\mathrm{n}\text{-}1}(Q_1) - P_n(Q_1)}{P_n(Q_1)}\mathrm{q}_1$$

$$= \frac{1 - \sum_{i=2^{\mathrm{n}-1}+1}^{2^{\mathrm{n}}}(P_{\mathrm{i}-1}(Q_{\mathrm{i}})P_{\mathrm{Q_1Q_i}})}{\sum_{i=2^{\mathrm{n}-1}+1}^{2^{\mathrm{n}}}(P_{\mathrm{i}-1}(Q_{\mathrm{i}})P_{\mathrm{Q_1Q_i}})}\mathrm{q}_1$$

藉由假想選手 X_n，我們可以判斷在未進行任何賽程之前，第 n 場賽程可能遇上的選手對我所造成的威脅。

3.包含 8 人賽程之威脅門檻

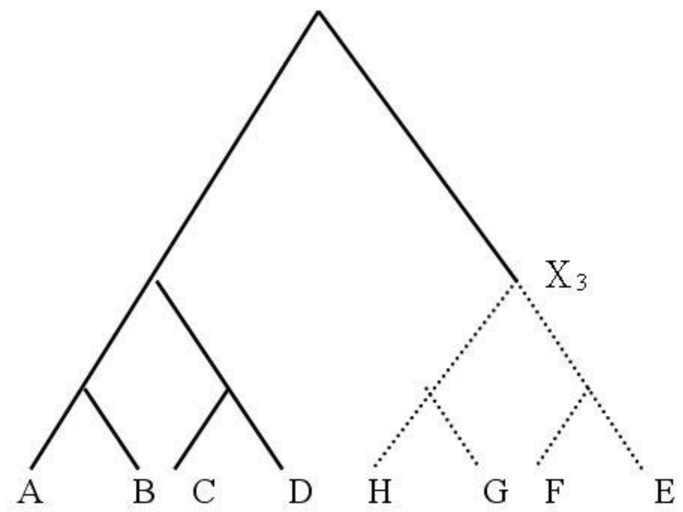


圖 1.8

得知包含 4 人的賽程中，第 2 場賽程所可能遇上的對手存在著威脅門檻後，那麼，包含 8 人的賽程中，第 3 場賽程所可能遇上的對手，是否也存在著威脅門檻呢?

定理 6 若選手 A、F、G、H 的實力 a、f、g、h 滿足

$$\frac{(g-h)^2-\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}<f<\frac{(g-h)^2+\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)},\ f>\sqrt{gh}$$

則存在一威脅門檻 e_0，若 E 選手之實力 $e>e_0$，則 e 增加將使 $P_3(A)$下降；若 E 選手之實力 $e<e_0$，則 e 增加將使 $P_3(A)$上升。

證明：

由定理，我們可以得到第三場賽程所會遇上的假想選手 X_3 的實力為:

$$x_3=\frac{1-(P_2(E)P_{AE}+P_2(F)P_{AF}+P_2(G)P_{AG}+P_2(H)P_{AH})}{(P_2(E)P_{AE}+P_2(F)P_{AF}+P_2(G)P_{AG}+P_2(H)P_{AH})}a$$

接下來，我們檢驗 X_3 的實力 x_3 是否為選手 E 的實力 e 之嚴格遞增函數。由於將其展開非常龐大，在此以另二的 e 的多項式之比值表示。則我們有:

$$x_3=\frac{1-(P_2(E)P_{AE}+P_2(F)P_{AF}+P_2(G)P_{AG}+P_2(H)P_{AH})}{(P_2(E)P_{AE}+P_2(F)P_{AF}+P_2(G)P_{AG}+P_2(H)P_{AH})}a=\frac{f(e)}{g(e)}$$

$$\frac{dx_3}{de}=\frac{\frac{df(e)}{de}g(x)-f(x)\frac{dg(e)}{de}}{(g(e))^2}=\frac{h(e)}{(g(e))^2}$$

我們解出 h(e)，得知其根與另一 e 的多項式 m(e)的根相同，為:

$$m(e)=a_6e^6+a_5e^5+a_4e^4+a_3e^3+a_2e^2+a_1e^1+a_0$$

其中:

$a_6=$

$$\begin{aligned}&2a^2gh^3+2ag^3h^2+2ag^2h^3+2a^2g^3h+2a^2g^2h^2+8h^2g^2f^2+2hg^3f^2+4h^2gf^3\\&+2h^3g^2f+2a^2gf^3+4hg^2f^3+2a^2hf^3+a^3hf^2+2h^2g^3f+2h^3gf^2\\&+2ah^3f^2+4ah^2f^3+4a^2h^2f^2+2a^2h^3f+2a^2g^3f+4a^2g^2f^2+4ag^2f^3\\&+2ag^3f^2+a^3h^2g+a^3g^2h+a^3gf^2+a^3g^2f+10ahg^2f^2+4ahg^3f\\&+4ahgf^3+6a^2hg^2f+6a^2hgf^2+2a^3hgf+8ah^2g^2f+6a^2h^2gf\\&+10ah^2gf^2+4ah^3gf+a^3h^2f\end{aligned}$$

$a_5 =$

$$8 f^3 a^2 g^2 + 12 f a^2 g h^3 + 8 f a^3 h^2 g + 28 f a^2 g^2 h^2 + 20 f a g^2 h^3 + 20 f a g^3 h^2 + 8 f a^3 g^2 h + 12 f a^2 g^3 h + 4 f^3 h g^3 + 16 f^2 a h^3 g + 16 f^3 h^2 g^2 + 2 f^3 a^3 g + 4 f^3 h^3 g + 2 f^3 a^3 h + 8 f^3 a^2 h^2 + 4 f^3 a h^3 + 16 f^2 h^3 g^2 + 4 f^2 a^3 h^2 + 16 f^2 h^2 g^3 + 4 f^2 a^3 g^2 + 4 f^2 a^2 g^3 + 2 f a^3 h^3 + 4 f^2 a^2 h^3 + 28 f^2 a^2 h^2 g + 8 f g^3 h^3 + 8 f^2 a^3 h g + 2 f a^3 g^3 + 4 a g^3 f^3 + 2 a^3 g^3 h + 8 a^2 g^2 h^3 + 8 a^2 g^3 h^2 + 4 a^3 g^2 h^2 + 8 a g^3 h^3 + 2 a^3 g h^3 + 20 f^3 a h g^2 + 48 f^2 a h^2 g^2 + 28 f^2 a^2 h g^2 + 12 f^3 a^2 h g + 20 f^3 a h^2 g + 16 f^2 a h g^3$$

a4 =

$$-2 f^4 g a h^2 - 2 f^4 g^2 a h - 2 f^4 g a^2 h + 2 f a g^2 h^4 - 2 f a^2 g h^4 + 2 f a g^4 h^2 - 2 f a^2 g^4 h + 62 f^2 h^2 a^2 g^2 + 46 f^2 g^3 a h^2 + 20 f^2 h^3 a^2 g + 22 f^3 g a^2 h^2 + 24 g^3 h^3 a f + 22 f^3 g^2 a^2 h + 6 f^3 g a^3 h + 28 g^2 h^3 a^2 f + 12 f^3 h^3 a g + 36 f^3 g^2 a h^2 + h^3 a^3 f^2 + 5 h^3 g^2 a^3 + 5 g^3 h^2 a^3 + g^3 a^3 f^2 + 3 f^3 h^2 a^3 + 3 f^3 g^2 a^3 + 12 g^3 h^3 a^2 + 4 f^3 g^3 a^2 + 20 f^2 g^3 h^3 + 10 f^3 g^3 h^2 + 4 f^3 h^3 a^2 + 10 f^3 h^3 g^2 + 18 g^2 h^2 a^3 f + 6 g^3 a^3 f h + 12 f^3 g^3 h a + 46 f^2 h^3 a g^2 + 6 h^3 g a^3 f + 16 f^2 g^2 a^3 h + 20 f^2 g^3 a^2 h + 16 f^2 h^2 a^3 g + 28 g^3 a^2 f h^2 - 2 f^4 g h^3 - 2 f^4 g^3 h + 2 g^3 a h^4 + 2 g^4 a h^3 - f a^3 g^4 - 2 f^2 g^4 a^2 - f^4 g a^3 - 2 f^4 g^3 a - 2 f^4 g^2 a^2 + 2 f g^4 h^3 + 2 f g^3 h^4 + 4 f^2 g^2 h^4 + 4 f^2 g^4 h^2 - g a^3 h^4 - 2 a h^3 f^4 - a^3 h^4 f - a^3 f^4 h - 2 a^2 f^4 h^2 - 2 a^2 f^2 h^4 - g^4 a^3 h$$

a3 =

$$-8 g f^4 a h^3 + 8 h^3 g^2 a^3 f + 4 h^3 g a^3 f^2 + 8 g^3 h^2 a^3 f + 4 f^3 g^2 h a^3 + 4 f^3 g h^2 a^3 + 4 g^3 h a^3 f^2 + 16 g^3 h^3 a^2 f + 32 f^2 g^3 h^2 a^2 + 8 f^3 g^3 h^2 a + 20 f^2 g^2 h^2 a^3 + 4 f^3 g^3 h a^2 + 32 f^2 g^2 h^3 a^2 + 32 f^2 g^3 h^3 a + 4 f^3 g h^3 a^2 + 8 f^3 g^2 h^3 a + 16 f^3 g^2 h^2 a^2 - 4 g^3 f^4 h^2 - 4 g^2 f^4 h^3 - 2 h^2 f^4 a^3 - 4 h^3 f^4 a^2 - 4 g^3 f^4 a^2 - 2 g^2 f^4 a^3 - 4 g^4 a^3 f h - 4 f^2 h^4 a^2 g + 4 f^2 g^4 a h^2 - 8 g^2 f^4 a h^2 - 8 g^2 f^4 a^2 h - 4 f^2 g^4 a^2 h - 8 g^3 f^4 h a - 2 h^4 a^3 f^2 - 2 h^4 g^2 a^3 - 2 g^4 h^2 a^3 - 2 g^4 a^3 f^2 + 4 f^2 g^4 h^3 + 4 f^2 g^3 h^4 - 4 h^4 g a^3 f - 4 g f^4 a^3 h - 4 g^4 a^2 f h^2 + 4 f^2 h^4 a g^2 - 8 h^2 f^4 a^2 g - 4 g^2 h^4 a^2 f + 4 g^3 a^3 h^3$$

a2 =

$$-2g^3h^4f^3-4g^3h^3f^4-2g^4h^3f^3-10g^3hf^4a^2-12g^3h^2f^4a-12g^2h^3f^4a-5g^2hf^4a^3-5gh^2f^4a^3-10gh^3f^4a^2-3h^4ga^3f^2-5h^4g^2a^3f-5g^4h^2a^3f-6g^4h^3a^2f-6g^3h^4a^2f+2f^2g^4h^3a+2f^2g^3h^4a-2f^3g^2h^4a-3g^4ha^3f^2-2f^3g^4h^2a-12g^2h^2f^4a^2-2f^2g^2h^4a^2-2f^3g^2h^2a^3-8f^3g^2h^3a^2-2f^2g^4h^2a^2-12f^3g^3h^3a-8f^3g^3h^2a^2-a^3g^3h^4-a^3g^4h^3-2g^4h^4a^2-f^4g^3a^3-2f^3g^3ha^3-2f^3h^3ga^3+f^3g^4a^3+2fa^3g^3h^3-4fag^4h^4-a^3f^4h^3+a^3f^3h^4+4f^2g^3a^3h^2+4f^2g^2a^3h^3+10f^2g^3a^2h^3$$

a1 =

$$-2f^4h^3ga^3-4f^3g^3h^2a^3-8f^4g^3h^3a-4f^3h^3g^2a^3-2f^4g^3ha^3+2f^3g^4ha^3-8f^4g^2h^3a^2-2fa^3g^3h^4-4fa^2g^4h^4-8f^4g^3h^2a^2+2f^3h^4ga^3-4f^4g^2h^2a^3-2fa^3g^4h^3-12f^3g^3h^3a^2-4f^3g^3h^4a-4f^3g^4h^3a$$

a0 =

$$-a^3g^2h^3f^4-2a^2g^3h^3f^4+2a^2g^4h^4f^2-a^3g^3h^2f^4-2a^3f^3g^3h^3+a^3f^3g^4h^2+a^3f^3g^2h^4+a^3f^2g^3h^4+a^3f^2g^4h^3$$

此為六次多項式，由其領導係數恆正，可得知其圖型兩端向上。觀察其常數項，**若常數項小於零，則此多項式必有一個以上的正根，即威脅門檻存在。**

分解 a_0，試求出使其為負的區域:

$$a_0=a^2g^2h^2f^2((-hf^2-gf^2-2fgh+fg^2+fh^2+gh^2+g^2h)a+(-2ghf^2+2g^2h^2))$$

若 a0 小於零，則必滿足:

$$-hf^2-gf^2-2fgh+fg^2+fh^2+gh^2+g^2h<0\ldots(1)$$
$$-2ghf^2+2g^2h^2<0\ldots(2)$$

首先，解出滿足(1)的區域，則:

$$-hf^2-gf^2-2fgh+fg^2+fh^2+gh^2+g^2h<0$$
$$-(g+h)f^2+(g-h)^2f+(g+h)gh<0$$
$$\frac{(g-h)^2-\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}<f<\frac{(g-h)^2+\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}$$

再解出滿足(2)的區域，則:

$$
\begin{aligned}
&-2ghf^2 + 2g^2h^2 < 0 \\
&-2gh(f^2 - gh) < 0 \\
&f > \sqrt{gh}\ , f < -\sqrt{gh}\ (\text{不合，因}f > 0)
\end{aligned}
$$

綜合(1)、(2)，我們可得若選手 A、F、G、H 的實力 a、f、g、h 滿足

$$
\frac{(g-h)^2 - \sqrt{(g^2+h^2)^2 + 4g^2h^2}}{2(g+h)} < f < \frac{(g-h)^2 + \sqrt{(g^2+h^2)^2 + 4g^2h^2}}{2(g+h)}\ , f > \sqrt{gh}
$$

選手 E 的實力 e 存在一威脅門檻 e_0，當 $e > e_0$ 時，E 選手的實力增加，對 A 選手才會造成威脅(即使 P_3 (A)下降)；而當 $e < e_0$ 時，E 選手實力增加卻對 A 選手之勝率 P_3(A)有提升的作用。

3.包含 2^n 人賽程之威脅門檻

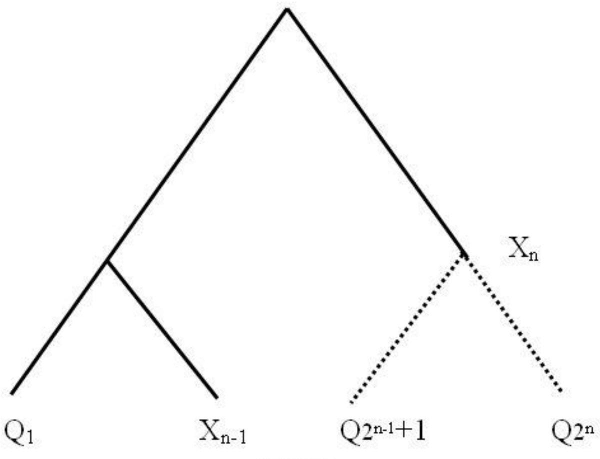


圖 1.7

在包含 2^n 人賽程中，由於假想選手 X_n 選手的實力 x_n 展開過於龐大，無法對其微分。然而，我們仍能寫出當包含 2^n 人賽程之威脅門檻存在時，所須滿足的條件。則我們有:

定理 7 若選手 $Q_{2^{n-1}+2}, Q_{2^{n-1}+3}, \dots , Q_{2^n}$ 的實力存在 $q_{2^{n-1}+2}, q_{2^{n-1}+3}, \dots , q_{2^n}>0$ 滿足 $\frac{dx_n}{dq_{2^n-1}}<0$，則存在一威脅門檻 $q_{(2^{n-1}+1)0}$，若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}> q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$下降；若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}< q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$上升。

證明：

選手 Q_1 於第 n 場賽程所遇上的假想選手 X_n 之實力 x_n 為:

$$x_n = \frac{1-\sum_{i=2^{n-1}+1}^{2^n}\left(P_{i-1}(Q_i)P_{Q_1Q_i}\right)}{\sum_{i=2^{n-1}+1}^{2^n}\left(P_{i-1}(Q_i)P_{Q_1Q_i}\right)}q_1$$

對假想選手 X_n 之實力 x_n 作選手 $Q_{2^{n-1}+1}$ 之實力 $q_{2^{n-1}+1}$ 的一次導函數:

$$\frac{dx_n}{dq_{2^n-1}}$$

若存在 $q_{2^{n-1}+2}, q_{2^{n-1}+3}, \dots , q_{2^n}>0$ 使其一次導函數< 0，則至少存在一威脅門檻 $q_{(2^{n-1}+1)0}$，若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}> q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$下降；若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}< q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$上升。

4.勝率一般式(2)

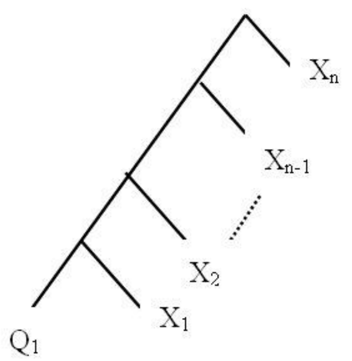


圖 1.9

藉由假想敵人的概念，我們可以將賽程表簡化如圖 1.9。則我們有:

定理 8 有一多項式 $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_1, x_2, x_3, \ldots, x_n$，

則選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1) = \dfrac{q_1^n}{\sum_{k=0}^{n} (-1)^k \dfrac{a_{n-k}}{a_n} q_1^{n-k}}$

證明：

根據圖 1.9，我們可以得到選手則選手 Q_1 晉升至第 n 節點的機率為:

$$P_n(Q_1) = \prod_{k=1}^{n} P_{Q_1 X_K}$$

假設有一多項式 $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_1, x_2, x_3, \ldots, x_n$。由根與係數的關係我們知道:

$$-\frac{a_{n-1}}{a_n} = x_1 + x_2 + \ldots + x_n$$

$$\frac{a_{n-2}}{a_n} = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n$$

.
.
.

$$(-1)^k \frac{a_{n-k}}{a_n} = \text{n 個根取 k 個相乘之和}$$

.
.
.

$$(-1)^n \frac{a_0}{a_n} = x_1 x_2 x_3 \ldots x_{n-1} x_n$$

回到選手 Q_1 晉升至第 n 節點的機率，我們可以得到:

$$P_n(Q_1)=\prod_{k=1}^{n}P_{Q_1X_K}$$

$$=\frac{q_1^n}{(q_1+x_1)(q_1+x_2)\ldots(q_1+x_n)}$$

$$=\frac{q_1^n}{q_1^n+(x_1+x_2+\ldots+x_n)q_1^{n-1}+(x_1x_2+x_1x_3+\ldots+x_{n-1}x_n)q_1^{n-2}+\ldots+x_1x_2x_3\ldots x_{n-1}x_n}$$

$$=\frac{q_1^n}{\sum_{k=0}^{n}(-1)^k\frac{a_{n-k}}{a_n}q_1^{n-k}}$$

由定理 8 與條件機率，我們可以進一步得到:

推論 3 有一多項式 $a_{m-n}x^{m-n}+a_{m-n-1}x^{m-n-1}+\ldots+a_1x+a_0=0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$ 則選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1)=\frac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\frac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

證明:

令 A 事件為選手 Q_1 晉升至第 n 節點的機率；B 事件為選手 Q_1 晉升至第 m 節點的機率: C 事件為選手 Q_1 由第 n 節點晉升至第 m 節點的機率。透過條件機率，且 B 事件完全包含於 A 事件中，我們可以得到:

$$P(C)=P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{P(B)}{P(A)}$$

$$P_{n,m}(Q_1)=\frac{P_m(Q_1)}{P_n(Q_1)}=\frac{\dfrac{q_1^m}{\sum_{k=0}^{m}(-1)^k\dfrac{a_{m-k}}{a_m}q_1^{m-k}}}{\dfrac{q_1^n}{\sum_{k=0}^{n}(-1)^k\dfrac{a_{n-k}}{a_n}q_1^{n-k}}}=\frac{\prod_{k=1}^{m}P_{Q_1X_K}}{\prod_{k=1}^{n}P_{Q_1X_K}}=\prod_{k=n}^{n}P_{Q_1X_K}$$

$$=\frac{q_1^{m-n}}{(q_1+x_n)(q_1+x_{n+1})\dots(q_1+x_m)}$$

$$=\frac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\dfrac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

6.勝率一般式(3)

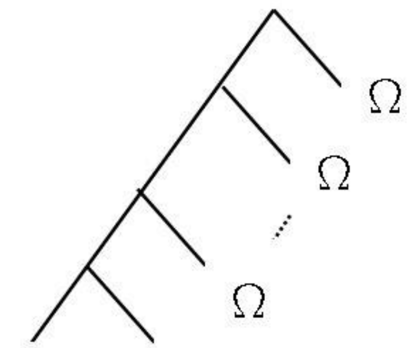


圖 1.10

進一步的，我們思考，贏得冠軍的機率若是打贏同一個對手 n 次，則此對手的實力如何呢?因此，我們有了:

定理 9　選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1)=\frac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k}\Omega^k}$，其中

$\Omega=\sqrt[n]{\prod_{k=1}^{n}(q_1+x_k)}-q_1$

證明：

根據圖 1.9 與圖 1.10，我們可以得到選手則選手 Q_1 晉升至第 n 節點的機率為:

$$P_n(Q_1)=\prod_{k=1}^{n} P_{Q_1 x_K}=(P_{Q_1\Omega})^n$$

$$\frac{q_1^n}{\prod_{k=1}^{n}(q_1+x_k)}=\frac{q_1^n}{(q_1+\Omega)^n}$$

$$\sqrt[n]{\prod_{k=1}^{n}(q_1+x_k)}=\Omega+q_1$$

$$\Omega=\sqrt[n]{\prod_{k=1}^{n}(q_1+x_k)}-q_1$$

得到Ω的實力後，再求出包含Ω的勝率一般式:

$$P_n(Q_1)\ =(P_{Q_1\Omega})^n=\frac{q_1^n}{(q_1+\Omega)^n}=\frac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k}\Omega^k}$$

由定理 9 與條件機率，我們可以進一步得到:

推論 4 選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1) = \frac{q_1^{m-n}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k}，其中 \Omega = \sqrt[m-n]{\prod_{k=n}^{m}(q_1 + x_k)} - q_1$$

證明:

令 A 事件為選手 Q_1 晉升至第 n 節點的機率；B 事件為選手 Q_1 晉升至第 m 節點的機率: C 事件為選手 Q_1 由第 n 節點晉升至第 m 節點的機率。透過條件機率，且 B 事件完全包含於 A 事件中，我們可以得到:

$$P(C) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$$

$$P_{n,m}(Q_1) = \frac{P_m(Q_1)}{P_n(Q_1)} = \frac{\prod_{k=1}^{m} P_{Q_1X_K}}{\prod_{k=1}^{n} P_{Q_1X_K}} = \prod_{k=n}^{m} P_{Q_1X_K} = (P_{Q_1\Omega})^{m-n}$$

$$= \frac{q_1^{m-n}}{(q_1 + \Omega)^{m-n}}$$

$$= \frac{q_1^{m-n}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k}$$

其中:

$$\frac{q_1^{m-n}}{\prod_{k=n}^{m}(q_1 + x_k)} = \frac{q_1^{m-n}}{(q_1 + \Omega)^{m-n}}$$

$$\sqrt[m-n]{\prod_{k=n}^{m}(q_1 + x_k)} = \Omega + q_1$$

$$\Omega = \sqrt[m-n]{\prod_{k=n}^{m}(q_1 + x_k)} - q_1$$

7.勝率一般式(4)

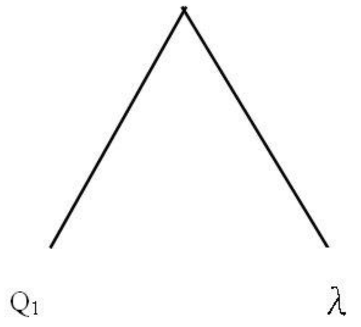


圖 1.11

另外，如果我把整場賽程，看作一個敵人，這個敵人的實力如何呢?因此，我們有了:

定理 10 選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1)=\dfrac{q_1}{q_1+\lambda}$，其中

$$\lambda=\frac{1-\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

證明：

由圖 3.5，我們可以得到選手 Q_1 晉升至第 n 節點的機率為:

$$P_n(Q_1)=\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})=P_{Q_1\lambda}$$

$$\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})=\frac{q_1}{q_1+\lambda}$$

$$\lambda=\frac{1-\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

由定理 10 與條件機率，我們可以進一步得到:

推論 5 選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_n(Q_1)=\frac{q_1}{q_1+\lambda}\text{，其中 }\lambda=\frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

證明:

令 A 事件為選手 Q_1 晉升至第 n 節點的機率；B 事件為選手 Q_1 晉升至第 m 節點的機率: C 事件為選手 Q_1 由第 n 節點晉升至第 m 節點的機率。透過條件機率，且 B 事件完全包含於 A 事件中，我們可以得到:

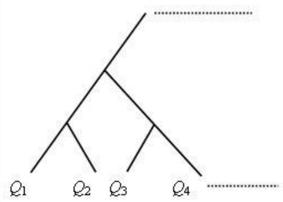
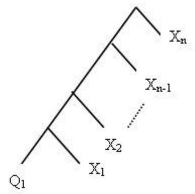
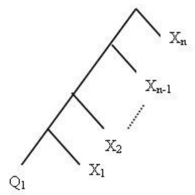
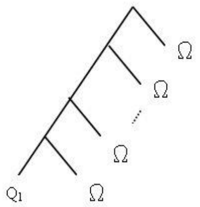
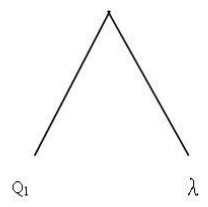
$$P(C)=P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{P(B)}{P(A)}$$

$$P_{n,m}(Q_1)=\frac{P_m(Q_1)}{P_n(Q_1)}=\frac{\prod_{j=1}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}=\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})=P_{Q_1\lambda}$$

$$\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})=\frac{q_1}{q_1+\lambda}$$

$$\lambda=\frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

經過推廣，我們利用不同角度觀看賽程，分別得到了勝率一般式(1)~(4)。下以表格比較其觀看角度與選手實力:

賽程表	勝率一般式	選手實力
 圖 3.1	$P_n(Q_1)=\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$	
 圖 3.2	$P_n(Q_1)=\prod_{k=1}^{n}P_{Q_1X_K}$	$x_n=\dfrac{1-\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})}{\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})}q_1$
 圖 3.3	$P_n(Q_1)=\dfrac{q_1^n}{\sum_{k=0}^{n}(-1)^k\frac{a_{n-k}}{a_n}q_1^{n-k}}$	多項式 $a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0=$ ，其 n 個根為 $x_1, x_2, x_3, \dots, x_n$
 圖 3.4	$P_n(Q_1)=\dfrac{q_1^n}{\sum_{k=0}^{n}C_k^nq_1^{n-k}\Omega^k}$	$\Omega=\sqrt[n]{\sum_{k=0}^{n}(-1)^k\frac{a_{n-k}}{a_n}q_1^{n-k}}-q_1$
 圖 3.5	$P_n(Q_1)=\dfrac{q_1}{q_1+\lambda}$	$\lambda=\dfrac{1-\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$

由不同角度觀看賽程，我們可以得到個體與群體的實力。

二、賽程安排對勝率的影響

(一)勝率實力比(Ratio of Winning Probability to Contestant Strength)

1.勝率實力比(1)

我們常說:「有一分實力，就有一分勝率」，彷彿意味著實力與勝率有正比的關係。我們定義勝率實力比為選手於賽程的勝率與其實力的比值。以 $F_n(Q_1)$表示選手 Q_1 晉升至第 n 節點的勝率實力比，則我們可以得:

定理 11 選手 Q_1 晉升至第 n 節點的勝率實力比為 $F_n(Q_1)=q_1^{\,n-1}\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}\frac{P_{j-1}(Q_i)}{q_1+q_i}$ 。

證明：

由勝率實力比的定義，我們可得:

$$F_n(Q_1)=\frac{P_n(Q_1)}{q_1}=\frac{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{q_1}=q_1^{\,n-1}\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}\frac{P_{j-1}(Q_i)}{q_1+q_i}$$

由推論 2 與定理 6，我們可以進一步得到一個推論:

推論 6 選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=q_1^{\,m-n-1}\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

證明:

由推論 2 我們可得選手 Q_1 由第 n 節點晉升至第 m 節點的機率。進一步地，由推論 2 與勝率實力比的定義，我們可得:

$$F_{n,m}(Q_1)=\frac{P_{n,m}(Q_1)}{q_1}=\frac{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{q_1}=q_1^{m-n-1}\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

2.勝率實力比(2)

同樣地，我們亦可以由勝率一般式(2)求出勝率實力比(2):

> 定理 12 有一多項式 $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$
>
> 則選手 Q_1 晉升至第 n 節點的勝率實力比為 $F_n(Q_1)=\dfrac{q_1^{n-1}}{\sum_{k=0}^{n}(-1)^k \dfrac{a_{n-k}}{a_n} q_1^{n-k}}$。

證明：

由勝率實力比的定義，我們可得:

$$F_n(Q_1)=\frac{P_n(Q_1)}{q_1}=\frac{\dfrac{q_1^n}{\sum_{k=0}^{n}(-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}}{q_1}=\frac{q_1^{n-1}}{\sum_{k=0}^{n}(-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}$$

由推論 3 與定理 8，我們可以進一步得到一個推論:

> 推論 7 有一多項式 $a_{m-n}x^{m-n} + a_{m-n-1}x^{m-n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$ 則選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:
>
> $$F_{n,m}(Q_1)=\frac{q_1^{m-n-1}}{\sum_{k=0}^{(m-n)}(-1)^k \frac{a_{(m-n)-k}}{a_{(m-n)}} q_1^{n-k}}$$

證明:

由推論 3 我們可得選手 Q_1 由第 n 節點晉升至第 m 節點的機率。進一步地，由推論 3 與勝率實力比的定義，我們可得:

$$F_{n,m}(Q_1)=\frac{P_{n,m}(Q_1)}{q_1}=\frac{\dfrac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\dfrac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}}{q_1}=\frac{q_1^{m-n-1}}{\sum_{k=0}^{(m-n)}(-1)^k\dfrac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

3.勝率實力比(3)

同樣地，我們還可以由勝率一般式(3)求出勝率實力比(3):

定理 13 選手 Q1 晉升至第 n 節點的勝率實力比為

$$F_n(Q_1)=\frac{q_1^{n-1}}{\sum_{k=0}^{n}C_k^n q_1^{n-k}\Omega^k}\text{，其中 }\Omega=\sqrt[n]{\prod_{k=1}^{n}(q_1+x_k)}-q_1$$

證明：

由勝率實力比的定義，我們可得:

$$F_n(Q_1)=\frac{P_n(Q_1)}{q_1}=\frac{\dfrac{q_1^n}{\sum_{k=0}^{n}C_k^n q_1^{n-k}\Omega^k}}{q_1}=\frac{q_1^{n-1}}{\sum_{k=0}^{n}C_k^n q_1^{n-k}\Omega^k}$$

由推論 4 與定理 9，我們可以進一步得到一個推論:

推論 8 選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=\frac{q_1^{m-n-1}}{\sum_{k=0}^{m-n}C_k^{m-n}q_1^{m-n-k}\Omega^k}\quad\text{其中: }\Omega=\sqrt[m-n]{\prod_{k=n}^{m}(q_1+x_k)}-q_1$$

證明:

由推論 4 我們可得選手 Q_1 由第 n 節點晉升至第 m 節點的機率。進一步地，由推論 4 與勝率實力比的定義，我們可得:

$$\mathrm{F}_{\mathrm{n,m}}(Q_1)=\frac{P_{\mathrm{n,m}}(Q_1)}{\mathrm{q}_1}=\frac{\dfrac{\mathrm{q}_1^{\mathrm{m-n}}}{\sum_{\mathrm{k=0}}^{\mathrm{m-n}}\mathrm{C}_{\mathrm{k}}^{\mathrm{m-n}}\mathrm{q}_1^{\mathrm{m-n-k}}\Omega^{\mathrm{k}}}}{\mathrm{q}_1}=\frac{\mathrm{q}_1^{\mathrm{m-n-1}}}{\sum_{\mathrm{k=0}}^{\mathrm{m-n}}\mathrm{C}_{\mathrm{k}}^{\mathrm{m-n}}\mathrm{q}_1^{\mathrm{m-n-k}}\Omega^{\mathrm{k}}}$$

4.勝率實力比(4)

同樣地，我們還可以由勝率一般式(4)求出勝率實力比(4):

定理 14 選手 Q_1 晉升至第 n 節點的勝率實力比為

$$\mathrm{F}_n(Q_1)=\frac{1}{\mathrm{q}_1+\lambda}\ \text{其中}\ \lambda=\frac{1-\prod_{\mathrm{j=1}}^{\mathrm{n}}\sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}}(P_{\mathrm{j-1}}(Q_{\mathrm{i}})P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j=1}}^{\mathrm{n}}\sum_{i=2^{\mathrm{j-1}}+1}^{2^{\mathrm{j}}}(P_{\mathrm{j-1}}(Q_{\mathrm{i}})P_{\mathrm{Q_1Q_i}})}\mathrm{q}_1$$

證明：

由勝率實力比的定義，我們可得:

$$\mathrm{F}_n(Q_1)=\frac{\mathrm{P}_n(Q_1)}{\mathrm{q}_1}=\frac{\dfrac{\mathrm{q}_1}{\mathrm{q}_1+\lambda}}{\mathrm{q}_1}=\frac{1}{\mathrm{q}_1+\lambda}$$

由推論 5 與定理 10，我們可以進一步得到一個推論:

推論 9　選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=\frac{1}{q_1+\lambda}\quad \text{，其中:}\lambda=\frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

證明:

由推論 2 我們可得選手 Q_1 由第 n 節點晉升至第 m 節點的機率。進一步地，由推論 2 與勝率實力比的定義，我們可得:

$$F_{n,m}(Q_1)=\frac{P_{n,m}(Q_1)}{q_1}=\frac{\frac{q_1}{q_1+\lambda}}{q_1}=\frac{1}{q_1+\lambda}$$

(二)賽程表現率(Rate of Winning Probability at Normal Scheme to Optimum Scheme)

1.賽程安排之影響

各位選手的實力不同，故其勝率亦有所不同。但一個賽制要公平，我們認為所有選手的勝率實力比得要相同。由定理 11 的勝率實力比，我們可以得知選手實力於賽程中的發揮情形。當參賽選手的實力固定時，賽程的安排是否會影響選手的勝率實力比?

包含 4 位選手的賽程表中，共有 3 種不同的賽程安排，如圖 2.1、圖 2.2、圖 2.3:

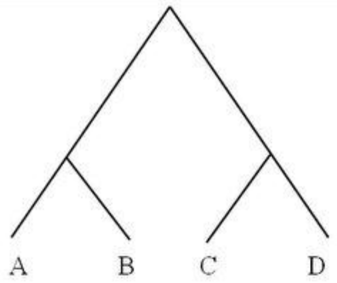


圖 2.1 賽程表 T_1

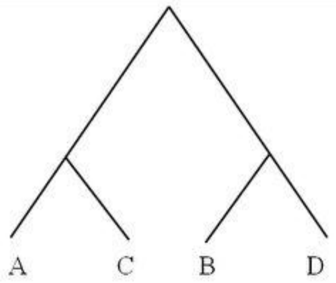


圖 2.2 賽程表 T_2

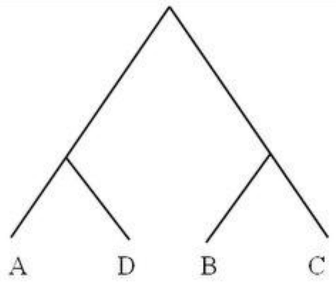


圖 2.3 賽程表 T_3

依序命名賽程表圖 2.1、圖 2.2、圖 2.3 為 T_1、T_2、T_3。並規定選手 A、B、C、D 間實力大小關係為 $a > b > c > d$ 。利用定理 11 的勝率實力比計算各選手於不同賽程中的勝率實力比，比較找出對選手最有利的賽程安排。

首先，我們將賽程表變數加入勝率實力比，令 $F_n(A, T_j)$為選手 A 於賽程表 T_j 時，晉升至第 n 節點的勝率實力比。對於 4 位選手，我們可以分別得到下面的引理:

引理 2 選手 A 的勝率實力比 $F_2(A, T_3) > F_2(A, T_2) > F_2(A, T_1)$。

證明:

列出選手 A 於各賽程表中的勝率實力比:

$$F_2(A,T_1) = a\frac{P_0(B)}{a+b}\left(\frac{P_1(C)}{a+c}+\frac{P_1(D)}{a+d}\right)$$

$$F_2(A,T_2) = a\frac{P_0(C)}{a+c}\left(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d}\right)$$

$$F_2(A,T_3) = a\frac{P_0(D)}{a+d}\left(\frac{P_1(B)}{a+b}+\frac{P_1(C)}{a+c}\right)$$

先比較 $F_2(A, T_3)$與 $F_2(A, T_2)$間大小:

$$\begin{aligned}
&F_2(A, T_3) - F_2(A, T_2)\\
&= a\frac{P_0(D)}{a+d}\left(\frac{P_1(B)}{a+b}+\frac{P_1(C)}{a+c}\right) - a\frac{P_0(C)}{a+c}\left(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d}\right)\\
&= a\left(\left(\frac{P_0(D)}{a+d}-\frac{P_0(C)}{a+c}\right)\frac{P_1(B)}{a+b}+\frac{P_0(D)P_1(C)-P_0(C)P_1(D)}{(a+c)(a+d)}\right)\\
&= a\left(\left(\frac{1}{a+d}-\frac{1}{a+c}\right)\frac{P_1(B)}{a+b}+\frac{P_1(C)-P_1(D)}{(a+c)(a+d)}\right)>0
\end{aligned}$$

得: $F_2(A, T_3) > F_2(A, T_2)$

再比較 $F_2(A, T_2)$與 $F_2(A, T_1)$間大小:

$$F_2(A, T_2) - F_2(A, T_1)$$

$$= a\frac{P_0(C)}{a+c}\left(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d}\right) - a\frac{P_0(B)}{a+b}\left(\frac{P_1(C)}{a+c}+\frac{P_1(D)}{a+d}\right)$$

$$= a\left(\left(\frac{P_0(C)}{a+c}-\frac{P_0(B)}{a+b}\right)\frac{P_1(D)}{a+d}+\frac{P_0(C)P_1(B)-P_0(B)P_1(C)}{(a+b)(a+c)}\right)$$

$$= a\left(\left(\frac{1}{a+c}-\frac{1}{a+b}\right)\frac{P_1(D)}{a+d}+\frac{P_1(B)-P_1(C)}{(a+b)(a+c)}\right) > 0$$

得: $F_2(A, T_2) > F_2(A, T_1)$

綜合得: $F_2(A, T_3) > F_2(A, T_2) > F_2(A, T_1)$

引理 3 選手 B 的勝率實力比 $F_2(B, T_2) > F_2(B, T_3) > F_2(B, T_1)$。

證明:

列出選手 B 於各賽程表中的勝率實力比:

$$F_2(B, T_1) = b\frac{P_0(A)}{a+b}\left(\frac{P_1(C)}{b+c}+\frac{P_1(D)}{b+d}\right)$$

$$F_2(B, T_2) = b\frac{P_0(D)}{b+d}\left(\frac{P_1(A)}{a+b}+\frac{P_1(C)}{b+c}\right)$$

$$F_2(B, T_3) = b\frac{P_0(C)}{b+c}\left(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d}\right)$$

先比較 $F_2(B, T_2)$與 $F_2(B, T_3)$間大小:

$$F_2(B, T_2) - F_2(B, T_3)$$

$$= b\frac{P_0(D)}{b+d}\left(\frac{P_1(A)}{a+b}+\frac{P_1(C)}{b+c}\right) - b\frac{P_0(C)}{b+c}\left(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d}\right)$$

$$= b\left(\left(\frac{P_0(D)}{b+d}-\frac{P_0(C)}{b+c}\right)\frac{P_1(A)}{a+b}+\frac{P_0(D)P_1(C)-P_0(C)P_1(D)}{(b+c)(b+d)}\right)$$

$$= b\left(\left(\frac{1}{b+d}-\frac{1}{b+c}\right)\frac{P_1(A)}{a+b}+\frac{P_1(C)-P_1(D)}{(b+c)(b+d)}\right) > 0$$

得: $F_2(B, T_2) > F_2(B, T_3)$

再比較 $F_2(B, T_3)$與 $F_2(B, T_1)$間大小:

$$F_2(B, T_3) > F_2(B, T_1)$$

$$= b\frac{P_0(C)}{b+c}\left(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d}\right) - b\frac{P_0(A)}{a+b}\left(\frac{P_1(C)}{b+c}+\frac{P_1(D)}{b+d}\right)$$

$$= b\left(\left(\frac{P_0(C)}{b+c}-\frac{P_0(A)}{a+b}\right)\frac{P_1(D)}{b+d}+\frac{P_0(C)P_1(A)-P_0(A)P_1(C)}{(a+b)(b+c)}\right)$$

$$= b\left(\left(\frac{1}{b+c}-\frac{1}{a+b}\right)\frac{P_1(D)}{b+d}+\frac{P_1(A)-P_1(C)}{(a+b)(b+c)}\right) > 0$$

得: $F_2(B, T_3) > F_2(B, T_1)$

綜合得: $F_2(B, T_2) > F_2(B, T_3) > F_2(B, T_1)$

引理 4 選手 C 的勝率實力比 $F_2(C, T_1) > F_2(C, T_3) > F_2(C, T_2)$。

證明:

列出選手 C 於各賽程表中的勝率實力比:

$$F_2(C, T_1) = c\frac{P_0(D)}{c+d}\left(\frac{P_1(A)}{a+c}+\frac{P_1(B)}{b+c}\right)$$

$$F_2(C, T_2) = c\frac{P_0(A)}{a+c}\left(\frac{P_1(B)}{b+c}+\frac{P_1(D)}{c+d}\right)$$

$$F_2(C, T_3) = c\frac{P_0(B)}{b+c}\left(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d}\right)$$

先比較 $F_2(C, T_1)$與 $F_2(C, T_3)$間大小:

$$F_2(C, T_1) - F_2(C, T_3)$$

$$= c\frac{P_0(D)}{c+d}\left(\frac{P_1(A)}{a+c}+\frac{P_1(B)}{b+c}\right) - c\frac{P_0(B)}{b+c}\left(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d}\right)$$

$$= c\left(\left(\frac{P_0(D)}{c+d}-\frac{P_0(B)}{b+c}\right)\frac{P_1(A)}{a+c}+\frac{P_0(D)P_1(B)-P_0(B)P_1(D)}{(b+c)(c+d)}\right)$$

$$= c\left(\left(\frac{1}{c+d}-\frac{1}{b+c}\right)\frac{P_1(A)}{a+c}+\frac{P_1(B)-P_1(D)}{(b+c)(c+d)}\right) > 0$$

得: $F_2(C, A_1) > F_2(C, A_3)$

再比較 $F_2(C, T_3)$與 $F_2(C, T_2)$間大小:

$$F_2(C, T_3) - F_2(C, T_2)$$

$$= c\frac{P_0(B)}{b+c}(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d}) - c\frac{P_0(A)}{a+c}(\frac{P_1(B)}{b+c}+\frac{P_1(D)}{c+d})$$

$$= c((\frac{P_0(B)}{b+c} - \frac{P_0(A)}{a+c})\frac{P_1(D)}{c+d}+\frac{P_0(B)P_1(A) - P_0(A)P_1(B)}{(a+c)(b+c)})$$

$$= (c(\frac{1}{b+c} - \frac{1}{a+c})\frac{P_1(D)}{c+d}+\frac{P_1(A) - P_1(B)}{(a+c)(b+c)}) > 0$$

得: $F_2(C, T_3) > F_2(C, T_2)$

綜合得: $F_2(C, T_1) > F_2(C, T_3) > F_2(C, T_2)$

引理 5 選手 D 的勝率實力比 $F_2(D, T_1) > F_2(D, T_2) > F_2(D, T_3)$。

證明:

列出選手 D 於各賽程表中的勝率實力比:

$$F_2(D, T_1) = d\frac{P_0(C)}{c+d}(\frac{P_1(A)}{a+d}+\frac{P_1(B)}{b+d})$$

$$F_2(D, T_2) = d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d}+\frac{P_1(C)}{c+d})$$

$$F_2(D, T_3) = d\frac{P_0(A)}{a+d}(\frac{P_1(B)}{b+d}+\frac{P_1(C)}{c+d})$$

先比較 $F_2(D, T_1)$與 $F_2(D, T_2)$間大小:

$$F_2(D, T_1) - F_2(D, T_2)$$

$$= d\frac{P_0(C)}{c+d}(\frac{P_1(A)}{a+d}+\frac{P_1(B)}{b+d}) - d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d}+\frac{P_1(C)}{c+d})$$

$$= d((\frac{P_0(C)}{c+d} - \frac{P_0(B)}{b+d})\frac{P_1(A)}{a+d}+\frac{P_0(C)P_1(B) - P_0(B)P_1(C)}{(b+d)(c+d)})$$

$$= d((\frac{1}{c+d} - \frac{1}{b+d})\frac{P_1(A)}{a+d}+\frac{P_1(B) - P_1(C)}{(b+d)(c+d)}) > 0$$

得: $F_2(D, T_1) > F_2(D, T_2)$

再比較 $F_2(D, T_2)$與 $F_2(D, T_3)$間大小:

$F_2(D, T_2) - F_2(D, T_3)$

$$= d\frac{P_0(B)}{b+d}\left(\frac{P_1(A)}{a+d}+\frac{P_1(C)}{c+d}\right) - d\frac{P_0(A)}{a+d}\left(\frac{P_1(B)}{b+d}+\frac{P_1(C)}{c+d}\right)$$

$$= d\left(\left(\frac{P_0(B)}{b+d}-\frac{P_0(A)}{a+d}\right)\frac{P_1(C)}{c+d}+\frac{P_0(B)P_1(A)-P_0(A)P_1(B)}{(a+d)(b+d)}\right)$$

$$= d\left(\left(\frac{1}{b+d}-\frac{1}{a+d}\right)\frac{P_1(C)}{c+d}+\frac{P_1(A)-P_1(B)}{(a+d)(b+d)}\right)>0$$

得: $F_2(D, T_2) > F_2(D, T_3)$

綜合得: $F_2(D, T_1) > F_2(D, T_2) > F_2(D, T_3)$

綜合引理 2~5，我們可以得到於包含 4 位選手賽程表中極重要的定理:

定理 15 在包含 4 位選手的賽程中，對自己的最有利賽程表為與除自己以外之最弱選手於第

一場賽程比賽。

證明:

以下二表格表示定理 9~12 的結果:

	A	B	C	D
優	T_3	T_2	T_1	T_1
中	T_2	T_3	T_3	T_2
劣	T_1	T_1	T_2	T_3

	A	B	C	D
T1	劣	劣	優	優
T2	中	優	劣	中
T3	優	中	中	劣

觀察各選手的最有利賽程表，發現在包含 4 位選手的賽程中，自己的最有利賽程表為與除自己以外之最弱選手於第一場賽程比賽。由此規則，在包含 4 位選手的賽程中，對於每一位選手，我們能直接地排出其最有利賽程。

2. 包含 4 人賽程之賽程表現率

由定理 7，我們知道因為**賽程表不同使選手會有不同的勝率**，即賽程表亦是影響選手的勝率的一個因素。為了解各賽程表對選手勝率造成的影響，我們定義賽程表現率 $S_n(Q_i,T_j)=\dfrac{P_n(Q_i,T_j)}{P_n(Q_i,T_k)}$，其中 T_k 為選手有最高勝率的賽程，我們稱其為選手之最有利賽程表。則 $S_n(Q_i,T_j)$ 為選手 Q_i 於賽程表 T_j 與最有利賽程表 T_k 中晉升至第 n 節點的機率比值。

另外，由於 $F_n(Q_i,T_j)=\dfrac{P_n(Q_i,T_j)}{q_i}$，可得 $S_n(Q_i,T_j)=\dfrac{P_n(Q_i,T_j)}{P_n(Q_i,T_k)}=\dfrac{F_n(Q_i,T_j)}{F_n(Q_i,T_k)}$。則我們有了:

> 定理 16 選手 Q_i 賽程表現率 $S_2(Q_i,A_j)=\dfrac{q_i+H_2(Q_i,A_j)}{q_i+H_2(Q_i,A_k)}$，其中 $H_2(Q_i,A_j)$ 為選手 Q_i 於賽程表 A_j 中第 2 場賽程與可能遇上的選手之調和平均。

證明:

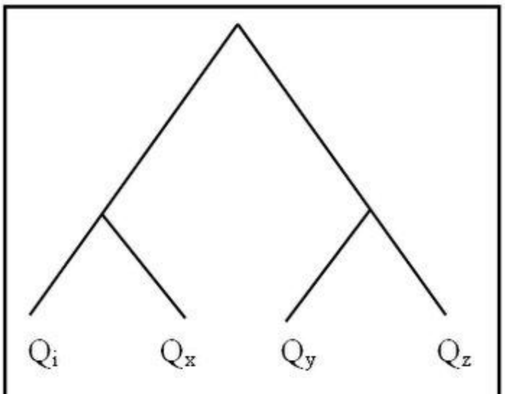


圖 2.4 普通賽程表 T_j

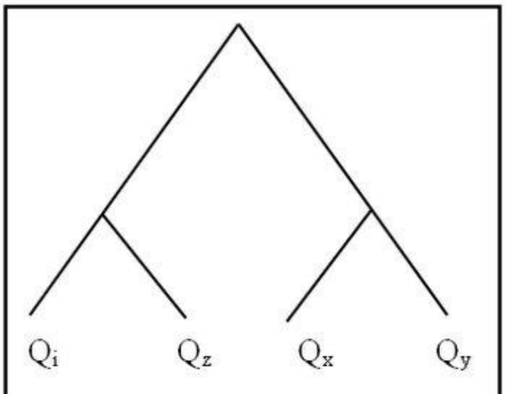


圖 2.5 最佳賽程表 T_k

命名賽程表圖 2.4、圖 2.5 為選手 Q_1 之普通賽程表 T_j 與最佳賽程表 T_k，列出選手 Q_i 於二賽程表中的勝率:

$$F_2(Q_i, T_j) = q_i \frac{P_0(Q_x)}{q_i + q_x}\left(\frac{P_1(Q_y)}{q_i + q_y} + \frac{P_1(Q_z)}{q_i + q_z}\right) = q_i \frac{q_i(P_1(Q_y) + P_1(Q_z)) + P_1(Q_y)q_z + P_1(Q_z)q_y}{(q_i + q_x)(q_i + q_y)(q_i + q_z)}$$

$$= \frac{q_i\left(q_i + \dfrac{2q_y q_z}{q_y + q_z}\right)}{(q_i + q_x)(q_i + q_y)(q_i + q_z)}$$

$$F_2(Q_i, T_k) = q_i \frac{P_0(Q_z)}{q_i + q_z}\left(\frac{P_1(Q_x)}{q_i + q_x} + \frac{P_1(Q_y)}{q_i + q_y}\right) = q_i \frac{q_i(P_1(Q_x) + P_1(Q_y)) + P_1(Q_x)q_y + P_1(Q_y)q_x}{(q_i + q_x)(q_i + q_y)(q_i + q_z)}$$

$$= \frac{q_i\left(q_i + \dfrac{2q_x q_y}{q_x + q_y}\right)}{(q_i + q_x)(q_i + q_y)(q_i + q_z)}$$

$$S_2(Q_i, T_j) = \frac{F_2(Q_i, T_j)}{F_2(Q_i, T_k)} = \frac{q_i + \dfrac{2q_y q_z}{q_y + q_z}}{q_i + \dfrac{2q_x q_y}{q_x + q_y}} = \frac{q_i + H_2(Q_i, A_j)}{q_i + H_2(Q_i, A_k)}$$

由定理 16，我們可以得到每個選手於各賽程表的賽程表現率，如推論 4~7 表示:

推論 10　選手 A 賽程表現率 $S_2(A,T_1) = \dfrac{a + \dfrac{2cd}{c+d}}{a + \dfrac{2bc}{b+c}}$ 、$S_2(A,T_2) = \dfrac{a + \dfrac{2bd}{b+d}}{a + \dfrac{2bc}{b+c}}$ 、

$S_2(AT_3) = \dfrac{a + \dfrac{2bc}{b+c}}{a + \dfrac{2bc}{b+c}}$ 。

證明:

$$F_2(A,T_1)=a\frac{P_0(B)}{a+b}\left(\frac{P_1(C)}{a+c}+\frac{P_1(D)}{a+d}\right)=\frac{a(a+\frac{2cd}{c+d})}{(a+b)(a+c)(a+d)}$$

$$F_2(A,T_2)=a\frac{P_0(C)}{a+c}\left(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d}\right)=\frac{a(a+\frac{2bd}{b+d})}{(a+b)(a+c)(a+d)}$$

$$F_2(A,T_3)=a\frac{P_0(D)}{a+d}\left(\frac{P_1(B)}{a+b}+\frac{P_1(C)}{a+c}\right)=\frac{a(a+\frac{2bc}{b+c})}{(a+b)(a+c)(a+d)}$$

$$S_2(A,T_1)=\frac{a+\frac{2cd}{c+d}}{a+\frac{2bc}{b+c}}\qquad S_2(A,T_2)=\frac{a+\frac{2bd}{b+d}}{a+\frac{2bc}{b+c}}\qquad S_2(A,T_3)=\frac{a+\frac{2bc}{b+c}}{a+\frac{2bc}{b+c}}$$

推論 11　選手 B 賽程表現率 $S_2(B,T_1)=\frac{b+\frac{2cd}{c+d}}{b+\frac{2ac}{a+c}}$、$S_2(B,T_2)=\frac{b+\frac{2ac}{a+c}}{b+\frac{2ac}{a+c}}$、

$$S_2(B,T_3)=\frac{b+\frac{2ad}{a+d}}{b+\frac{2ac}{a+c}}\text{。}$$

證明:

$$F_2(B,T_1)=b\frac{P_0(A)}{a+b}\left(\frac{P_1(C)}{b+c}+\frac{P_1(D)}{b+d}\right)=\frac{b(b+\frac{2cd}{c+d})}{(a+b)(b+c)(b+d)}$$

$$F_2(B,T_2)=b\frac{P_0(D)}{b+d}\left(\frac{P_1(A)}{a+b}+\frac{P_1(C)}{b+c}\right)=\frac{b(b+\frac{2ac}{a+c})}{(a+b)(b+c)(b+d)}$$

$$F_2(B,T_3)=b\frac{P_0(C)}{b+c}\left(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d}\right)=\frac{b(b+\frac{2ad}{a+d})}{(a+b)(b+c)(b+d)}$$

$$S_2(B,T_1)=\frac{b+\frac{2cd}{c+d}}{b+\frac{2ac}{a+c}}\qquad S_2(B,T_2)=\frac{b+\frac{2ac}{a+c}}{b+\frac{2ac}{a+c}}\qquad S_2(B,T_3)=\frac{b+\frac{2ad}{a+d}}{b+\frac{2ac}{a+c}}$$

推論 12　選手 C 賽程表現率 $S_2(C,T_1)=\dfrac{c+\dfrac{2ab}{a+b}}{c+\dfrac{2ab}{a+b}}$ 、$S_2(C,T_2)=\dfrac{c+\dfrac{2bd}{b+d}}{c+\dfrac{2ab}{a+b}}$ 、

$S_2(CT_3)=\dfrac{c+\dfrac{2ad}{a+d}}{c+\dfrac{2ab}{a+b}}$ 。

證明:

$$F_2(C,T_1)=c\frac{P_0(D)}{c+d}\left(\frac{P_1(A)}{a+c}+\frac{P_1(B)}{b+c}\right)=\frac{c\left(c+\dfrac{2ab}{a+b}\right)}{(a+c)(b+c)(c+d)}$$

$$F_2(C,T_2)=c\frac{P_0(A)}{a+c}\left(\frac{P_1(B)}{b+c}+\frac{P_1(D)}{c+d}\right)=\frac{c\left(c+\dfrac{2bd}{b+d}\right)}{(a+c)(b+c)(c+d)}$$

$$F_2(C,T_3)=c\frac{P_0(B)}{b+c}\left(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d}\right)=\frac{c\left(c+\dfrac{2ad}{a+d}\right)}{(a+c)(b+c)(c+d)}$$

$$S_2(C,T_1)=\frac{c+\dfrac{2ab}{a+b}}{c+\dfrac{2ab}{a+b}}\qquad S_2(CT_2)=\frac{c+\dfrac{2bd}{b+d}}{c+\dfrac{2ab}{a+b}}\qquad S_2(CT_3)=\frac{c+\dfrac{2ad}{a+d}}{c+\dfrac{2ab}{a+b}}$$

推論 13　選手 D 賽程表現率 $S_2(D,T_1)=\dfrac{d+\dfrac{2ab}{a+b}}{d+\dfrac{2ab}{a+b}}$ 、$S_2(D,T_2)=\dfrac{d+\dfrac{2ac}{a+c}}{d+\dfrac{2ab}{a+b}}$ 、

$S_2(DT_3)=\dfrac{d+\dfrac{2bc}{b+c}}{d+\dfrac{2ab}{a+b}}$ 。

證明:

$$F_2(D,T_1)=d\frac{P_0(C)}{c+d}(\frac{P_1(A)}{a+d}+\frac{P_1(B)}{b+d})=\frac{d(d+\frac{2ab}{a+b})}{(a+d)(b+d)(c+d)}$$

$$F_2(D,T_2)=d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d}+\frac{P_1(C)}{c+d})=\frac{d(d+\frac{2ac}{a+c})}{(a+d)(b+d)(c+d)}$$

$$F_2(D,T_3)=d\frac{P_0(A)}{a+d}(\frac{P_1(B)}{b+d}+\frac{P_1(C)}{c+d})=\frac{d(d+\frac{2bc}{b+c})}{(a+d)(b+d)(c+d)}$$

$$S_2(D,T_1)=\frac{d+\frac{2ab}{a+b}}{d+\frac{2ab}{a+b}}\qquad S_2(A,T_2)=\frac{d+\frac{2ac}{a+c}}{d+\frac{2ab}{a+b}}\qquad S_2(A,T_3)=\frac{d+\frac{2bc}{b+c}}{d+\frac{2ab}{a+b}}$$

賽程的安排對選手的影響到底有多大呢?下舉兩個例子，分別是選手間實力成等差與成等比，讓我們看看此兩種情況下賽程安排對選手的影響。

(一)選手間實力成等差

選手 A、B、C、D 的實力關係分別為: a = 4d、b = 3d、c = 2d、d = d。則各選手於各賽程表中的賽程表現率如下表示:

等差	A	B	C	D
T_1	83.333%	76.471%	100%	100%
T_2	85.938%	100%	64.474%	82.764%
T_3	100%	81.176%	66.316%	76.744%

(二)選手間實力成等比

選手 A、B、C、D 的實力關係分別為: a = 8d、b = 4d、c = 2d、d = d。則各選手於各賽程表中的賽程表現率如下表示:

等比	A	B	C	D
T_1	70%	77.193%	100%	100%
T_2	72%	100%	49.091%	66.316%
T_3	100%	83.626%	51.52%	57.895%

國際上包含 4 位選手的單淘汰制賽程採用 T_3 賽程，由賽程表現率可發現 T_3 賽程能使擁有最高實力的選手有最大的勝率，其餘選手的勝率則受到賽程影響而非該選手應有的最大勝率。有趣的是，**受到此賽程安排影響最大的，並不是直觀上的最弱選手，反而是次弱選手**，這真是一個有趣的發現。

3.包含 8 人賽程之賽程表現率

接下來，我們討論包含 8 人賽程之賽程表現率。經過組合的計算，我們得知其總共有 315 種可能的賽程。在討論之前，我們先介紹標記賽程的方式。

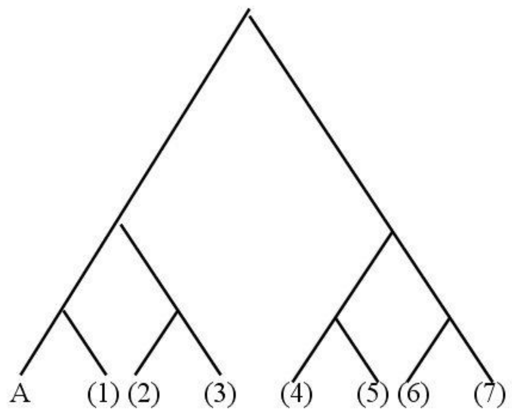


圖 2.6

如圖 ，我們利用 Tr,xy,z 表示賽程，並規定選手間實力的關係為 a>b>c>d>e>f>g>h。其中 r = 1~7 依序表示(1)的位置為選手 B~H。xy 表示除了選手 A 與位置(1)的選手以外第 x 強的選手對上第 y 強的選手，且 $x<y$；例如若位置(1)為選手 B，則以 xy=12 表示位置(2)、(3)為選手 C、D。然後我們將剩下的 4 位選手中，最強的選手置於位置(4)，且位置(5)為其餘 3 位選手中，第 z 強的選手；例如剩下的 4 位選手為 E、F、G、H，則將選手 E 置於位置(4)，以 z = 1~3 依序表示位置(5)為選手 F、G、H。

根據此標記的方式，我們令賽程選換為選手位置的方式為

$T_{r,xy,z}$=A(1)(2)(3)(4)(5)(6)(7)。例如 $T_{1,12,1}$=ABCDEFGH、$T_{1,12,2}$=ABCDEGFH、$T_{1,12,3}$=ABCDEHFG、$T_{7,12,1}$=AHBCDEFG。

我們令各選手 A~H 的實力依序為 8~1，分別求出各選手於各場賽程的勝率，並根據勝率排序得到各賽程對各選手的優劣順序。由於其非常龐大，詳細請見<附錄一~四>。

在這裡，我們來看看幾個我們關心的賽程:

(1)各選手之最有利賽程

觀察各賽程對於選手的優劣順序，我們可以得到各選手的最有利賽程，其結果如下:

選手	A	B	C	D
最有利賽程	T7.56.1	T2.23.3	T1.23.3	T1.13.3
選手位置	AHFGBCDE	ACDEBHFG	ABDECGFH	ABCEDHFG
選手	E	F	G	H
最有利賽程	T1.12.3	T1.12.2	T1.12.1	T1.12.1
選手位置	ABCDEHGF	ABCDFHEG	ABCDEFGH	ABCDEFGH

由此結果，我們可以歸納出，一個選手最有利賽程的安排方式為「**第一場賽程與除了自己以外最弱的選手對上，第二場賽程可能遇上的選手為次弱與第三弱的選手，第三場賽程使第四弱選手最容易遇上**」。

(2)各選手之最不利賽程

同樣地，我們可以得到各選手的最不利賽程，其結果如下:

選手	A	B	C	D
最有利賽程	T1.16.3	T1.16.3	T2.16.3	T3.16.3
選手位置	ABCHDGEF	ABCHDGEF	ACBHDGEF	ADBHCGEF
選手	E	F	G	H
最有利賽程	T4.23.3	T5.23.3	T6.23.3	T7.13.3
選手位置	AECDBHFG	AFCDBHEG	AGCDBHEF	AHBDCGEF

由此結果，我們可以歸納出，一個選手最不利賽程的安排方式為「**第一場賽程與除了自己以外最強的選手對上，第二場賽程可能遇上的選手為次強與最弱的選手，第三場賽程使第三強選手最容易遇上**」。

(3)國際上 8 人賽程安排

國際賽事的選手安排通常為 AHDEBGCF，即為賽程表 T7.34.3。觀察其排名，可發現:

選手	A	B	C	D	E	F	G	H
排名	31	96	152	222	238	253	273	297
賽程表現率	84.73%	71.18%	58.93%	47.82%	39.24%	35.76%	29.93%	33.07%

國際賽事的選手安排保障了愈強的選手有對其愈好的賽程。另外，觀察賽程表現率，可以發現受此安排影響最大的，並非直觀上的最弱選手，而是次弱選手。其實最差並不是最吃虧的呢!

(三)勝率不等式

1.晉級第二節點勝率的最大值

當賽程排定，已知所有參賽選手的實力時，選手晉級第二節點勝率的最大值如何呢?

引理 6　選手 Q_1 於第 2 場賽程可能遇上之選手 Q_3、Q_4 為，則選手晉級的機率最大值為:

$$P_{1,2}(Q_1)=\frac{{P_{Q_1Q_3}}^2+{P_{Q_1Q_4}}^2}{P_{Q_1Q_3}+P_{Q_1Q_4}}$$

證明:

選手 Q_1 由第 1 場賽程晉級至第 2 場賽程的機率為:

$$P_{1,2}(Q_1)=P_1(Q_3)P_{Q_1Q_3}+P_1(Q_4)P_{Q_1Q_4}$$

由柯西不等式，我們可以知道:

$$(P_1(Q_3)^2+P_1(Q_4)^2)(P_{Q_1Q_3}{}^2+P_{Q_1Q_4}{}^2)\ge(P_1(Q_3)P_{Q_1Q_3}+P_1(Q_4)P_{Q_1Q_4})^2=P_{1,2}(Q_1)^2$$

等號成立時，我們知道:

$$\frac{P_1(Q_3)}{P_{Q_1Q_3}}=\frac{P_1(Q_4)}{P_{Q_1Q_4}}=t\ \text{......(1)}$$

此時 $P_{1,2}(Q_1)$有最大值。

另外，由(1)我們可得:

$$P_1(Q_3)=tP_{Q_1Q_3}\quad ,\quad P_1(Q_4)=tP_{Q_1Q_4}$$

由選手 Q_3、Q_4 晉升至第 1 場賽程的機率之和為 1，我們得:

$$P_1(Q_3)+P_1(Q_4)=1$$

$$t(P_{Q_1Q_3}+P_{Q_1Q_4})=1$$

$$t=\frac{1}{P_{Q_1Q_3}+P_{Q_1Q_4}}$$

即:

$$P_1(Q_3)=\frac{P_{Q_1Q_3}}{P_{Q_1Q_3}+P_{Q_1Q_4}},\ P_1(Q_4)=\frac{P_{Q_1Q_4}}{P_{Q_1Q_3}+P_{Q_1Q_4}}$$

因此，我們可知道選手 Q_1 由第 1 場賽程晉級至第 2 場賽程的機率最大值為:

$$P_{1,2}(Q_1)=P_1(Q_3)P_{Q_1Q_3}+P_1(Q_4)P_{Q_1Q_4}$$
$$=\frac{P_{Q_1Q_3}{}^2+P_{Q_1Q_4}{}^2}{P_{Q_1Q_3}+P_{Q_1Q_4}}$$

2.晉級第 n 節點勝率的最大值

當賽程排定，已知所有參賽選手的實力時，我們可知選手晉級第 n 點勝率的最大值:

定理 17　選手 Q_1 於第 n 場賽程可能遇上之選手為 $Q_{2^{n-1}+1}\ldots Q_{2^n}$，則選手晉級的機率最大值為:

$$P_{n-1,n}(Q_1)=\frac{\sum_{i=2^{n-1}+1}^{2^n} {P_{Q_1Q_i}}^2}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$$

證明:

選手 Q_1 由第 n-1 場賽程晉級至第 n 場賽程的機率為:

$$P_{n-1,n}(Q_1)=\sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i})$$

由柯西不等式，我們可以知道:

$$\sum_{i=2^{n-1}+1}^{2^n} P_{n-1}(Q_i)^2 \cdot \sum_{i=2^{n-1}+1}^{2^n} {P_{Q_1Q_i}}^2 \geq \left(\sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i})\right)^2=P_{n-1,n}(Q_1)^2$$

等號成立時，我們知道:

且當 $\dfrac{P_{n-1}(Q_i)}{P_{Q_1Qi}}=t\ ,i=2^{n-1}+1$ 至 2^n.....(1)時，等號成立。

此時 $P_{n-1,n}(Q_1)$有最大值。

另外，由(1)我們可得:

$$P_{n-1}(Q_i)=tP_{Q_1Q_i}\ ,i=2^{n-1}+1\sim 2^n$$

又選手 $Q_{2^{n-1}+1}\ldots Q_{2^n}$ 晉升至第 n-1 場賽程的機率之和為 1，我們得:

由於 $\sum_{i=2^{n-1}+1}^{2^n} P_{n-1}(Q_i)=1$ ，則 $\sum_{i=2^{n-1}+1}^{2^n} t\,P_{Q_1Q_i}=1$ ，

推得 $t=1\Big/\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}$ ，所以 $P_{n-1}(Q_i)=P_{Q_1Q_i}\Big/\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}$.

即: $P_{n-1}(Q_i) = \dfrac{P_{Q_1Q_i}}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$, $t = \dfrac{1}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$

因此，我們可知道選手 Q_1 由第 n-1 場賽程晉級至第 n 場賽程的機率最大值為:

$$P_{n-1,n}(Q_1) = \sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i}) = \frac{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}{}^2}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$$

3.勝率不等式

定理 18　選手 Q_1 晉升至第 n 場賽程的勝率不等式為:

$$P_n(Q_1) \le \prod_{j=1}^{n} \frac{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}{}^2}{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}}$$

證明:

綜合勝率一般式與晉級各場賽程的最大值，我們可得選手晉升至第 n 場賽程的機率最大值為:

$$P_n(Q_1) = \prod_{j=1}^{n} \sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i}) \le \prod_{j=1}^{n} \frac{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}{}^2}{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}}$$

定理 18 告訴我們，僅決定了各場賽程可能遇到的對手之後，選手勝率最大值已經因為對手的實力而降低，而在安排賽制後，選手的勝率又會因賽制的安排而再次降低。

參、研究結果與討論

一、定理:

定理 1　選手 A 於第 2 場賽程所遇上的假想選手 X 之實力 x 為:

$$x=\frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)}$$

定理 2　存在一威脅門檻 b_0，若 B 選手之實力 $b> b_0$，則 b 增加將使 $P_2(A)$下降；若 B 選手之實　力 $b< b_0$，則 b 增加將使 $P_2(A)$上升。其中:

$$b_0=\frac{c(\sqrt{2a(a+c)}-a)}{a+2c}$$

定理 3　選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1)=\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$

定理 4　於包含 2^m 位選手之完全二元樹賽程表中，自己的實力增加使自己的勝率增加。

定理 5　選手 Q_1 於第 n 場賽程所遇上的假想選手 X_n 之實力 x_n

為: $$x_n=\frac{1-\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})}{\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})}q_1$$

定理 6　若選手 A、F、G、H 的實力 a、f、g、h 滿足

$$\frac{(g-h)^2-\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}<f<\frac{(g-h)^2+\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)},\ f>\sqrt{gh}$$

則存在一威脅門檻 e_0，若 E 選手之實力 $e> e_0$，則 e 增加將使 $P_3(A)$下降；若 E 選手之實力 $e< e_0$，則 e 增加將使 $P_3(A)$上升。

定理 7　若選手 $Q_{2^{n-1}+2}, Q_{2^{n-1}+3}, \ldots, Q_{2^n}$ 的實力存在 $q_{2^{n-1}+2}, q_{2^{n-1}+3}, \ldots, q_{2^n}>0$ 滿足 $\frac{dx_n}{dq_{2^n-1}}<0$，則存在一威脅門檻 $q_{(2^{n-1}+1)0}$，若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}>q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$下降；若 $Q_{2^{n-1}+1}$ 選手之實力 $q_{2^{n-1}+1}<q_{(2^{n-1}+1)0}$，則 $q_{2^{n-1}+1}$ 增加將使 $P_n(Q_1)$上升。

定理 8　有一多項式 $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_1, x_2, x_3, \ldots, x_n$，

則選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1) = \dfrac{q_1^n}{\sum_{k=0}^{n}(-1)^k \dfrac{a_{n-k}}{a_n} q_1^{n-k}}$

定理 9　選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1) = \dfrac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k}\Omega^k}$，其中

$$\Omega = \sqrt[n]{\prod_{k=1}^{n}(q_1 + x_k)} - q_1$$

定理 10　選手 Q_1 晉升至第 n 節點的機率為 $P_n(Q_1) = \dfrac{q_1}{q_1 + \lambda}$，其中

$$\lambda = \frac{1 - \prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} (P_{j-1}(Q_i) P_{Q_1Q_i})}{\prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} (P_{j-1}(Q_i) P_{Q_1Q_i})} q_1$$

定理 11　選手 Q_1 晉升至第 n 節點的勝率實力比為 $F_n(Q_1) = q_1^{n-1} \prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} \dfrac{P_{j-1}(Q_i)}{q_1 + q_i}$

定理 12　有一多項式 $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$ 則選手 Q_1 晉升至第 n 節點的勝率實力比為 $F_n(Q_1) = \dfrac{q_1^{n-1}}{\sum_{k=0}^{n}(-1)^k \dfrac{a_{n-k}}{a_n} q_1^{n-k}}$

定理 13　選手 Q_1 晉升至第 n 節點的勝率實力比為

$$F_n(Q_1)=\frac{q_1^{n-1}}{\sum_{k=0}^{n}C_k^n q_1^{n-k}\Omega^k}\text{，其中 }\Omega=\sqrt[n]{\prod_{k=1}^{n}(q_1+x_k)}-q_1$$

定理 14　選手 Q_1 晉升至第 n 節點的勝率實力比為

$$F_n(Q_1)=\frac{1}{q_1+\lambda}\text{ 其中 }\lambda=\frac{1-\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

定理 15　在包含 4 位選手的賽程中，對自己的最有利賽程表為與除自己以外之最弱選手於第一場賽程比賽

定理 16　選手 Q_i 賽程表現率 $S_2(Q_i,A_j)=\dfrac{q_i+H_2(Q_i,A_j)}{q_i+H_2(Q_i,A_k)}$，其中 $H_2(Q_i,A_j)$ 為選手 Q_i 於賽程表 A_j 中第 2 場賽程與可能遇上的選手之調和平均

定理 17　選手 Q_1 於第 n 場賽程可能遇上之選手為 $Q_{2^{n-1}+1}\ldots Q_{2^n}$，則選手晉級的機率最大值為:

$$P_{n-1,n}(Q_1)=\frac{\sum_{i=2^{n-1}+1}^{2^n}P_{Q_1Q_i}^{\ 2}}{\sum_{i=2^{n-1}+1}^{2^n}P_{Q_1Q_i}}$$

定理 18　選手 Q_1 晉升至第 n 場賽程的機率最大值為

$$P_n(Q_1)\le\prod_{j=1}^{n}\frac{\sum_{i=2^{j-1}+1}^{2^j}P_{Q_1Q_i}^{\ 2}}{\sum_{i=2^{j-1}+1}^{2^j}P_{Q_1Q_i}}$$

二、引理與推論:

推論 1　若選手 B 為種子選手，則其威脅門檻 $b_0=0$

引理 1　於包含 4 位選手之完全二元樹賽程表中，自己的實力增加使自己的勝率增加。

推論 2　選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1)=\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

推論 3　有一多項式 $a_{m-n}x^{m-n}+a_{m-n-1}x^{m-n-1}+\ldots+a_1x+a_0=0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$ 則選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1)=\frac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\frac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

推論 4　選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_{n,m}(Q_1)=\frac{q_1^{m-n}}{\sum_{k=0}^{m-n}C_k^{m-n}q_1^{m-n-k}\Omega^k}\text{，其中 }\Omega=\sqrt[m-n]{\prod_{k=n}^{m}(q_1+x_k)}-q_1$$

推論 5　選手 Q_1 由第 n 節點晉升至第 m 節點的機率為

$$P_n(Q_1)=\frac{q_1}{q_1+\lambda}\text{，其中 }\lambda=\frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

推論 6　選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=q_1^{m-n-1}\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})$$

推論 7　有一多項式 $a_{m-n}x^{m-n}+a_{m-n-1}x^{m-n-1}+\ldots+a_1x+a_0=0$，其 n 個根為 $x_n, x_{n+1}, x_{n+2}, \ldots, x_m$ 則選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=\frac{q_1^{m-n-1}}{\sum_{k=0}^{(m-n)}(-1)^k\frac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

推論 8　選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=\frac{q_1^{m-n-1}}{\sum_{k=0}^{m-n}C_k^{m-n}q_1^{m-n-k}\Omega^k} \quad \text{其中: } \Omega=\sqrt[m-n]{\prod_{k=n}^{m}(q_1+x_k)}-q_1$$

推論 9　選手 Q_1 由第 n 節點晉升至第 m 節點的勝率實力比為:

$$F_{n,m}(Q_1)=\frac{1}{q_1+\lambda} \quad \text{，其中: } \lambda=\frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P_{j-1}(Q_i)P_{Q_1Q_i})}q_1$$

引理 2　選手 A 的勝率實力比 $F_2(A,T_3)>F_2(A,T_2)>F_2(A,T_1)$

引理 3　選手 B 的勝率實力比 $F_2(B,T_2)>F_2(B,T_3)>F_2(B,T_1)$

引理 4　選手 C 的勝率實力比 $F_2(C,T_1)>F_2(C,T_3)>F_2(C,T_2)$

引理 5　選手 D 的勝率實力比 $F_2(D,T_1)>F_2(D,T_2)>F_2(D,T_3)$

推論 10　選手 A 賽程表現率 $S_2(A,T_1)=\dfrac{a+\frac{2cd}{c+d}}{a+\frac{2bc}{b+c}}$、$S_2(A,T_2)=\dfrac{a+\frac{2bd}{b+d}}{a+\frac{2bc}{b+c}}$、

$$S_2(AT_3)=\frac{a+\frac{2bc}{b+c}}{a+\frac{2bc}{b+c}}$$

推論 11　選手 B 賽程表現率 $S_2(B,T_1)=\dfrac{b+\dfrac{2cd}{c+d}}{b+\dfrac{2ac}{a+c}}$、$S_2(B,T_2)=\dfrac{b+\dfrac{2ac}{a+c}}{b+\dfrac{2ac}{a+c}}$、

$$S_2(B,T_3)=\frac{b+\dfrac{2ad}{a+d}}{b+\dfrac{2ac}{a+c}}$$

推論 12　選手 C 賽程表現率 $S_2(C,T_1)=\dfrac{c+\dfrac{2ab}{a+b}}{c+\dfrac{2ab}{a+b}}$、$S_2(C,T_2)=\dfrac{c+\dfrac{2bd}{b+d}}{c+\dfrac{2ab}{a+b}}$、

$$S_2(C,T_3)=\frac{c+\dfrac{2ad}{a+d}}{c+\dfrac{2ab}{a+b}}$$

推論 13　選手 D 賽程表現率 $S_2(D,T_1)=\dfrac{d+\dfrac{2ab}{a+b}}{d+\dfrac{2ab}{a+b}}$、$S_2(D,T_2)=\dfrac{d+\dfrac{2ac}{a+c}}{d+\dfrac{2ab}{a+b}}$、

$$S_2(D,T_3)=\frac{d+\dfrac{2bc}{b+c}}{d+\dfrac{2ab}{a+b}}$$

引理 6　選手 Q_1 於第 2 場賽程可能遇上之選手 Q_3、Q_4 為，則選手晉級的機率最大值為 $P_{1,2}(Q_1)=\dfrac{{P_{Q_1Q_3}}^2+{P_{Q_1Q_4}}^2}{P_{Q_1Q_3}+P_{Q_1Q_4}}$

肆、討論與展望

在這趟於淘汰賽世界的探索中，我們得到了許多文獻中未曾提及的概念方法，幫助我們澄清先有的迷思，更帶領我們更了解淘汰賽的結構。

第一部份對選手實力變化與勝率的影響所做的討論裡，我們有了「假想選手」

的概念，得到各場賽程可能遇上的選手對我的威脅。我們甚至發現，未來的敵人實力增強，竟也會讓我們的勝率增加。在「勝率一般式」的討論中，我們從許多角度觀看賽程，甚至能夠知道一個群體、一場賽程的實力。這部份所得到的驚喜，如此令人著迷。

第二部份對選手位置安排與勝率的影響所做的討論當中，我們首先知道，選手位置安排亦是影響勝率的一個因素。有了「勝率實力比」，我們便能得知一份實力，於賽程能夠兌換幾份勝率。再透過「賽程表現率」，我們得以比較所有可能的賽程，進而找出選手的最有利賽程。

透過「假想選手」、「威脅門檻」、「勝率一般式」、「勝率實力比」、「賽程表現率」，我們能夠分析選手實力變化與選手安排所造成勝率的影響，進一步地判斷選手賽況的優劣。

這趟旅程中，我們仍尚有幾個想法，希望有興趣的你(妳)，能夠一起討論:

(1) 威脅門檻在包含 2^n 人賽程之中，是否皆存在?

(2) 最有利賽程與最不利賽成的安排規則，是否適用於所有包含 2^n 人賽程之中?

(3) 包含 2^n 人賽程之賽程表現率，是否存在同包含 4 人賽程之賽程表現率的調和平均關係?

(4) 是否存在著所有選手的勝率實力比皆相同的賽程安排方式?

「數學家與過去的數學家討論、與現在的數學家討論、與未來的數學家討論。」研究的喜悅，是我們無法言詮的快意。閱讀到此的你(妳)，是否，在我們一起討論之後，也和我一樣，愛上了，淘汰賽的世界呢?

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二、 于小涵 (2006) 輸贏一線間-淘汰賽的相關探討台灣 2006 年國際科學展覽會。

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<附錄一>選手賽程勝率

賽程	A	B	C	D	E	F	G	H
T1.12.1	0.230924	0.184079	0.163152	0.115095	0.153421	0.087218	0.05633	0.009781
T1.12.2	0.228068	0.18155	0.160631	0.113065	0.158216	0.114055	0.036318	0.008098
T1.12.3	0.226632	0.180299	0.159409	0.112104	0.184106	0.092586	0.037355	0.007509
T1.13.1	0.227926	0.181575	0.170577	0.205287	0.076101	0.077484	0.052098	0.00895
T1.13.2	0.224797	0.178803	0.167645	0.21069	0.074354	0.104731	0.031712	0.007268
T1.13.3	0.222178	0.176531	0.165302	0.239862	0.073054	0.083444	0.033222	0.006406
T1.14.1	0.228042	0.181745	0.184507	0.183066	0.121393	0.043831	0.049095	0.008322
T1.14.2	0.222435	0.176793	0.178871	0.190774	0.156077	0.041729	0.027425	0.005895
T1.14.3	0.22117	0.175702	0.177665	0.221606	0.12898	0.041339	0.027933	0.005604
T1.15.1	0.226482	0.180442	0.203473	0.163323	0.106584	0.095036	0.018433	0.006227
T1.15.2	0.22363	0.17794	0.200335	0.165835	0.150187	0.058724	0.017916	0.005433
T1.15.3	0.221665	0.176243	0.198243	0.211781	0.109814	0.059575	0.017614	0.005065
T1.16.1	0.223262	0.177775	0.233778	0.156933	0.101543	0.074483	0.028595	0.00363
T1.16.2	0.221319	0.176076	0.231299	0.162589	0.122962	0.056696	0.025513	0.003545
T1.16.3	0.220535	0.175397	0.230321	0.178746	0.108796	0.058602	0.024087	0.003515
T1.23.1	0.227976	0.181646	0.256839	0.125637	0.079423	0.070898	0.049182	0.008399
T1.23.2	0.224626	0.178675	0.262676	0.122985	0.077462	0.098167	0.02869	0.006719
T1.23.3	0.221	0.175538	0.294197	0.120363	0.075613	0.077131	0.030446	0.005711
T1.24.1	0.231114	0.184393	0.232365	0.138168	0.11248	0.046437	0.047093	0.00795
T1.24.2	0.225051	0.179026	0.240815	0.133023	0.14763	0.044042	0.024932	0.005483
T1.24.3	0.222663	0.176971	0.274613	0.131199	0.120514	0.043292	0.025723	0.005026
T1.25.1	0.23231	0.185454	0.210463	0.155011	0.099092	0.09185	0.019835	0.005985
T1.25.2	0.229196	0.182712	0.213269	0.152128	0.144134	0.054168	0.019234	0.005159
T1.25.3	0.225415	0.179449	0.264582	0.148873	0.102953	0.055514	0.018645	0.00457
T1.26.1	0.230999	0.184451	0.203957	0.182004	0.094699	0.072246	0.027681	0.003964
T1.26.2	0.228622	0.182366	0.210432	0.179442	0.118374	0.052486	0.024424	0.003854
T1.26.3	0.227099	0.181048	0.228932	0.177877	0.103577	0.055573	0.022098	0.003796
T1.34.1	0.237649	0.190016	0.21512	0.156925	0.095914	0.050928	0.045761	0.007687
T1.34.2	0.229214	0.182555	0.224899	0.198221	0.090387	0.047452	0.022508	0.004764
T1.34.3	0.228047	0.181554	0.259821	0.166364	0.089743	0.047077	0.022801	0.004593

T1.35.1	0.242092	0.193894	0.195326	0.140819	0.10988	0.089956	0.022203	0.005829
T1.35.2	0.236026	0.18856	0.199667	0.194981	0.105596	0.049653	0.021002	0.004515
T1.35.3	0.234131	0.186928	0.253924	0.145462	0.104403	0.05016	0.020703	0.00429
T1.36.1	0.243451	0.195238	0.189698	0.136023	0.132869	0.071014	0.02717	0.004538
T1.36.2	0.238894	0.191232	0.200022	0.165433	0.129036	0.049197	0.021869	0.004317
T1.36.3	0.238119	0.190562	0.220186	0.147051	0.12844	0.050397	0.020957	0.004288
T1.45.1	0.252287	0.202726	0.180379	0.129018	0.138667	0.06675	0.025307	0.004867
T1.45.2	0.249102	0.199931	0.181961	0.191572	0.083111	0.065292	0.024658	0.004373
T1.45.3	0.246647	0.197805	0.24789	0.131252	0.08381	0.064272	0.02422	0.004105
T1.46.1	0.258164	0.208089	0.178544	0.127334	0.114447	0.084609	0.023451	0.005361
T1.46.2	0.255819	0.206025	0.182469	0.162766	0.083006	0.083282	0.021392	0.005242
T1.46.3	0.25442	0.204806	0.215169	0.13287	0.084735	0.082547	0.020273	0.00518
T1.56.1	0.278543	0.226207	0.172428	0.122533	0.098407	0.052485	0.042377	0.00702
T1.56.2	0.276755	0.224626	0.178306	0.14247	0.080665	0.048422	0.041842	0.006914
T1.56.3	0.276093	0.224044	0.191006	0.13084	0.083267	0.046215	0.041656	0.006878
T2.12.1	0.237425	0.204809	0.143364	0.109468	0.152587	0.086693	0.055949	0.009705
T2.12.2	0.234489	0.201996	0.141149	0.107537	0.157355	0.113367	0.036072	0.008036
T2.12.3	0.233013	0.200604	0.140075	0.106623	0.183104	0.092028	0.037103	0.007451
T2.13.1	0.233106	0.21323	0.140484	0.203853	0.072056	0.076822	0.051597	0.008851
T2.13.3	0.229906	0.209975	0.138069	0.209218	0.070401	0.103836	0.031407	0.007188
T2.13.3	0.227228	0.207306	0.136139	0.238187	0.069171	0.082731	0.032902	0.006336
T2.14.1	0.231843	0.228951	0.13975	0.181368	0.120151	0.041279	0.04846	0.008198
T2.14.2	0.226143	0.222714	0.135481	0.189005	0.15448	0.039299	0.027071	0.005808
T2.14.3	0.224856	0.221339	0.134568	0.219551	0.127661	0.038932	0.027572	0.005521
T2.15.1	0.22877	0.250041	0.137663	0.161368	0.105175	0.093631	0.017246	0.006106
T2.15.2	0.225889	0.246574	0.13554	0.163851	0.148201	0.057856	0.016763	0.005327
T2.15.3	0.223904	0.244222	0.134125	0.209247	0.108362	0.058694	0.01648	0.004966
T2.16.1	0.224083	0.282692	0.134538	0.154488	0.099799	0.073057	0.027975	0.003369
T2.16.2	0.222133	0.27999	0.133112	0.160055	0.12085	0.05561	0.02496	0.00329
T2.16.3	0.221346	0.27891	0.132548	0.175961	0.106927	0.057479	0.023565	0.003262
T2.23.1	0.233171	0.305321	0.140624	0.123117	0.077763	0.065549	0.04655	0.007906
T2.23.2	0.229579	0.311428	0.1379	0.120383	0.075741	0.092441	0.026278	0.00625
T2.23.3	0.225007	0.344402	0.134639	0.117241	0.073535	0.071852	0.028155	0.005169
T2.24.1	0.2376	0.279036	0.143826	0.136253	0.104902	0.04577	0.045045	0.007568
T2.24.2	0.23108	0.287965	0.138899	0.130927	0.13987	0.043283	0.022869	0.005108
T2.24.3	0.227651	0.323681	0.136484	0.128438	0.113098	0.042271	0.023822	0.004556

T2.25.1	0.239681	0.255333	0.145347	0.153531	0.092475	0.088267	0.019661	0.005704
T2.25.2	0.236328	0.258337	0.142841	0.150537	0.137872	0.050183	0.019031	0.00487
T2.25.3	0.230848	0.313266	0.138955	0.146051	0.096679	0.05182	0.018234	0.004147
T2.26.1	0.23842	0.248372	0.14471	0.180365	0.088323	0.069385	0.026485	0.003939
T2.26.2	0.235664	0.255418	0.142671	0.177513	0.113164	0.048596	0.023157	0.003817
T2.26.3	0.233451	0.275549	0.141091	0.175351	0.098137	0.052407	0.020276	0.003739
T2.34.1	0.245469	0.260393	0.149375	0.147519	0.095123	0.050521	0.044206	0.007394
T2.34.2	0.236365	0.270809	0.142501	0.188928	0.089364	0.046892	0.020692	0.004449
T2.34.3	0.23411	0.308008	0.140924	0.157225	0.088187	0.04621	0.021162	0.004174
T2.35.1	0.250847	0.238868	0.153285	0.132503	0.109437	0.087293	0.022152	0.005616
T2.35.2	0.244289	0.243552	0.148391	0.187542	0.104951	0.046117	0.020884	0.004274
T2.35.3	0.24059	0.302098	0.14578	0.137599	0.102749	0.04694	0.02034	0.003904
T2.36.1	0.25234	0.23287	0.154697	0.127965	0.132406	0.068901	0.02628	0.004541
T2.36.2	0.247048	0.244195	0.15076	0.159091	0.128125	0.045687	0.020801	0.004294
T2.36.3	0.245528	0.266315	0.149678	0.140261	0.127019	0.047668	0.019291	0.00424
T2.45.1	0.262233	0.222504	0.161454	0.121681	0.135252	0.066809	0.025363	0.004704
T2.45.2	0.258772	0.224228	0.158883	0.186042	0.077942	0.065267	0.024673	0.004193
T2.45.3	0.253943	0.296096	0.15543	0.124447	0.079088	0.063379	0.023869	0.003748
T2.46.1	0.268273	0.220745	0.166347	0.120138	0.111678	0.084668	0.022767	0.005385
T2.46.2	0.26555	0.225095	0.164314	0.158126	0.077877	0.083178	0.02061	0.00525
T2.46.3	0.262794	0.261342	0.162318	0.127102	0.080761	0.081815	0.018731	0.005137
T2.56.1	0.289729	0.214366	0.182699	0.115962	0.09635	0.05129	0.042539	0.007065
T2.56.2	0.287463	0.220973	0.180991	0.138833	0.075937	0.046979	0.041888	0.006936
T2.56.3	0.286162	0.235248	0.18003	0.126575	0.080346	0.043226	0.041544	0.00687
T3.12.1	0.248207	0.194415	0.147247	0.105816	0.152301	0.086514	0.05582	0.00968
T3.12.2	0.245137	0.191744	0.144972	0.103949	0.157061	0.113133	0.03599	0.008015
T3.12.3	0.243594	0.190423	0.143869	0.103065	0.182761	0.091838	0.037018	0.007432
T3.13.1	0.239818	0.208777	0.253392	0.101695	0.07033	0.069614	0.048174	0.008201
T3.13.2	0.236295	0.205363	0.259151	0.099548	0.068593	0.096388	0.028101	0.00656
T3.13.3	0.232481	0.201757	0.290249	0.097426	0.066956	0.075734	0.029822	0.005576
T3.14.1	0.239966	0.225625	0.228269	0.101963	0.110066	0.040613	0.045801	0.007696
T3.14.2	0.23367	0.219058	0.23657	0.098166	0.144462	0.038517	0.024248	0.005308
T3.14.3	0.23119	0.216544	0.269773	0.09682	0.117928	0.037862	0.025018	0.004865
T3.15.1	0.237744	0.247479	0.205728	0.100854	0.096367	0.089007	0.017084	0.005736
T3.15.2	0.234557	0.24382	0.208471	0.098978	0.140171	0.052491	0.016567	0.004945
T3.15.3	0.230687	0.239466	0.25863	0.09686	0.100122	0.053795	0.016059	0.00438

T3.16.1	0.23299	0.279951	0.198029	0.098545	0.091345	0.06937	0.026421	0.003349
T3.16.2	0.230593	0.276786	0.204315	0.097158	0.114181	0.050397	0.023313	0.003257
T3.16.3	0.229057	0.274785	0.222277	0.096311	0.099908	0.053362	0.021093	0.003208
T3.23.1	0.239833	0.303302	0.163639	0.101757	0.072726	0.064907	0.046032	0.007804
T3.23.2	0.236138	0.309368	0.16047	0.099497	0.070835	0.091536	0.025986	0.00617
T3.23.3	0.231435	0.342124	0.156675	0.0969	0.068772	0.071149	0.027842	0.005103
T3.24.1	0.242656	0.27663	0.179286	0.103421	0.103693	0.042504	0.044374	0.007436
T3.24.2	0.235997	0.285482	0.173144	0.099378	0.138257	0.040195	0.022529	0.005019
T3.24.3	0.232495	0.32089	0.170134	0.097489	0.111794	0.039255	0.023467	0.004477
T3.25.1	0.242834	0.252548	0.199447	0.103587	0.091114	0.086796	0.0181	0.005574
T3.25.2	0.239438	0.255519	0.196008	0.101567	0.135843	0.049347	0.01752	0.004759
T3.25.3	0.233885	0.309848	0.190675	0.098541	0.095256	0.050956	0.016786	0.004053
T3.26.1	0.239595	0.24489	0.229407	0.102135	0.086655	0.067901	0.025831	0.003587
T3.26.2	0.236825	0.251837	0.226174	0.100519	0.111026	0.047556	0.022586	0.003476
T3.26.3	0.234601	0.271686	0.223669	0.099295	0.096283	0.051286	0.019776	0.003404
T3.34.1	0.257758	0.244566	0.189865	0.111558	0.095347	0.050699	0.043044	0.007164
T3.34.2	0.246171	0.255746	0.235893	0.104547	0.088439	0.046363	0.018895	0.003947
T3.34.3	0.245053	0.293479	0.200653	0.103955	0.087894	0.04605	0.019076	0.00384
T3.35.1	0.264144	0.224539	0.172703	0.115214	0.110141	0.085425	0.022383	0.005451
T3.35.2	0.254479	0.2302	0.234876	0.109495	0.103935	0.042564	0.020652	0.003799
T3.35.3	0.252621	0.29058	0.17846	0.108497	0.102902	0.042886	0.020401	0.003653
T3.36.1	0.265807	0.218868	0.167669	0.116654	0.133337	0.067423	0.025631	0.004611
T3.36.2	0.258154	0.232839	0.203598	0.112122	0.127476	0.042756	0.018781	0.004275
T3.36.3	0.257385	0.256122	0.181863	0.111703	0.126954	0.043548	0.018175	0.004251
T3.45.1	0.276871	0.209593	0.160193	0.122341	0.133059	0.06762	0.025741	0.004582
T3.45.2	0.269939	0.212378	0.233775	0.118278	0.072679	0.064744	0.024466	0.00374
T3.45.3	0.267488	0.287581	0.163355	0.116932	0.073134	0.063857	0.024092	0.003562
T3.46.1	0.283199	0.20801	0.158679	0.126711	0.109929	0.085687	0.022297	0.005488
T3.46.2	0.277825	0.215183	0.203065	0.12352	0.073555	0.082926	0.018685	0.005242
T3.46.3	0.276422	0.253915	0.166816	0.122733	0.074723	0.082283	0.017918	0.005189
T3.56.1	0.306112	0.202559	0.154139	0.140936	0.095199	0.050591	0.043249	0.007215
T3.56.2	0.301686	0.213659	0.181315	0.138255	0.072799	0.043262	0.042046	0.006978
T3.56.3	0.301025	0.229197	0.166751	0.13787	0.074619	0.041707	0.041885	0.006947
T4.12.1	0.257513	0.189782	0.143555	0.203099	0.06943	0.076479	0.05134	0.008802
T4.12.2	0.253978	0.186885	0.141087	0.208444	0.067836	0.103373	0.03125	0.007147
T4.12.3	0.251019	0.18451	0.139115	0.237306	0.066651	0.082362	0.032738	0.0063

T4.13.1	0.253419	0.195754	0.252918	0.104138	0.06811	0.069444	0.048042	0.008175
T4.13.3	0.249696	0.192553	0.258666	0.101939	0.066428	0.096153	0.028025	0.00654
T4.13.3	0.245666	0.189172	0.289706	0.099766	0.064842	0.075548	0.029741	0.005559
T4.14.1	0.253691	0.224722	0.209219	0.152178	0.068589	0.040548	0.04376	0.007292
T4.14.2	0.244686	0.215898	0.218729	0.192225	0.064637	0.03778	0.021524	0.004519
T4.14.3	0.24344	0.214714	0.252694	0.161332	0.064177	0.037481	0.021804	0.004358
T4.15.1	0.252179	0.247382	0.188601	0.135455	0.068143	0.0856	0.0172	0.00544
T4.15.2	0.245861	0.240576	0.192792	0.187553	0.065486	0.047249	0.016269	0.004214
T4.15.3	0.243886	0.238494	0.245181	0.13992	0.064746	0.047731	0.016037	0.004004
T4.16.1	0.247255	0.279992	0.181402	0.129443	0.066627	0.06665	0.025236	0.003394
T4.16.2	0.242627	0.274247	0.191274	0.157431	0.064706	0.046174	0.020312	0.003229
T4.16.3	0.241839	0.273286	0.210556	0.139939	0.064407	0.0473	0.019465	0.003207
T4.23.1	0.252097	0.302209	0.152592	0.107295	0.067717	0.064573	0.045766	0.007753
T4.23.2	0.248213	0.308253	0.149637	0.104912	0.065957	0.091064	0.025835	0.006129
T4.23.3	0.24327	0.340892	0.146098	0.102174	0.064036	0.070781	0.027681	0.005069
T4.24.1	0.257742	0.255693	0.179551	0.144298	0.07009	0.042711	0.042798	0.007116
T4.24.2	0.248183	0.265921	0.171289	0.184803	0.065847	0.039643	0.020033	0.004281
T4.24.3	0.245816	0.302448	0.169393	0.153791	0.06498	0.039067	0.020488	0.004017
T4.25.1	0.258685	0.233513	0.20045	0.128879	0.070558	0.084232	0.018341	0.005342
T4.25.2	0.251923	0.238092	0.194049	0.182414	0.067666	0.0445	0.017291	0.004065
T4.25.3	0.248109	0.295326	0.190636	0.133836	0.066246	0.045295	0.01684	0.003713
T4.26.1	0.255344	0.226302	0.230688	0.123553	0.069687	0.065846	0.024919	0.00366
T4.26.2	0.249989	0.237307	0.224818	0.153607	0.067434	0.043661	0.019723	0.003461
T4.26.3	0.248451	0.258803	0.223204	0.135426	0.066852	0.045554	0.018292	0.003417
T4.34.1	0.265008	0.242168	0.187824	0.136769	0.072774	0.046155	0.042288	0.007014
T4.34.2	0.253094	0.253239	0.233357	0.128173	0.067502	0.042208	0.018563	0.003865
T4.34.3	0.251945	0.290602	0.198495	0.127448	0.067086	0.041922	0.018741	0.00376
T4.35.1	0.268861	0.221837	0.170411	0.155288	0.074381	0.083789	0.020131	0.005303
T4.35.2	0.259023	0.227429	0.231759	0.14758	0.07019	0.041749	0.018574	0.003696
T4.35.3	0.257132	0.287083	0.176091	0.146235	0.069492	0.042065	0.018348	0.003554
T4.36.1	0.267653	0.215604	0.164905	0.18264	0.074413	0.0658	0.024903	0.004082
T4.36.2	0.259946	0.229366	0.200242	0.175545	0.071142	0.041727	0.018247	0.003785
T4.36.3	0.259172	0.252303	0.178865	0.174888	0.070851	0.0425	0.017658	0.003763
T4.45.1	0.293176	0.196347	0.149225	0.18032	0.083828	0.067465	0.025688	0.003952
T4.45.2	0.289519	0.197426	0.230326	0.105633	0.082309	0.066084	0.025084	0.003619
T4.45.3	0.286543	0.282163	0.150847	0.106195	0.081142	0.065048	0.024643	0.003419

T4.46.1	0.300573	0.197262	0.149826	0.153427	0.087828	0.085884	0.01968	0.00552
T4.46.2	0.297786	0.200121	0.20017	0.10709	0.08664	0.084569	0.018219	0.005405
T4.46.3	0.295832	0.249242	0.154122	0.108577	0.085847	0.083709	0.017338	0.005334
T4.56.1	0.325726	0.194745	0.147734	0.135647	0.099569	0.04558	0.043683	0.007317
T4.56.2	0.323467	0.199274	0.179302	0.106515	0.098568	0.042554	0.043116	0.007205
T4.56.3	0.322211	0.225624	0.154553	0.108882	0.098032	0.040729	0.042821	0.007149
T5.12.1	0.27482	0.187753	0.141992	0.180271	0.119356	0.039622	0.048065	0.008122
T5.12.2	0.268063	0.182638	0.137655	0.187861	0.153458	0.037722	0.02685	0.005754
T5.12.3	0.266538	0.181511	0.136727	0.218222	0.126816	0.037369	0.027347	0.00547
T5.13.1	0.272092	0.194918	0.227324	0.103777	0.109528	0.039191	0.045526	0.007644
T5.13.2	0.264954	0.189245	0.235591	0.099913	0.143755	0.037169	0.024102	0.005271
T5.13.3	0.262142	0.187073	0.268656	0.098542	0.117351	0.036536	0.024867	0.004832
T5.14.1	0.272217	0.207054	0.208732	0.151795	0.069958	0.039368	0.043613	0.007264
T5.14.2	0.262554	0.198923	0.218219	0.191741	0.065927	0.036681	0.021452	0.004502
T5.14.3	0.261218	0.197833	0.252104	0.160925	0.065457	0.036391	0.021731	0.004341
T5.15.1	0.269538	0.245075	0.172034	0.122367	0.130598	0.038973	0.017012	0.004404
T5.15.2	0.266135	0.241696	0.173543	0.181697	0.078275	0.038122	0.016575	0.003958
T5.15.3	0.263512	0.239126	0.236421	0.124486	0.078933	0.037526	0.016281	0.003715
T5.16.1	0.264958	0.278194	0.168403	0.119259	0.106232	0.038397	0.02116	0.003396
T5.16.2	0.262551	0.275435	0.172104	0.152444	0.077048	0.037795	0.019302	0.003321
T5.16.3	0.261115	0.273805	0.202947	0.124444	0.078653	0.037462	0.018293	0.003281
T5.23.1	0.272068	0.27443	0.152878	0.1076	0.10263	0.039257	0.043808	0.007328
T5.23.2	0.264602	0.283212	0.147641	0.103393	0.13684	0.037124	0.022241	0.004946
T5.23.3	0.260676	0.318339	0.145074	0.101428	0.110648	0.036256	0.023168	0.004411
T5.24.1	0.274925	0.254545	0.164189	0.143538	0.073276	0.039978	0.042492	0.007058
T5.24.2	0.264728	0.264726	0.156633	0.18383	0.06884	0.037106	0.01989	0.004246
T5.24.3	0.262203	0.30109	0.1549	0.152981	0.067934	0.036566	0.020342	0.003984
T5.25.1	0.277362	0.214887	0.199196	0.116544	0.128789	0.040666	0.018223	0.004332
T5.25.2	0.273702	0.216553	0.196024	0.178188	0.074217	0.039727	0.017727	0.003861
T5.25.3	0.268593	0.285961	0.191764	0.119193	0.075309	0.038578	0.017149	0.003452
T5.26.1	0.274324	0.211606	0.229861	0.113961	0.105148	0.040493	0.020931	0.003676
T5.26.2	0.27154	0.215776	0.227053	0.149996	0.073323	0.039781	0.018947	0.003584
T5.26.3	0.268722	0.250522	0.224295	0.120568	0.076039	0.039129	0.01722	0.003507
T5.34.1	0.28067	0.24049	0.186412	0.123934	0.078536	0.041231	0.041805	0.006922
T5.34.2	0.268053	0.251483	0.231602	0.116145	0.072846	0.037705	0.018351	0.003814
T5.34.3	0.266835	0.288588	0.197003	0.115488	0.072397	0.037451	0.018527	0.003711

T5.35.1	0.289041	0.20435	0.155759	0.154805	0.128378	0.043267	0.02009	0.004311
T5.35.2	0.281804	0.207066	0.227303	0.149663	0.070122	0.041427	0.019094	0.003519
T5.35.3	0.279246	0.280388	0.158833	0.147961	0.07056	0.040859	0.018802	0.003351
T5.36.1	0.288168	0.201838	0.153441	0.182535	0.105253	0.043684	0.020966	0.004115
T5.36.2	0.282699	0.208798	0.196362	0.177939	0.070427	0.042276	0.01757	0.00393
T5.36.3	0.281271	0.246382	0.16131	0.176804	0.071544	0.041948	0.016849	0.003891
T5.45.1	0.300693	0.193628	0.146934	0.177211	0.109547	0.045876	0.022292	0.003819
T5.45.2	0.296943	0.194692	0.226789	0.103812	0.107562	0.044937	0.021767	0.003497
T5.45.3	0.29389	0.278256	0.148531	0.104364	0.106038	0.044233	0.021384	0.003303
T5.46.1	0.303737	0.194117	0.147155	0.150328	0.133557	0.047408	0.019023	0.004675
T5.46.2	0.30092	0.19693	0.196602	0.104927	0.13175	0.046682	0.017611	0.004578
T5.46.3	0.298946	0.245268	0.151375	0.106384	0.130544	0.046207	0.016759	0.004517
T5.56.1	0.353382	0.189901	0.143724	0.121913	0.076977	0.061796	0.044737	0.00757
T5.56.2	0.351638	0.195801	0.162836	0.10424	0.072355	0.061306	0.044333	0.007491
T5.56.3	0.351019	0.206662	0.15262	0.107334	0.069568	0.061138	0.044196	0.007464
T6.12.1	0.297753	0.184051	0.139006	0.159922	0.104147	0.092623	0.016476	0.006022
T6.12.2	0.294003	0.181499	0.136863	0.162383	0.146752	0.057233	0.016014	0.005254
T6.12.3	0.29142	0.179767	0.135434	0.207372	0.107303	0.058062	0.015744	0.004899
T6.13.1	0.295992	0.191904	0.204288	0.101959	0.095573	0.088201	0.016413	0.005671
T6.13.2	0.292023	0.189067	0.207011	0.100063	0.139015	0.052016	0.015915	0.004888
T6.13.3	0.287206	0.185691	0.256819	0.097922	0.099297	0.053308	0.015428	0.00433
T6.14.1	0.297113	0.204592	0.187592	0.134673	0.068995	0.085012	0.016631	0.005394
T6.14.2	0.289669	0.198963	0.19176	0.18647	0.066305	0.046924	0.015731	0.004178
T6.14.3	0.287343	0.197241	0.243869	0.139112	0.065555	0.047403	0.015507	0.003969
T6.15.1	0.295952	0.219979	0.171496	0.121955	0.130121	0.03954	0.016574	0.004383
T6.15.2	0.292216	0.216946	0.173001	0.181085	0.077989	0.038676	0.016149	0.003938
T6.15.3	0.289336	0.214639	0.235682	0.124067	0.078645	0.038072	0.015862	0.003697
T6.16.1	0.290978	0.276878	0.161149	0.113469	0.090031	0.047226	0.016787	0.003482
T6.16.2	0.28911	0.274942	0.166643	0.131931	0.0738	0.04357	0.016575	0.00343
T6.16.3	0.288419	0.27423	0.178512	0.121161	0.07618	0.041584	0.016501	0.003412
T6.23.1	0.29702	0.249155	0.150974	0.106177	0.089545	0.085143	0.016548	0.005438
T6.23.2	0.292866	0.252086	0.148371	0.104106	0.133503	0.048407	0.016018	0.004643
T6.23.3	0.286074	0.305686	0.144334	0.101004	0.093615	0.049986	0.015347	0.003954
T6.24.1	0.301016	0.231105	0.162718	0.127327	0.072574	0.08301	0.017007	0.005243
T6.24.2	0.293147	0.235636	0.157522	0.180217	0.069599	0.043855	0.016034	0.00399
T6.24.3	0.288708	0.29228	0.154751	0.132225	0.068139	0.044638	0.015616	0.003644

T6.25.1	0.302576	0.213584	0.17683	0.115722	0.127799	0.042022	0.017182	0.004286
T6.25.2	0.298584	0.215239	0.174014	0.17693	0.073647	0.041052	0.016714	0.00382
T6.25.3	0.293011	0.284226	0.170233	0.118352	0.07473	0.039864	0.016169	0.003415
T6.26.1	0.301555	0.203366	0.229204	0.108415	0.089111	0.046734	0.017839	0.003776
T6.26.2	0.299196	0.209634	0.227062	0.129797	0.070231	0.042806	0.017566	0.003708
T6.26.3	0.297842	0.223177	0.225855	0.118337	0.074309	0.039386	0.017422	0.003672
T6.34.1	0.308168	0.218315	0.167483	0.123228	0.0781	0.08187	0.017691	0.005145
T6.34.2	0.296892	0.223819	0.227777	0.117112	0.0737	0.040793	0.016323	0.003586
T6.34.3	0.294724	0.282525	0.173066	0.116044	0.072967	0.041102	0.016124	0.003448
T6.35.1	0.312811	0.202432	0.154179	0.135567	0.126823	0.045799	0.018152	0.004236
T6.35.2	0.304979	0.205122	0.224998	0.131064	0.069273	0.043852	0.017253	0.003458
T6.35.3	0.30221	0.277756	0.157222	0.129573	0.069706	0.043251	0.016989	0.003293
T6.36.1	0.317045	0.19396	0.1468	0.182475	0.08922	0.046815	0.019448	0.004236
T6.36.2	0.312461	0.204588	0.172682	0.179004	0.068227	0.040034	0.018907	0.004097
T6.36.3	0.311775	0.219467	0.158811	0.178506	0.069933	0.038594	0.018835	0.004078
T6.45.1	0.322171	0.191165	0.144901	0.174524	0.094306	0.050194	0.019016	0.003723
T6.45.2	0.318153	0.192215	0.223652	0.102238	0.092597	0.049167	0.018569	0.003409
T6.45.3	0.314882	0.274716	0.146476	0.102782	0.091285	0.048396	0.018242	0.00322
T6.46.1	0.335032	0.188604	0.142494	0.13015	0.134418	0.043006	0.021444	0.004852
T6.46.2	0.332709	0.19299	0.172941	0.102198	0.133067	0.040151	0.021166	0.004778
T6.46.3	0.331417	0.21851	0.14907	0.104469	0.132343	0.038429	0.021021	0.004741
T6.56.1	0.359693	0.186522	0.140836	0.119103	0.074897	0.088416	0.024605	0.005928
T6.56.2	0.357917	0.192317	0.159565	0.101837	0.0704	0.087715	0.024383	0.005866
T6.56.3	0.357287	0.202985	0.149554	0.10486	0.067688	0.087474	0.024308	0.005845
T7.12.1	0.332367	0.179071	0.135021	0.152502	0.098404	0.071937	0.027499	0.003199
T7.12.2	0.329475	0.177359	0.133589	0.157997	0.119161	0.054758	0.024536	0.003124
T7.12.3	0.328308	0.176675	0.133023	0.173699	0.105433	0.056599	0.023164	0.003098
T7.13.1	0.330567	0.186816	0.19587	0.098951	0.09018	0.068403	0.026014	0.003199
T7.13.2	0.327166	0.184704	0.202088	0.097557	0.112726	0.049695	0.022954	0.003111
T7.13.3	0.324986	0.183369	0.219854	0.096707	0.098635	0.052618	0.020768	0.003064
T7.14.1	0.331978	0.199274	0.179675	0.128117	0.066949	0.065843	0.024899	0.003264
T7.14.2	0.325764	0.195185	0.189453	0.155818	0.065018	0.045615	0.020041	0.003106
T7.14.3	0.324707	0.194501	0.208551	0.138505	0.064718	0.046728	0.019206	0.003084
T7.15.1	0.331538	0.214889	0.167131	0.118291	0.1053	0.038622	0.020938	0.003292
T7.15.2	0.328526	0.212758	0.170804	0.151206	0.076371	0.038016	0.0191	0.003219
T7.15.3	0.326728	0.211499	0.201414	0.123434	0.077962	0.037681	0.018101	0.003181

T7.16.1	0.331599	0.238272	0.160405	0.112905	0.089548	0.046951	0.016911	0.003409
T7.16.2	0.32947	0.236606	0.165873	0.131276	0.073404	0.043316	0.016697	0.003357
T7.16.3	0.328683	0.235994	0.177687	0.12056	0.075772	0.041342	0.016623	0.00334
T7.23.1	0.331845	0.239757	0.146807	0.1031	0.084349	0.065914	0.02499	0.003236
T7.23.2	0.328009	0.246559	0.144739	0.10147	0.108073	0.046165	0.02185	0.003136
T7.23.3	0.324929	0.265992	0.143135	0.100234	0.093722	0.049785	0.019131	0.003071
T7.24.1	0.336453	0.22213	0.158315	0.120909	0.070462	0.064165	0.024213	0.003352
T7.24.2	0.329397	0.232933	0.154287	0.15032	0.068183	0.042546	0.019165	0.003169
T7.24.3	0.327371	0.254032	0.153179	0.132528	0.067595	0.044391	0.017774	0.00313
T7.25.1	0.338871	0.208484	0.172508	0.112014	0.103196	0.041039	0.020462	0.003427
T7.25.2	0.335432	0.212592	0.1704	0.147433	0.071963	0.040316	0.018523	0.003341
T7.25.3	0.33195	0.246826	0.16833	0.118507	0.074628	0.039656	0.016834	0.003269
T7.26.1	0.341467	0.201511	0.193772	0.10727	0.08809	0.04615	0.018144	0.003597
T7.26.2	0.338795	0.207722	0.19196	0.128427	0.069427	0.042271	0.017866	0.003531
T7.26.3	0.337262	0.221141	0.19094	0.117088	0.073458	0.038893	0.01772	0.003497
T7.34.1	0.344565	0.209479	0.15985	0.119796	0.075872	0.063149	0.023779	0.003509
T7.34.2	0.334644	0.22285	0.194103	0.115142	0.072537	0.040046	0.017424	0.003254
T7.34.3	0.333647	0.245135	0.173382	0.114712	0.07224	0.040788	0.016862	0.003235
T7.35.1	0.350273	0.197207	0.149646	0.132126	0.102169	0.044724	0.020211	0.003644
T7.35.2	0.343626	0.204007	0.191505	0.128799	0.068363	0.043282	0.016937	0.00348
T7.35.3	0.34189	0.240728	0.15732	0.127978	0.069448	0.042947	0.016243	0.003445
T7.36.1	0.35609	0.191172	0.144526	0.150826	0.087586	0.045872	0.020038	0.00389
T7.36.2	0.350941	0.201647	0.170008	0.147957	0.066978	0.039227	0.01948	0.003762
T7.36.3	0.350171	0.216312	0.156351	0.147545	0.068652	0.037817	0.019406	0.003746
T7.45.1	0.361592	0.188054	0.142173	0.14477	0.092264	0.04924	0.018048	0.00386
T7.45.2	0.358238	0.190779	0.189946	0.101048	0.091015	0.048486	0.016708	0.00378
T7.45.3	0.355888	0.237606	0.146249	0.102451	0.090183	0.047993	0.0159	0.00373
T7.46.1	0.372258	0.184853	0.139423	0.12709	0.107866	0.041771	0.022503	0.004236
T7.46.2	0.369676	0.189152	0.169214	0.099796	0.106782	0.038997	0.022211	0.004172
T7.46.3	0.368241	0.214163	0.145858	0.102013	0.106201	0.037325	0.022059	0.004139
T7.56.1	0.393769	0.181851	0.136985	0.115517	0.072388	0.068134	0.026558	0.004799
T7.56.2	0.391825	0.1875	0.155201	0.098771	0.068041	0.067594	0.026318	0.004748
T7.56.3	0.391136	0.197901	0.145464	0.101703	0.06542	0.067408	0.026237	0.004731

<附錄二>選手賽程排名(依賽程為序)

賽程	A	B	C	D	E	F	G	H
T1.12.1	273	280	191	221	10	28	1	1
T1.12.2	284	292	198	225	4	1	49	13
T1.12.3	292	297	204	228	1	13	46	25
T1.13.1	288	290	166	13	198	55	4	4
T1.13.2	300	302	177	8	213	4	55	35
T1.13.3	308	310	185	1	226	44	52	58
T1.14.1	286	288	126	31	56	205	8	8
T1.14.2	307	308	140	21	7	239	73	69
T1.14.3	313	313	147	4	43	245	67	80
T1.15.1	293	296	81	65	91	10	226	62
T1.15.2	304	304	88	62	13	107	244	92
T1.15.3	310	311	96	7	79	106	252	117
T1.16.1	305	305	30	79	114	61	62	243
T1.16.2	312	312	36	67	55	114	93	251
T1.16.3	315	315	39	46	83	109	120	254
T1.23.1	287	289	13	172	180	68	7	7
T1.23.2	301	303	8	183	193	7	61	55
T1.23.3	314	314	1	196	204	56	58	76
T1.24.1	270	279	33	127	70	176	13	16
T1.24.2	297	301	23	139	15	202	101	87
T1.24.3	306	307	4	148	58	211	90	118
T1.25.1	266	274	69	83	120	16	190	67
T1.25.2	280	283	67	94	18	120	201	111
T1.25.3	296	299	7	103	109	118	220	144
T1.26.1	272	278	79	36	137	63	70	197
T1.26.2	283	286	70	44	62	126	112	211
T1.26.3	291	295	43	50	107	117	146	218
T1.34.1	241	256	66	80	131	134	18	21
T1.34.2	279	285	54	16	148	160	137	132
T1.34.3	285	291	9	61	152	166	133	141
T1.35.1	228	242	104	119	78	20	144	72
T1.35.2	249	264	92	17	95	146	163	150
T1.35.3	255	269	15	113	103	142	171	166

T1.36.1	222	233	120	132	35	67	75	147
T1.36.2	237	250	91	63	42	149	148	163
T1.36.3	239	254	61	110	45	139	165	167
T1.45.1	192	207	134	156	23	84	95	126
T1.45.2	201	217	129	20	173	92	107	158
T1.45.3	209	224	20	147	171	98	116	186
T1.46.1	174	189	142	166	65	39	126	96
T1.46.2	181	196	128	66	174	45	156	106
T1.46.3	184	200	65	140	168	50	182	109
T1.56.1	123	133	160	185	123	127	37	46
T1.56.2	127	138	145	117	178	153	42	52
T1.56.3	129	141	114	151	172	178	44	53
T2.12.1	243	199	273	233	11	30	2	2
T2.12.2	254	210	281	239	5	2	50	14
T2.12.3	262	215	288	244	2	15	47	28
T2.13.1	261	176	287	14	239	57	5	5
T2.13.3	277	182	295	10	249	5	56	39
T2.13.3	290	192	303	2	267	49	53	59
T2.14.1	267	130	289	38	59	246	9	10
T2.14.2	294	145	305	22	8	274	76	73
T2.14.3	299	149	309	5	49	283	71	84
T2.15.1	282	85	297	69	99	11	262	65
T2.15.2	295	92	304	64	14	111	279	99
T2.15.3	303	100	311	9	84	108	292	120
T2.16.1	302	34	310	85	117	62	66	278
T2.16.2	309	39	313	72	57	116	100	289
T2.16.3	311	42	315	55	89	112	124	293
T2.23.1	260	14	286	182	192	91	14	17
T2.23.2	278	8	296	195	203	14	83	61
T2.23.3	298	1	308	207	220	65	63	110
T2.24.1	242	41	270	131	102	190	20	24
T2.24.2	271	27	294	150	21	212	132	114
T2.24.3	289	4	302	159	68	231	122	145
T2.25.1	234	74	261	88	142	23	194	77
T2.25.2	246	68	275	98	25	141	208	125
T2.25.3	274	7	293	112	127	130	234	183

T2.26.1	238	88	266	40	159	72	79	202
T2.26.2	251	73	276	51	67	151	130	215
T2.26.3	259	49	282	57	125	128	181	228
T2.34.1	215	66	242	108	136	137	24	31
T2.34.2	245	60	277	23	155	171	172	153
T2.34.3	256	12	284	78	160	179	159	181
T2.35.1	198	112	228	142	82	27	145	79
T2.35.2	219	103	246	26	101	185	168	171
T2.35.3	230	18	259	129	111	169	179	205
T2.36.1	191	124	221	164	36	74	82	146
T2.36.2	208	101	236	74	47	191	169	165
T2.36.3	214	61	238	120	51	158	200	174
T2.45.1	163	146	195	190	28	83	94	137
T2.45.2	172	140	205	28	190	93	106	179
T2.45.3	186	20	217	174	181	101	121	225
T2.46.1	143	152	183	197	72	38	134	95
T2.46.2	152	136	187	75	191	46	174	104
T2.46.3	160	65	194	168	177	54	218	113
T2.56.1	102	172	127	215	129	131	35	44
T2.56.2	111	151	133	124	200	167	40	50
T2.56.3	115	121	135	171	179	217	45	54
T3.12.1	204	239	250	249	12	31	3	3
T3.12.2	216	249	263	261	6	3	51	15
T3.12.3	221	255	269	268	3	17	48	30
T3.13.1	233	187	16	282	251	70	10	9
T3.13.2	247	197	10	295	272	8	64	56
T3.13.3	265	212	2	309	291	59	59	81
T3.14.1	231	134	44	276	76	256	16	20
T3.14.2	258	155	24	305	17	289	115	100
T3.14.3	269	161	5	313	63	295	98	127
T3.15.1	240	89	76	288	128	21	267	75
T3.15.2	253	102	74	299	20	125	289	122
T3.15.3	275	110	12	312	115	121	302	157
T3.16.1	263	40	97	302	144	73	80	282
T3.16.2	276	47	77	310	66	138	127	294
T3.16.3	281	52	60	315	116	122	161	301

T3.23.1	232	15	189	280	231	95	15	18
T3.23.2	248	10	199	296	244	18	86	63
T3.23.3	268	2	213	311	270	66	68	115
T3.24.1	226	48	139	265	106	227	22	29
T3.24.2	250	31	153	297	24	260	136	119
T3.24.3	264	5	170	308	71	276	125	152
T3.25.1	225	79	93	264	146	29	241	82
T3.25.2	236	72	102	283	27	147	256	133
T3.25.3	257	9	116	304	134	133	278	191
T3.26.1	235	98	41	274	165	77	88	246
T3.26.2	244	82	51	289	73	159	135	261
T3.26.3	252	59	57	298	130	132	191	275
T3.34.1	176	99	119	231	133	135	32	40
T3.34.2	210	70	26	254	158	177	213	201
T3.34.3	217	22	86	260	162	186	207	212
T3.35.1	158	139	157	219	75	35	139	89
T3.35.2	183	127	29	232	105	224	173	217
T3.35.3	190	25	144	236	110	220	177	240
T3.36.1	151	156	176	211	32	81	92	140
T3.36.2	175	125	80	227	50	222	216	170
T3.36.3	179	69	130	230	52	209	237	172
T3.45.1	126	184	201	187	34	78	89	142
T3.45.2	138	179	31	206	232	96	111	227
T3.45.3	147	28	190	210	225	100	119	249
T3.46.1	117	190	208	170	77	33	140	86
T3.46.2	124	167	82	179	219	48	219	107
T3.46.3	128	77	180	184	208	52	243	108
T3.56.1	66	208	225	118	135	136	30	37
T3.56.2	71	174	132	126	229	215	39	48
T3.56.3	73	129	181	128	210	241	41	49
T4.12.1	178	258	272	15	264	58	6	6
T4.12.2	185	270	283	11	282	6	57	42
T4.12.3	197	277	291	3	294	51	54	60
T4.13.1	188	232	17	258	279	71	12	11
T4.13.3	200	245	11	278	296	9	65	57
T4.13.3	213	260	3	294	308	60	60	83

T4.14.1	187	137	71	93	273	257	27	34
T4.14.2	218	163	63	18	312	298	153	148
T4.14.3	223	170	18	70	314	303	150	159
T4.15.1	193	90	122	135	277	34	264	90
T4.15.2	211	107	108	25	303	164	297	178
T4.15.3	220	113	21	122	309	157	303	193
T4.16.1	207	38	131	155	295	85	96	277
T4.16.2	227	55	113	77	311	181	180	298
T4.16.3	229	58	68	121	313	163	196	302
T4.23.1	194	17	232	241	283	97	17	19
T4.23.2	203	11	240	252	299	19	87	64
T4.23.3	224	3	257	273	315	69	69	116
T4.24.1	177	71	137	115	255	223	34	43
T4.24.2	205	63	164	29	301	269	188	169
T4.24.3	212	16	172	86	307	280	175	192
T4.25.1	173	122	87	157	246	41	229	97
T4.25.2	196	115	106	35	285	199	260	190
T4.25.3	206	21	117	138	298	195	275	232
T4.26.1	182	132	37	178	259	88	102	239
T4.26.2	199	117	55	87	288	207	192	262
T4.26.3	202	67	59	136	293	194	231	270
T4.34.1	153	104	123	130	230	183	38	47
T4.34.2	189	78	32	161	287	232	223	208
T4.34.3	195	24	95	165	289	236	217	224
T4.35.1	140	148	167	82	212	42	184	101
T4.35.2	171	131	34	106	253	238	221	236
T4.35.3	180	29	149	111	262	233	228	250
T4.36.1	146	165	186	32	211	90	103	188
T4.36.2	169	128	89	56	242	240	232	219
T4.36.3	170	80	141	58	243	228	251	222
T4.45.1	94	230	243	41	170	80	91	200
T4.45.2	104	225	38	250	175	86	97	244
T4.45.3	114	36	235	247	176	94	108	269
T4.46.1	77	226	237	89	163	32	193	85
T4.46.2	82	216	90	243	166	40	236	93
T4.46.3	90	86	226	235	167	43	259	98

T4.56.1	52	236	248	133	118	193	28	33
T4.56.2	56	219	138	245	122	225	31	38
T4.56.3	57	135	222	234	126	254	33	41
T5.12.1	131	266	280	42	60	270	11	12
T5.12.2	144	284	298	24	9	299	77	74
T5.12.3	149	293	301	6	54	306	74	88
T5.13.1	135	235	46	263	81	278	19	22
T5.13.2	155	259	28	292	19	308	118	102
T5.13.3	165	268	6	303	64	313	105	129
T5.14.1	134	194	72	95	256	273	29	36
T5.14.2	161	221	64	19	300	311	154	151
T5.14.3	166	223	19	71	304	314	152	160
T5.15.1	139	97	162	186	39	282	268	155
T5.15.2	150	105	151	37	186	292	286	198
T5.15.3	159	111	25	173	182	302	296	231
T5.16.1	154	44	174	199	92	291	160	276
T5.16.2	162	50	161	92	194	297	199	285
T5.16.3	167	57	83	175	183	304	230	290
T5.23.1	136	54	230	238	112	275	26	32
T5.23.2	157	33	249	266	26	309	142	121
T5.23.3	168	6	262	285	74	315	128	154
T5.24.1	130	75	188	116	224	264	36	45
T5.24.2	156	64	214	30	269	310	189	173
T5.24.3	164	19	219	90	281	312	178	195
T5.25.1	125	169	94	212	44	255	235	161
T5.25.2	133	160	101	48	215	267	248	209
T5.25.3	142	30	110	200	205	288	266	264
T5.26.1	132	180	40	223	100	258	167	237
T5.26.2	137	164	49	101	223	266	211	248
T5.26.3	141	84	56	193	199	279	263	256
T5.34.1	121	108	125	177	185	247	43	51
T5.34.2	145	83	35	213	228	300	227	216
T5.34.3	148	26	98	218	235	305	224	233
T5.35.1	107	203	216	84	46	214	185	164
T5.35.2	119	193	47	102	254	243	206	253
T5.35.3	122	37	206	104	245	251	215	281

T5.36.1	110	211	227	33	98	206	164	185
T5.36.2	118	186	100	49	248	229	254	204
T5.36.3	120	94	196	54	241	235	274	206
T5.45.1	76	243	252	52	80	187	141	214
T5.45.2	86	237	50	262	87	196	151	257
T5.45.3	93	43	245	256	94	201	157	286
T5.46.1	68	240	251	99	30	161	209	138
T5.46.2	75	229	99	251	38	175	253	143
T5.46.3	79	95	233	246	40	180	280	149
T5.56.1	14	257	271	189	195	103	21	23
T5.56.2	15	231	192	257	237	104	23	26
T5.56.3	16	195	231	240	261	105	25	27
T6.12.1	83	281	292	73	104	12	293	66
T6.12.2	92	294	300	68	16	113	306	103
T6.12.3	100	298	306	12	88	110	310	123
T6.13.1	88	248	78	277	132	24	294	78
T6.13.2	99	262	75	291	22	129	307	124
T6.13.3	113	273	14	306	119	123	314	162
T6.14.1	84	201	124	137	268	37	284	94
T6.14.2	103	220	111	27	297	170	311	180
T6.14.3	112	227	22	123	302	162	313	196
T6.15.1	89	153	163	188	41	271	288	156
T6.15.2	98	159	155	39	188	285	300	203
T6.15.3	105	171	27	176	184	293	309	235
T6.16.1	101	46	197	224	151	165	277	259
T6.16.2	106	51	182	145	216	208	287	267
T6.16.3	109	56	143	191	197	242	291	272
T6.23.1	85	87	234	248	154	36	290	91
T6.23.2	97	81	247	259	31	154	305	139
T6.23.3	116	13	268	287	140	143	315	199
T6.24.1	74	126	193	167	233	47	269	105
T6.24.2	95	120	210	43	260	203	304	194
T6.24.3	108	23	220	143	278	198	312	241
T6.25.1	69	175	148	216	48	234	265	168
T6.25.2	80	166	150	53	218	249	281	213
T6.25.3	96	32	169	203	207	265	299	271

T6.26.1	72	205	42	237	157	173	246	221
T6.26.2	78	183	48	153	252	221	255	234
T6.26.3	81	143	52	204	214	272	258	238
T6.34.1	65	158	178	181	187	53	250	112
T6.34.2	87	142	45	208	217	252	295	247
T6.34.3	91	35	154	214	227	248	301	265
T6.35.1	62	209	224	134	53	189	238	175
T6.35.2	67	198	53	149	266	204	261	263
T6.35.3	70	45	212	154	258	216	270	287
T6.36.1	60	241	254	34	156	172	197	177
T6.36.2	63	202	158	45	275	263	212	187
T6.36.3	64	154	207	47	257	287	214	189
T6.45.1	58	252	264	59	138	140	210	230
T6.45.2	59	247	58	271	141	150	222	273
T6.45.3	61	53	255	269	145	155	233	299
T6.46.1	29	263	278	152	29	218	155	128
T6.46.2	32	244	156	272	33	261	158	131
T6.46.3	39	157	244	255	37	290	162	135
T6.56.1	8	272	285	201	206	22	109	68
T6.56.2	10	246	203	279	250	25	113	70
T6.56.3	11	206	241	253	284	26	114	71
T7.12.1	33	300	307	91	124	64	72	304
T7.12.2	41	306	312	76	61	119	110	309
T7.12.3	46	309	314	60	96	115	129	312
T7.13.1	40	271	103	300	150	75	85	303
T7.13.2	49	276	84	307	69	145	131	310
T7.13.3	53	282	62	314	121	124	170	315
T7.14.1	34	218	136	162	292	89	104	292
T7.14.2	51	234	121	81	306	192	186	311
T7.14.3	55	238	73	125	310	174	202	313
T7.15.1	38	168	179	205	97	286	166	288
T7.15.2	45	177	165	96	196	294	205	300
T7.15.3	50	181	85	180	189	301	240	305
T7.16.1	37	114	200	226	153	168	272	274
T7.16.2	42	118	184	146	222	210	283	279
T7.16.3	44	119	146	194	202	244	285	284

T7.23.1	36	109	253	267	169	87	99	296
T7.23.2	47	93	265	284	85	182	149	307
T7.23.3	54	62	274	290	139	144	204	314
T7.24.1	27	147	209	192	247	99	117	280
T7.24.2	43	123	223	100	276	226	203	306
T7.24.3	48	76	229	141	286	200	247	308
T7.25.1	24	188	159	229	108	250	176	268
T7.25.2	28	178	168	109	240	259	225	283
T7.25.3	35	91	175	202	209	268	276	291
T7.26.1	23	214	107	242	161	184	239	245
T7.26.2	25	191	109	160	265	230	245	252
T7.26.3	26	150	115	209	221	284	249	258
T7.34.1	20	185	202	198	201	102	123	255
T7.34.2	30	144	105	220	234	262	257	295
T7.34.3	31	96	152	222	238	253	273	297
T7.35.1	18	228	239	144	113	197	183	242
T7.35.2	21	204	112	158	274	213	271	260
T7.35.3	22	106	211	163	263	219	298	266
T7.36.1	12	251	267	97	164	188	187	207
T7.36.2	17	213	171	105	290	277	195	223
T7.36.3	19	162	215	107	271	296	198	226
T7.45.1	7	265	279	114	143	148	242	210
T7.45.2	9	253	118	286	147	152	282	220
T7.45.3	13	116	256	270	149	156	308	229
T7.46.1	4	275	290	169	86	237	138	176
T7.46.2	5	261	173	293	90	281	143	182
T7.46.3	6	173	258	275	93	307	147	184
T7.56.1	1	287	299	217	236	76	78	130
T7.56.2	2	267	218	301	280	79	81	134
T7.56.3	3	222	260	281	305	82	84	136

<附錄三>選手賽程排名(依名次為序)

名次	A	B	C	D	E	F	G	H
1	T7.56.1	T2.23.3	T1.23.3	T1.13.3	T1.12.3	T1.12.2	T1.12.1	T1.12.1
2	T7.56.2	T3.23.3	T3.13.3	T2.13.3	T2.12.3	T2.12.2	T2.12.1	T2.12.1
3	T7.56.3	T4.23.3	T4.13.3	T4.12.3	T3.12.3	T3.12.2	T3.12.1	T3.12.1
4	T7.46.1	T2.24.3	T1.24.3	T1.14.3	T1.12.2	T1.13.2	T1.13.1	T1.13.1
5	T7.46.2	T3.24.3	T3.14.3	T2.14.3	T2.12.2	T2.13.2	T2.13.1	T2.13.1
6	T7.46.3	T5.23.3	T5.13.3	T5.12.3	T3.12.2	T4.12.2	T4.12.1	T4.12.1
7	T7.45.1	T2.25.3	T1.25.3	T1.15.3	T1.14.2	T1.23.2	T1.23.1	T1.23.1
8	T6.56.1	T2.23.2	T1.23.2	T1.13.2	T2.14.2	T3.13.2	T1.14.1	T1.14.1
9	T7.45.2	T3.25.3	T1.34.3	T2.15.3	T5.12.2	T4.13.2	T2.14.1	T3.13.1
10	T6.56.2	T3.23.2	T3.13.2	T2.13.2	T1.12.1	T1.15.1	T3.13.1	T2.14.1
11	T6.56.3	T4.23.2	T4.13.2	T4.12.2	T2.12.1	T2.15.1	T5.12.1	T4.13.1
12	T7.36.1	T2.34.3	T3.15.3	T6.12.3	T3.12.1	T6.12.1	T4.13.1	T5.12.1
13	T7.45.3	T6.23.3	T1.23.1	T1.13.1	T1.15.2	T1.12.3	T1.24.1	T1.12.2
14	T5.56.1	T2.23.1	T6.13.3	T2.13.1	T2.15.2	T2.23.2	T2.23.1	T2.12.2
15	T5.56.2	T3.23.1	T1.35.3	T4.12.1	T1.24.2	T2.12.3	T3.23.1	T3.12.2
16	T5.56.3	T4.24.3	T3.13.1	T1.34.2	T6.12.2	T1.25.1	T3.14.1	T1.24.1
17	T7.36.2	T4.23.1	T4.13.1	T1.35.2	T3.14.2	T3.12.3	T4.23.1	T2.23.1
18	T7.35.1	T2.35.3	T4.14.3	T4.14.2	T1.25.2	T3.23.2	T1.34.1	T3.23.1
19	T7.36.3	T5.24.3	T5.14.3	T5.14.2	T5.13.2	T4.23.2	T5.13.1	T4.23.1
20	T7.34.1	T2.45.3	T1.45.3	T1.45.2	T3.15.2	T1.35.1	T2.24.1	T3.14.1
21	T7.35.2	T4.25.3	T4.15.3	T1.14.2	T2.24.2	T3.15.1	T5.56.1	T1.34.1
22	T7.35.3	T3.34.3	T6.14.3	T2.14.2	T6.13.2	T6.56.1	T3.24.1	T5.13.1
23	T7.26.1	T6.24.3	T1.24.2	T2.34.2	T1.45.1	T2.25.1	T5.56.2	T5.56.1
24	T7.25.1	T4.34.3	T3.14.2	T5.12.2	T3.24.2	T6.13.1	T2.34.1	T2.24.1
25	T7.26.2	T3.35.3	T5.15.3	T4.15.2	T2.25.2	T6.56.2	T5.56.3	T1.12.3
26	T7.26.3	T5.34.3	T3.34.2	T2.35.2	T5.23.2	T6.56.3	T5.23.1	T5.56.2
27	T7.24.1	T2.24.2	T6.15.3	T6.14.2	T3.25.2	T2.35.1	T4.14.1	T5.56.3
28	T7.25.2	T3.45.3	T5.13.2	T2.45.2	T2.45.1	T1.12.1	T4.56.1	T2.12.3
29	T6.46.1	T4.35.3	T3.35.2	T4.24.2	T6.46.1	T3.25.1	T5.14.1	T3.24.1
30	T7.34.2	T5.25.3	T1.16.1	T5.24.2	T5.46.1	T2.12.1	T3.56.1	T3.12.3
31	T7.34.3	T3.24.2	T3.45.2	T1.14.1	T6.23.2	T3.12.1	T4.56.2	T2.34.1
32	T6.46.2	T6.25.3	T4.34.2	T4.36.1	T3.36.1	T4.46.1	T3.34.1	T5.23.1

33	T7.12.1	T5.23.2	T1.24.1	T5.36.1	T6.46.2	T3.46.1	T4.56.3	T4.56.1
34	T7.14.1	T2.16.1	T4.35.2	T6.36.1	T3.45.1	T4.15.1	T4.24.1	T4.14.1
35	T7.25.3	T6.34.3	T5.34.2	T4.25.2	T1.36.1	T3.35.1	T2.56.1	T1.13.2
36	T7.23.1	T4.45.3	T1.16.2	T1.26.1	T2.36.1	T6.23.1	T5.24.1	T5.14.1
37	T7.16.1	T5.35.3	T4.26.1	T5.15.2	T6.46.3	T6.14.1	T1.56.1	T3.56.1
38	T7.15.1	T4.16.1	T4.45.2	T2.14.1	T5.46.2	T2.46.1	T4.34.1	T4.56.2
39	T6.46.3	T2.16.2	T1.16.3	T6.15.2	T5.15.1	T1.46.1	T3.56.2	T2.13.10
40	T7.13.1	T3.16.1	T5.26.1	T2.26.1	T5.46.3	T4.46.2	T2.56.2	T3.34.1
41	T7.12.2	T2.24.1	T3.26.1	T4.45.1	T6.15.1	T4.25.1	T3.56.3	T4.56.3
42	T7.16.2	T2.16.3	T6.26.1	T5.12.1	T1.36.2	T4.35.1	T1.56.2	T4.12.2
43	T7.24.2	T5.45.3	T1.26.3	T6.24.2	T1.14.3	T4.46.3	T5.34.1	T4.24.1
44	T7.16.3	T5.16.1	T3.14.1	T1.26.2	T5.25.1	T1.13.3	T1.56.3	T2.56.1
45	T7.15.2	T6.35.3	T6.34.2	T6.36.2	T1.36.3	T1.46.2	T2.56.3	T5.24.1
46	T7.12.3	T6.16.1	T5.13.1	T1.16.3	T5.35.1	T2.46.2	T1.12.3	T1.56.1
47	T7.23.2	T3.16.2	T5.35.2	T6.36.3	T2.36.2	T6.24.1	T2.12.3	T4.34.1
48	T7.24.3	T3.24.1	T6.26.2	T5.25.2	T6.25.1	T3.46.2	T3.12.3	T3.56.2
49	T7.13.2	T2.26.3	T5.26.2	T5.36.2	T2.14.3	T2.13.3	T1.12.2	T3.56.3
50	T7.15.3	T5.16.2	T5.45.2	T1.26.3	T3.36.2	T1.46.3	T2.12.2	T2.56.2
51	T7.14.2	T6.16.2	T3.26.2	T2.26.2	T2.36.3	T4.12.3	T3.12.2	T5.34.1
52	T4.56.1	T3.16.3	T6.26.3	T5.45.1	T3.36.3	T3.46.3	T1.13.3	T1.56.2
53	T7.13.3	T6.45.3	T6.35.2	T6.25.2	T6.35.1	T6.34.1	T2.13.3	T1.56.3
54	T7.23.3	T5.23.1	T1.34.2	T5.36.3	T5.12.3	T2.46.3	T4.12.3	T2.56.3
55	T7.14.3	T4.16.2	T4.26.2	T2.16.3	T1.16.2	T1.13.1	T1.13.2	T1.23.2
56	T4.56.2	T6.16.3	T5.26.3	T4.36.2	T1.14.1	T1.23.3	T2.13.2	T3.13.2
57	T4.56.3	T5.16.3	T3.26.3	T2.26.3	T2.16.2	T2.13.1	T4.12.2	T4.13.2
58	T6.45.1	T4.16.3	T6.45.2	T4.36.3	T1.24.3	T4.12.1	T1.23.3	T1.13.3
59	T6.45.2	T3.26.3	T4.26.3	T6.45.1	T2.14.1	T3.13.3	T3.13.3	T2.13.3
60	T6.36.1	T2.34.2	T3.16.3	T7.12.3	T5.12.1	T4.13.3	T4.13.3	T4.12.3
61	T6.45.3	T2.36.3	T1.36.3	T1.34.3	T7.12.2	T1.16.1	T1.23.2	T2.23.2
62	T6.35.1	T7.23.3	T7.13.3	T1.15.2	T1.26.2	T2.16.1	T1.16.1	T1.15.1
63	T6.36.2	T4.24.2	T4.14.2	T1.36.2	T3.14.3	T1.26.1	T2.23.3	T3.23.2
64	T6.36.3	T5.24.2	T5.14.2	T2.15.2	T5.13.3	T7.12.1	T3.13.2	T4.23.2
65	T6.34.1	T2.46.3	T1.46.3	T1.15.1	T1.46.1	T2.23.3	T4.13.2	T2.15.1
66	T3.56.1	T2.34.1	T1.34.1	T1.46.2	T3.16.2	T3.23.3	T2.16.1	T6.12.1
67	T6.35.2	T4.26.3	T1.25.2	T1.16.2	T2.26.2	T1.36.1	T1.14.3	T1.25.1
68	T5.46.1	T2.25.2	T4.16.3	T6.12.2	T2.24.3	T1.23.1	T3.23.3	T6.56.1

69	T6.25.1	T3.36.3	T1.25.1	T2.15.1	T7.13.2	T4.23.3	T4.23.3	T1.14.2
70	T6.35.3	T3.34.2	T1.26.2	T4.14.3	T1.24.1	T3.13.1	T1.26.1	T6.56.2
71	T3.56.2	T4.24.1	T4.14.1	T5.14.3	T3.24.3	T4.13.1	T2.14.3	T6.56.3
72	T6.26.1	T3.25.2	T5.14.1	T2.16.2	T2.46.1	T2.26.1	T7.12.1	T1.35.1
73	T3.56.3	T2.26.2	T7.14.3	T6.12.1	T3.26.2	T3.16.1	T1.14.2	T2.14.2
74	T6.24.1	T2.25.1	T3.15.2	T2.36.2	T5.23.3	T2.36.1	T5.12.3	T5.12.2
75	T5.46.2	T5.24.1	T6.13.2	T2.46.2	T3.35.1	T7.13.1	T1.36.1	T3.15.1
76	T5.45.1	T7.24.3	T3.15.1	T7.12.2	T3.14.1	T7.56.1	T2.14.2	T1.23.3
77	T4.46.1	T3.46.3	T3.16.2	T4.16.2	T3.46.1	T3.26.1	T5.12.2	T2.25.1
78	T6.26.2	T4.34.2	T6.13.1	T2.34.3	T1.35.1	T3.45.1	T7.56.1	T6.13.1
79	T5.46.3	T3.25.1	T1.26.1	T1.16.1	T1.15.3	T7.56.2	T2.26.1	T2.35.1
80	T6.25.2	T4.36.3	T3.36.2	T1.34.1	T5.45.1	T4.45.1	T3.16.1	T1.14.3
81	T6.26.3	T6.23.2	T1.15.1	T7.14.2	T5.13.1	T3.36.1	T7.56.2	T3.13.3
82	T4.46.2	T3.26.2	T3.46.2	T4.35.1	T2.35.1	T7.56.3	T2.36.1	T3.25.1
83	T6.12.1	T5.34.2	T5.16.3	T1.25.1	T1.16.3	T2.45.1	T2.23.2	T4.13.3
84	T6.14.1	T5.26.3	T7.13.2	T5.35.1	T2.15.3	T1.45.1	T7.56.3	T2.14.3
85	T6.23.1	T2.15.1	T7.15.3	T2.16.1	T7.23.2	T4.16.1	T7.13.1	T4.46.1
86	T5.45.2	T4.46.3	T3.34.3	T4.24.3	T7.46.1	T4.45.2	T3.23.2	T3.46.1
87	T6.34.2	T6.23.1	T4.25.1	T4.26.2	T5.45.2	T7.23.1	T4.23.2	T1.24.2
88	T6.13.1	T2.26.1	T1.15.2	T2.25.1	T6.12.3	T4.26.1	T3.26.1	T5.12.3
89	T6.15.1	T3.15.1	T4.36.2	T4.46.1	T2.16.3	T7.14.1	T3.45.1	T3.35.1
90	T4.46.3	T4.15.1	T4.46.2	T5.24.3	T7.46.2	T4.36.1	T1.24.3	T4.15.1
91	T6.34.3	T7.25.3	T1.36.2	T7.12.1	T1.15.1	T2.23.1	T4.45.1	T6.23.1
92	T6.12.2	T2.15.2	T1.35.2	T5.16.2	T5.16.1	T1.45.2	T3.36.1	T1.15.2
93	T5.45.3	T7.23.2	T3.25.1	T4.14.1	T7.46.3	T2.45.2	T1.16.2	T4.46.2
94	T4.45.1	T5.36.3	T5.25.1	T1.25.2	T5.45.3	T4.45.3	T2.45.1	T6.14.1
95	T6.24.2	T5.46.3	T4.34.3	T5.14.1	T1.35.2	T3.23.1	T1.45.1	T2.46.1
96	T6.25.3	T7.34.3	T1.15.3	T7.15.2	T7.12.3	T3.45.2	T4.16.1	T1.46.1
97	T6.23.2	T5.15.1	T3.16.1	T7.36.1	T7.15.1	T4.23.1	T4.45.2	T4.25.1
98	T6.15.2	T3.26.1	T5.34.3	T2.25.2	T5.36.1	T1.45.3	T3.14.3	T4.46.3
99	T6.13.2	T3.34.1	T5.46.2	T5.46.1	T2.15.1	T7.24.1	T7.23.1	T2.15.2
100	T6.12.3	T2.15.3	T5.36.2	T7.24.2	T5.26.1	T3.45.3	T2.16.2	T3.14.2
101	T6.16.1	T2.36.2	T5.25.2	T5.26.2	T2.35.2	T2.45.3	T1.24.2	T4.35.1
102	T2.56.1	T3.15.2	T3.25.2	T5.35.2	T2.24.1	T7.34.1	T4.26.1	T5.13.2
103	T6.14.2	T2.35.2	T7.13.1	T1.25.3	T1.35.3	T5.56.1	T4.36.1	T6.12.2
104	T4.45.2	T4.34.1	T1.35.1	T5.35.3	T6.12.1	T5.56.2	T7.14.1	T2.46.2

105	T6.15.3	T5.15.2	T7.34.2	T7.36.2	T3.35.2	T5.56.3	T5.13.3	T6.24.1
106	T6.16.2	T7.35.3	T4.25.2	T4.35.2	T3.24.1	T1.15.3	T2.45.2	T1.46.2
107	T5.35.1	T4.15.2	T7.26.1	T7.36.3	T1.26.3	T1.15.2	T1.45.2	T3.46.2
108	T6.24.3	T5.34.1	T4.15.2	T2.34.1	T7.25.1	T2.15.3	T4.45.3	T3.46.3
109	T6.16.3	T7.23.1	T7.26.2	T7.25.2	T1.25.3	T1.16.3	T6.56.1	T1.46.3
110	T5.36.1	T3.15.3	T5.25.3	T1.36.3	T3.35.3	T6.12.3	T7.12.2	T2.23.3
111	T2.56.2	T5.15.3	T6.14.2	T4.35.3	T2.35.3	T2.15.2	T3.45.2	T1.25.2
112	T6.14.3	T2.35.1	T7.35.2	T2.25.3	T5.23.1	T2.16.3	T1.26.2	T6.34.1
113	T6.13.3	T4.15.3	T4.16.2	T1.35.3	T7.35.1	T6.12.2	T6.56.2	T2.46.3
114	T4.45.3	T7.16.1	T1.56.3	T7.45.1	T1.16.1	T1.16.2	T6.56.3	T2.24.2
115	T2.56.3	T4.25.2	T7.26.3	T4.24.1	T3.15.3	T7.12.3	T3.14.2	T3.23.3
116	T6.23.3	T7.45.3	T3.25.3	T5.24.1	T3.16.3	T2.16.2	T1.45.3	T4.23.3
117	T3.46.1	T4.26.2	T4.25.3	T1.56.2	T2.16.1	T1.26.3	T7.24.1	T1.15.3
118	T5.36.2	T7.16.2	T7.45.2	T3.56.1	T4.56.1	T1.25.3	T5.13.2	T1.24.3
119	T5.35.2	T7.16.3	T3.34.1	T1.35.1	T6.13.3	T7.12.2	T3.45.3	T3.24.2
120	T5.36.3	T6.24.2	T1.36.1	T2.36.3	T1.25.1	T1.25.2	T1.16.3	T2.15.3
121	T5.34.1	T2.56.3	T7.14.2	T4.16.3	T7.13.3	T3.15.3	T2.45.3	T5.23.2
122	T5.35.3	T4.25.1	T4.15.1	T4.15.3	T4.56.2	T3.16.3	T2.24.3	T3.15.2
123	T1.56.1	T7.24.2	T4.34.1	T6.14.3	T1.56.1	T6.13.3	T7.34.1	T6.12.3
124	T3.46.2	T2.36.1	T6.14.1	T2.56.2	T7.12.1	T7.13.3	T2.16.3	T6.13.2
125	T5.25.1	T3.36.2	T5.34.1	T7.14.3	T2.26.3	T3.15.2	T3.24.3	T2.25.2
126	T3.45.1	T6.24.1	T1.14.1	T3.56.2	T4.56.3	T1.26.2	T1.46.1	T1.45.1
127	T1.56.2	T3.35.2	T2.56.1	T1.24.1	T2.25.3	T1.56.1	T3.16.2	T3.14.3
128	T3.46.3	T4.36.2	T1.46.2	T3.56.3	T3.15.1	T2.26.3	T5.23.3	T6.46.1
129	T1.56.3	T3.56.3	T1.45.2	T2.35.3	T2.56.1	T6.13.2	T7.12.3	T5.13.3
130	T5.24.1	T2.14.1	T3.36.3	T4.34.1	T3.26.3	T2.25.3	T2.26.2	T7.56.1
131	T5.12.1	T4.35.2	T4.16.1	T2.24.1	T1.34.1	T2.56.1	T7.13.2	T6.46.2
132	T5.26.1	T4.26.1	T3.56.2	T1.36.1	T6.13.1	T3.26.3	T2.24.2	T1.34.2
133	T5.25.2	T1.56.1	T2.56.2	T4.56.1	T3.34.1	T3.25.3	T1.34.3	T3.25.2
134	T5.14.1	T3.14.1	T1.45.1	T6.35.1	T3.25.3	T1.34.1	T2.46.1	T7.56.2
135	T5.13.1	T4.56.3	T2.56.3	T4.15.1	T3.56.1	T3.34.1	T3.26.2	T6.46.3
136	T5.23.1	T2.46.2	T7.14.1	T4.26.3	T2.34.1	T3.56.1	T3.24.2	T7.56.3
137	T5.26.2	T4.14.1	T4.24.1	T6.14.1	T1.26.1	T2.34.1	T1.34.2	T2.45.1
138	T3.45.2	T1.56.2	T4.56.2	T4.25.3	T6.45.1	T3.16.2	T7.46.1	T5.46.1
139	T5.15.1	T3.35.1	T3.24.1	T1.24.2	T7.23.3	T1.36.3	T3.35.1	T6.23.2
140	T4.35.1	T2.45.2	T1.14.2	T1.46.3	T6.23.3	T6.45.1	T3.46.1	T3.36.1

141	T5.26.3	T1.56.3	T4.36.3	T7.24.3	T6.45.2	T2.25.2	T5.45.1	T1.34.3
142	T5.25.3	T6.34.2	T1.46.1	T2.35.1	T2.25.1	T1.35.3	T5.23.2	T3.45.1
143	T2.46.1	T6.26.3	T6.16.3	T6.24.3	T7.45.1	T6.23.3	T7.46.2	T5.46.2
144	T5.12.2	T7.34.2	T3.35.3	T7.35.1	T3.16.1	T7.23.3	T1.35.1	T1.25.3
145	T5.34.2	T2.14.2	T1.56.2	T6.16.2	T6.45.3	T7.13.2	T2.35.1	T2.24.3
146	T4.36.1	T2.45.1	T7.16.3	T7.16.2	T3.25.1	T1.35.2	T1.26.3	T2.36.1
147	T3.45.3	T7.24.1	T1.14.3	T1.45.3	T7.45.2	T3.25.2	T7.46.3	T1.36.1
148	T5.34.3	T4.35.1	T6.25.1	T1.24.3	T1.34.2	T7.45.1	T1.36.2	T4.14.2
149	T5.12.3	T2.14.3	T4.35.3	T6.35.2	T7.45.3	T1.36.2	T7.23.2	T5.46.3
150	T5.15.2	T7.26.3	T6.25.2	T2.24.2	T7.13.1	T6.45.2	T4.14.3	T1.35.2
151	T3.36.1	T2.56.2	T5.15.2	T1.56.3	T6.16.1	T2.26.2	T5.45.2	T5.14.2
152	T2.46.2	T2.46.1	T7.34.3	T6.46.1	T1.34.3	T7.45.2	T5.14.3	T3.24.3
153	T4.34.1	T6.15.1	T3.24.2	T6.26.2	T7.16.1	T1.56.2	T4.14.2	T2.34.2
154	T5.16.1	T6.36.3	T6.34.3	T6.35.3	T6.23.1	T6.23.2	T5.14.2	T5.23.3
155	T5.13.2	T3.14.2	T6.15.2	T4.16.1	T2.34.2	T6.45.3	T6.46.1	T5.15.1
156	T5.24.2	T3.36.1	T6.46.2	T1.45.1	T6.36.1	T7.45.3	T1.46.2	T6.15.1
157	T5.23.2	T6.46.3	T3.35.1	T4.25.1	T6.26.1	T4.15.3	T5.45.3	T3.15.3
158	T3.35.1	T6.34.1	T6.36.2	T7.35.2	T3.34.2	T2.36.3	T6.46.2	T1.45.2
159	T5.15.3	T6.15.2	T7.25.1	T2.24.3	T2.26.1	T3.26.2	T2.34.3	T4.14.3
160	T2.46.3	T5.25.2	T1.56.1	T7.26.2	T2.34.3	T1.34.2	T5.16.1	T5.14.3
161	T5.14.2	T3.14.3	T5.16.2	T4.34.2	T7.26.1	T5.46.1	T3.16.3	T5.25.1
162	T5.16.2	T7.36.3	T5.15.1	T7.14.1	T3.34.3	T6.14.3	T6.46.3	T6.13.3
163	T2.45.1	T4.14.2	T6.15.1	T7.35.3	T4.46.1	T4.16.3	T1.35.2	T1.36.2
164	T5.24.3	T5.26.2	T4.24.2	T2.36.1	T7.36.1	T4.15.2	T5.36.1	T5.35.1
165	T5.13.3	T4.36.1	T7.15.2	T4.34.3	T3.26.1	T6.16.1	T1.36.3	T2.36.2
166	T5.14.3	T6.25.2	T1.13.1	T1.46.1	T4.46.2	T1.34.3	T7.15.1	T1.35.3
167	T5.16.3	T3.46.2	T4.35.1	T6.24.1	T4.46.3	T2.56.2	T5.26.1	T1.36.3
168	T5.23.3	T7.15.1	T7.25.2	T2.46.3	T1.46.3	T7.16.1	T2.35.2	T6.25.1
169	T4.36.2	T5.25.1	T6.25.3	T7.46.1	T7.23.1	T2.35.3	T2.36.2	T4.24.2
170	T4.36.3	T4.14.3	T3.24.3	T3.46.1	T4.45.1	T6.14.2	T7.13.3	T3.36.2
171	T4.35.2	T6.15.3	T7.36.2	T2.56.3	T1.45.3	T2.34.2	T1.35.3	T2.35.2
172	T2.45.2	T2.56.1	T4.24.3	T1.23.1	T1.56.3	T6.36.1	T2.34.2	T3.36.3
173	T4.25.1	T7.46.3	T7.46.2	T5.15.3	T1.45.2	T6.26.1	T3.35.2	T5.24.2
174	T1.46.1	T3.56.2	T5.16.1	T2.45.3	T1.46.2	T7.14.3	T2.46.2	T2.36.3
175	T3.36.2	T6.25.1	T7.25.3	T5.16.3	T4.45.2	T5.46.2	T4.24.3	T6.35.1
176	T3.34.1	T2.13.1	T3.36.1	T6.15.3	T4.45.3	T1.24.1	T7.25.1	T7.46.1

177	T4.24.1	T7.15.2	T1.13.2	T5.34.1	T2.46.3	T3.34.2	T3.35.3	T6.36.1
178	T4.12.1	T7.25.2	T6.34.1	T4.26.1	T1.56.2	T1.56.3	T5.24.3	T4.15.2
179	T3.36.3	T3.45.2	T7.15.1	T3.46.2	T2.56.3	T2.34.3	T2.35.3	T2.45.2
180	T4.35.3	T5.26.1	T3.46.3	T7.15.3	T1.23.1	T5.46.3	T4.16.2	T6.14.2
181	T1.46.2	T7.15.3	T3.56.3	T6.34.1	T2.45.3	T4.16.2	T2.26.3	T2.34.3
182	T4.26.1	T2.13.2	T6.16.2	T2.23.1	T5.15.3	T7.23.2	T1.46.3	T7.46.2
183	T3.35.2	T6.26.2	T2.46.1	T1.23.2	T5.16.3	T4.34.1	T7.35.1	T2.25.3
184	T1.46.3	T3.45.1	T7.16.2	T3.46.3	T6.15.3	T7.26.1	T4.35.1	T7.46.3
185	T4.12.2	T7.34.1	T1.13.3	T1.56.1	T5.34.1	T2.35.2	T5.35.1	T5.36.1
186	T2.45.3	T5.36.2	T4.36.1	T5.15.1	T5.15.2	T3.34.3	T7.14.2	T1.45.3
187	T4.14.1	T3.13.1	T2.46.2	T3.45.1	T6.34.1	T5.45.1	T7.36.1	T6.36.2
188	T4.13.1	T7.25.1	T5.24.1	T6.15.1	T6.15.2	T7.36.1	T4.24.2	T4.36.1
189	T4.34.2	T1.46.1	T3.23.1	T5.56.1	T7.15.3	T6.35.1	T5.24.2	T6.36.3
190	T3.35.3	T3.46.1	T3.45.3	T2.45.1	T2.45.2	T2.24.1	T1.25.1	T4.25.2
191	T2.36.1	T7.26.2	T1.12.1	T6.16.3	T2.46.2	T2.36.2	T3.26.3	T3.25.3
192	T1.45.1	T2.13.3	T5.56.2	T7.24.1	T2.23.1	T7.14.2	T4.26.2	T4.24.3
193	T4.15.1	T5.35.2	T6.24.1	T5.26.3	T1.23.2	T4.56.1	T4.46.1	T4.15.3
194	T4.23.1	T5.14.1	T2.46.3	T7.16.3	T5.16.2	T4.26.3	T2.25.1	T6.24.2
195	T4.34.3	T5.56.3	T2.45.1	T2.23.2	T5.56.1	T4.25.3	T7.36.2	T5.24.3
196	T4.25.2	T1.46.2	T5.36.3	T1.23.3	T7.15.2	T5.45.2	T4.16.3	T6.14.3
197	T4.12.3	T3.13.2	T6.16.1	T2.46.1	T6.16.3	T7.35.1	T6.36.1	T1.26.1
198	T2.35.1	T6.35.2	T1.12.2	T7.34.1	T1.13.1	T6.24.3	T7.36.3	T5.15.2
199	T4.26.2	T2.12.1	T3.23.2	T5.16.1	T5.26.3	T4.25.2	T5.16.2	T6.23.3
200	T4.13.3	T1.46.3	T7.16.1	T5.25.3	T2.56.2	T7.24.3	T2.36.3	T4.45.1
201	T1.45.2	T6.14.1	T3.45.1	T6.56.1	T7.34.1	T5.45.3	T1.25.2	T3.34.2
202	T4.26.3	T6.36.2	T7.34.1	T7.25.3	T7.16.3	T1.24.2	T7.14.3	T2.26.1
203	T4.23.2	T5.35.1	T6.56.2	T6.25.3	T2.23.2	T6.24.2	T7.24.2	T6.15.2
204	T3.12.1	T7.35.2	T1.12.3	T6.26.3	T1.23.3	T6.35.2	T7.23.3	T5.36.2
205	T4.24.2	T6.26.1	T2.45.2	T7.15.1	T5.25.3	T1.14.1	T7.15.2	T2.35.3
206	T4.25.3	T6.56.3	T5.35.3	T3.45.2	T6.56.1	T5.36.1	T5.35.2	T5.36.3
207	T4.16.1	T1.45.1	T6.36.3	T2.23.3	T6.25.3	T4.26.2	T3.34.3	T7.36.1
208	T2.36.2	T3.56.1	T3.46.1	T6.34.2	T3.46.3	T6.16.2	T2.25.2	T4.34.2
209	T1.45.3	T6.35.1	T7.24.1	T7.26.3	T7.25.3	T3.36.3	T5.46.1	T5.25.2
210	T3.34.2	T2.12.2	T6.24.2	T3.45.3	T3.56.3	T7.16.2	T6.45.1	T7.45.1
211	T4.15.2	T5.36.1	T7.35.3	T3.36.1	T4.36.1	T1.24.3	T5.26.2	T1.26.2
212	T4.24.3	T3.13.3	T6.35.3	T5.25.1	T4.35.1	T2.24.2	T6.36.2	T3.34.3

213	T4.13.3	T7.36.2	T3.23.3	T5.34.2	T1.13.2	T7.35.2	T3.34.2	T6.25.2
214	T2.36.3	T7.26.1	T5.24.2	T6.34.3	T6.26.3	T5.35.1	T6.36.3	T5.45.1
215	T2.34.1	T2.12.3	T7.36.3	T2.56.1	T5.25.2	T3.56.2	T5.35.3	T2.26.2
216	T3.12.2	T4.46.2	T5.35.1	T6.25.1	T6.16.2	T6.35.3	T3.36.2	T5.34.2
217	T3.34.3	T1.45.2	T2.45.3	T7.56.1	T6.34.2	T2.56.3	T4.34.3	T3.35.2
218	T4.14.2	T7.14.1	T7.56.2	T5.34.3	T6.25.2	T6.46.1	T2.46.3	T1.26.3
219	T2.35.2	T4.56.2	T5.24.3	T3.35.1	T3.46.2	T7.35.3	T3.46.2	T4.36.2
220	T4.15.3	T6.14.2	T6.24.3	T7.34.2	T2.23.3	T3.35.3	T1.25.3	T7.45.2
221	T3.12.3	T5.14.2	T2.36.1	T1.12.1	T7.26.3	T6.26.2	T4.35.2	T6.26.1
222	T1.36.1	T7.56.3	T4.56.3	T7.34.3	T7.16.2	T3.36.2	T6.45.2	T4.36.3
223	T4.14.3	T5.14.3	T7.24.2	T5.26.1	T5.26.2	T4.24.1	T4.34.2	T7.36.2
224	T4.23.3	T1.45.3	T6.35.1	T6.16.1	T5.24.1	T3.35.2	T5.34.3	T4.34.3
225	T3.25.1	T4.45.2	T3.56.1	T1.12.2	T3.45.3	T4.56.2	T7.25.2	T2.45.3
226	T3.24.1	T4.46.1	T4.46.3	T7.16.1	T1.13.3	T7.24.2	T1.15.1	T7.36.3
227	T4.16.2	T6.14.3	T5.36.1	T3.36.2	T6.34.3	T3.24.1	T5.34.2	T3.45.2
228	T1.35.1	T7.35.1	T2.35.1	T1.12.3	T5.34.2	T4.36.3	T4.35.3	T2.26.3
229	T4.16.3	T5.46.2	T7.24.3	T7.25.1	T3.56.2	T5.36.2	T4.25.1	T7.45.3
230	T2.35.3	T4.45.1	T5.23.1	T3.36.3	T4.34.1	T7.26.2	T5.16.3	T6.45.1
231	T3.14.1	T5.56.2	T5.56.3	T3.34.1	T3.23.1	T2.24.3	T4.26.3	T5.15.3
232	T3.23.1	T4.13.1	T4.23.1	T3.35.2	T3.45.2	T4.34.2	T4.36.2	T4.25.3
233	T3.13.1	T1.36.1	T5.46.3	T2.12.1	T6.24.1	T4.35.3	T6.45.3	T5.34.3
234	T2.25.1	T7.14.2	T6.23.1	T4.56.3	T7.34.2	T6.25.1	T2.25.3	T6.26.2
235	T3.26.1	T5.13.1	T4.45.3	T4.46.3	T5.34.3	T5.36.3	T5.25.1	T6.15.3
236	T3.25.2	T4.56.1	T2.36.2	T3.35.3	T7.56.1	T4.34.3	T4.46.2	T4.35.2
237	T1.36.2	T5.45.2	T4.46.1	T6.26.1	T5.56.2	T7.46.1	T3.36.3	T5.26.1
238	T2.26.1	T7.14.3	T2.36.3	T5.23.1	T7.34.3	T4.35.2	T6.35.1	T6.26.3
239	T1.36.3	T3.12.1	T7.35.1	T2.12.2	T2.13.1	T1.14.2	T7.26.1	T4.26.1
240	T3.15.1	T5.46.1	T4.23.2	T5.56.3	T7.25.2	T4.36.2	T7.15.3	T3.35.3
241	T1.34.1	T6.36.1	T6.56.3	T4.23.1	T5.36.3	T3.56.3	T3.25.1	T6.24.3
242	T2.24.1	T1.35.1	T2.34.1	T7.26.1	T4.36.2	T6.16.3	T7.45.1	T7.35.1
243	T2.12.1	T5.45.1	T4.45.1	T4.46.2	T4.36.3	T5.35.2	T3.46.3	T1.16.1
244	T3.26.2	T6.46.2	T6.46.3	T2.12.3	T3.23.2	T7.16.3	T1.15.2	T4.45.2
245	T2.34.2	T4.13.2	T5.45.3	T4.56.2	T5.35.3	T1.14.3	T7.26.2	T7.26.1
246	T2.25.2	T6.56.2	T2.35.2	T5.46.3	T4.25.1	T2.14.1	T6.26.1	T3.26.1
247	T3.13.2	T6.45.2	T6.23.2	T4.45.3	T7.24.1	T5.34.1	T7.24.3	T6.34.2
248	T3.23.2	T6.13.1	T4.56.1	T6.23.1	T5.36.2	T6.34.3	T5.25.2	T5.26.2

249	T1.35.2	T3.12.2	T5.23.2	T3.12.1	T2.13.2	T6.25.2	T7.26.3	T3.45.3
250	T3.24.2	T1.36.2	T3.12.1	T4.45.2	T6.56.2	T7.25.1	T6.34.1	T4.35.3
251	T2.26.2	T7.36.1	T5.46.1	T5.46.2	T3.13.1	T5.35.3	T4.36.3	T1.16.2
252	T3.26.3	T6.45.1	T5.45.1	T4.23.2	T6.26.2	T6.34.2	T1.15.3	T7.26.2
253	T3.15.2	T7.45.2	T7.23.1	T6.56.3	T4.35.2	T7.34.3	T5.46.2	T5.35.2
254	T2.12.2	T1.36.3	T6.36.1	T3.34.2	T5.35.2	T4.56.3	T5.36.2	T1.16.3
255	T1.35.3	T3.12.3	T6.45.3	T6.46.3	T4.24.1	T5.25.1	T6.26.2	T7.34.1
256	T2.34.3	T1.34.1	T7.45.3	T5.45.3	T5.14.1	T3.14.1	T3.25.2	T5.26.3
257	T3.25.3	T5.56.1	T4.23.3	T5.56.2	T6.36.3	T4.14.1	T7.34.2	T5.45.2
258	T3.14.2	T4.12.1	T7.46.3	T4.13.1	T6.35.3	T5.26.1	T6.26.3	T7.26.3
259	T2.26.3	T5.13.2	T2.35.3	T6.23.2	T4.26.1	T7.25.2	T4.46.3	T6.16.1
260	T2.23.1	T4.13.3	T7.56.3	T3.34.3	T6.24.2	T3.24.2	T4.25.2	T7.35.2
261	T2.13.1	T7.46.2	T2.25.1	T3.12.2	T5.56.3	T6.46.2	T6.35.2	T3.26.2
262	T2.12.3	T6.13.2	T5.23.3	T5.45.2	T4.35.3	T7.34.2	T2.15.1	T4.26.2
263	T3.16.1	T6.46.1	T3.12.2	T5.13.1	T7.35.3	T6.36.2	T5.26.3	T6.35.2
264	T3.24.3	T1.35.2	T6.45.1	T3.25.1	T4.12.1	T5.24.1	T4.15.1	T5.25.3
265	T3.13.3	T7.45.1	T7.23.2	T3.24.1	T7.26.2	T6.25.3	T6.25.1	T6.34.3
266	T1.25.1	T5.12.1	T2.26.1	T5.23.2	T6.35.2	T5.26.2	T5.25.3	T7.35.3
267	T2.14.1	T7.56.2	T7.36.1	T7.23.1	T2.13.3	T5.25.2	T3.15.1	T6.16.2
268	T3.23.3	T5.13.3	T6.23.3	T3.12.3	T6.14.1	T7.25.3	T5.15.1	T7.25.1
269	T3.14.3	T1.35.3	T3.12.3	T6.45.3	T5.24.2	T4.24.2	T6.24.1	T4.45.3
270	T1.24.1	T4.12.2	T2.24.1	T7.45.3	T3.23.3	T5.12.1	T6.35.3	T4.26.3
271	T2.24.2	T7.13.1	T5.56.1	T6.45.2	T7.36.3	T6.15.1	T7.35.2	T6.25.3
272	T1.26.1	T6.56.1	T4.12.1	T6.46.2	T3.13.2	T6.26.3	T7.16.1	T6.16.3
273	T1.12.1	T6.13.3	T2.12.1	T4.23.3	T4.14.1	T5.14.1	T7.34.3	T6.45.2
274	T2.25.3	T1.25.1	T7.23.3	T3.26.1	T7.35.2	T2.14.2	T5.36.3	T7.16.1
275	T3.15.3	T7.46.1	T2.25.2	T7.46.3	T6.36.2	T5.23.1	T4.25.3	T3.26.3
276	T3.16.2	T7.13.2	T2.26.2	T3.14.1	T7.24.2	T3.24.3	T7.25.3	T5.16.1
277	T2.13.3	T4.12.3	T2.34.2	T6.13.1	T4.15.1	T7.36.2	T6.16.1	T4.16.1
278	T2.23.2	T1.26.1	T6.46.1	T4.13.2	T6.24.3	T5.13.1	T3.25.3	T2.16.1
279	T1.34.2	T1.24.1	T7.45.1	T6.56.2	T4.13.1	T5.26.3	T2.15.2	T7.16.2
280	T1.25.2	T1.12.1	T5.12.1	T3.23.1	T7.56.2	T4.24.3	T5.46.3	T7.24.1
281	T3.16.3	T6.12.1	T2.12.2	T7.56.3	T5.24.3	T7.46.2	T6.25.2	T5.35.3
282	T2.15.1	T7.13.3	T2.26.3	T3.13.1	T4.12.2	T5.15.1	T7.45.2	T3.16.1
283	T1.26.2	T1.25.2	T4.12.2	T3.25.2	T4.23.1	T2.14.3	T7.16.2	T7.25.2
284	T1.12.2	T5.12.2	T2.34.3	T7.23.2	T6.56.3	T7.26.3	T6.14.1	T7.16.3

285	T1.34.3	T1.34.2	T6.56.1	T5.23.3	T4.25.2	T6.15.2	T7.16.3	T5.16.2
286	T1.14.1	T1.26.2	T2.23.1	T7.45.2	T7.24.3	T7.15.1	T5.15.2	T5.45.3
287	T1.23.1	T7.56.1	T2.13.1	T6.23.3	T4.34.2	T6.36.3	T6.16.2	T6.35.3
288	T1.13.1	T1.14.1	T2.12.3	T3.15.1	T4.26.2	T5.25.3	T6.15.1	T7.15.1
289	T2.24.3	T1.23.1	T2.14.1	T3.26.2	T4.34.3	T3.14.2	T3.15.2	T2.16.2
290	T2.13.3	T1.13.1	T7.46.1	T7.23.3	T7.36.2	T6.46.3	T6.23.1	T5.16.3
291	T1.26.3	T1.34.3	T4.12.3	T6.13.2	T3.13.3	T5.16.1	T6.16.3	T7.25.3
292	T1.12.3	T1.12.2	T6.12.1	T5.13.2	T7.14.1	T5.15.2	T2.15.3	T7.14.1
293	T1.15.1	T5.12.3	T2.25.3	T7.46.2	T4.26.3	T6.15.3	T6.12.1	T2.16.3
294	T2.14.2	T6.12.2	T2.24.2	T4.13.3	T4.12.3	T7.15.2	T6.13.1	T3.16.2
295	T2.15.2	T1.26.3	T2.13.2	T3.13.2	T4.16.1	T3.14.3	T6.34.2	T7.34.2
296	T1.25.3	T1.15.1	T2.23.2	T3.23.2	T4.13.2	T7.36.3	T5.15.3	T7.23.1
297	T1.24.2	T1.12.3	T2.15.1	T3.24.2	T6.14.2	T5.16.2	T4.15.2	T7.34.3
298	T2.23.3	T6.12.3	T5.12.2	T3.26.3	T4.25.3	T4.14.2	T7.35.3	T4.16.2
299	T2.14.3	T1.25.3	T7.56.1	T3.15.2	T4.23.2	T5.12.2	T6.25.3	T6.45.3
300	T1.13.2	T7.12.1	T6.12.2	T7.13.1	T5.14.2	T5.34.2	T6.15.2	T7.15.2
301	T1.23.2	T1.24.2	T5.12.3	T7.56.2	T4.24.2	T7.15.3	T6.34.3	T3.16.3
302	T2.16.1	T1.13.2	T2.24.3	T3.16.1	T6.14.3	T5.15.3	T3.15.3	T4.16.3
303	T2.15.3	T1.23.2	T2.13.3	T5.13.3	T4.15.2	T4.14.3	T4.15.3	T7.13.1
304	T1.15.2	T1.15.2	T2.15.2	T3.25.3	T5.14.3	T5.16.3	T6.24.2	T7.12.1
305	T1.16.1	T1.16.1	T2.14.2	T3.14.2	T7.56.3	T5.34.3	T6.23.2	T7.15.3
306	T1.24.3	T7.12.2	T6.12.3	T6.13.3	T7.14.2	T5.12.3	T6.12.2	T7.24.2
307	T1.14.2	T1.24.3	T7.12.1	T7.13.2	T4.24.3	T7.46.3	T6.13.2	T7.23.2
308	T1.13.3	T1.14.2	T2.23.3	T3.24.3	T4.13.3	T5.13.2	T7.45.3	T7.24.3
309	T2.16.2	T7.12.3	T2.14.3	T3.13.3	T4.15.3	T5.23.2	T6.15.3	T7.12.2
310	T1.15.3	T1.13.3	T2.16.1	T3.16.2	T7.14.3	T5.24.2	T6.12.3	T7.13.2
311	T2.16.3	T1.15.3	T2.15.3	T3.23.3	T4.16.2	T5.14.2	T6.14.2	T7.14.2
312	T1.16.2	T1.16.2	T7.12.2	T3.15.3	T4.14.2	T5.24.3	T6.24.3	T7.12.3
313	T1.14.3	T1.14.3	T2.16.2	T3.14.3	T4.16.3	T5.13.3	T6.14.3	T7.14.3
314	T1.23.3	T1.23.3	T7.12.3	T7.13.3	T4.14.3	T5.14.3	T6.13.3	T7.23.3
315	T1.16.3	T1.16.3	T2.16.3	T3.16.3	T4.23.3	T5.23.3	T6.23.3	T7.13.3

<附錄四>選手賽程表現率(依賽程為序)

賽程	A	B	C	D	E	F	G	H
T1.12.1	58.64%	53.45%	55.46%	47.98%	83.33%	76.47%	100.00%	100.00%
T1.12.2	57.92%	52.71%	54.60%	47.14%	85.94%	100.00%	64.47%	82.80%
T1.12.3	57.55%	52.35%	54.18%	46.74%	100.00%	81.18%	66.32%	76.77%
T1.13.1	57.88%	52.72%	57.98%	85.59%	41.34%	67.94%	92.49%	91.51%
T1.13.2	57.09%	51.92%	56.98%	87.84%	40.39%	91.83%	56.30%	74.31%
T1.13.3	56.42%	51.26%	56.19%	100.00%	39.68%	73.16%	58.98%	65.50%
T1.14.1	57.91%	52.77%	62.72%	76.32%	65.94%	38.43%	87.16%	85.08%
T1.14.2	56.49%	51.33%	60.80%	79.53%	84.78%	36.59%	48.69%	60.28%
T1.14.3	56.17%	51.02%	60.39%	92.39%	70.06%	36.25%	49.59%	57.30%
T1.15.1	57.52%	52.39%	69.16%	68.09%	57.89%	83.33%	32.72%	63.67%
T1.15.2	56.79%	51.67%	68.10%	69.14%	81.58%	51.49%	31.81%	55.55%
T1.15.3	56.29%	51.17%	67.38%	88.29%	59.65%	52.23%	31.27%	51.79%
T1.16.1	56.70%	51.62%	79.46%	65.43%	55.15%	65.30%	50.76%	37.12%
T1.16.2	56.21%	51.13%	78.62%	67.78%	66.79%	49.71%	45.29%	36.25%
T1.16.3	56.01%	50.93%	78.29%	74.52%	59.09%	51.38%	42.76%	35.94%
T1.23.1	57.90%	52.74%	87.30%	52.38%	43.14%	62.16%	87.31%	85.88%
T1.23.2	57.05%	51.88%	89.29%	51.27%	42.07%	86.07%	50.93%	68.70%
T1.23.3	56.12%	50.97%	100.00%	50.18%	41.07%	67.63%	54.05%	58.40%
T1.24.1	58.69%	53.54%	78.98%	57.60%	61.10%	40.71%	83.60%	81.28%
T1.24.2	57.15%	51.98%	81.85%	55.46%	80.19%	38.61%	44.26%	56.06%
T1.24.3	56.55%	51.38%	93.34%	54.70%	65.46%	37.96%	45.67%	51.38%
T1.25.1	59.00%	53.85%	71.54%	64.63%	53.82%	80.53%	35.21%	61.19%
T1.25.2	58.21%	53.05%	72.49%	63.42%	78.29%	47.49%	34.15%	52.75%
T1.25.3	57.25%	52.10%	89.93%	62.07%	55.92%	48.67%	33.10%	46.72%
T1.26.1	58.66%	53.56%	69.33%	75.88%	51.44%	63.34%	49.14%	40.53%
T1.26.2	58.06%	52.95%	71.53%	74.81%	64.30%	46.02%	43.36%	39.41%
T1.26.3	57.67%	52.57%	77.82%	74.16%	56.26%	48.73%	39.23%	38.81%
T1.34.1	60.35%	55.17%	73.12%	65.42%	52.10%	44.65%	81.24%	78.59%
T1.34.2	58.21%	53.01%	76.44%	82.64%	49.10%	41.60%	39.96%	48.70%
T1.34.3	57.91%	52.72%	88.32%	69.36%	48.75%	41.28%	40.48%	46.97%
T1.35.1	61.48%	56.30%	66.39%	58.71%	59.68%	78.87%	39.42%	59.60%
T1.35.2	59.94%	54.75%	67.87%	81.29%	57.36%	43.53%	37.28%	46.17%
T1.35.3	59.46%	54.28%	86.31%	60.64%	56.71%	43.98%	36.75%	43.86%

T1.36.1	61.83%	56.69%	64.48%	56.71%	72.17%	62.26%	48.23%	46.39%
T1.36.2	60.67%	55.53%	67.99%	68.97%	70.09%	43.13%	38.82%	44.14%
T1.36.3	60.47%	55.33%	74.84%	61.31%	69.76%	44.19%	37.20%	43.84%
T1.45.1	64.07%	58.86%	61.31%	53.79%	75.32%	58.52%	44.93%	49.76%
T1.45.2	63.26%	58.05%	61.85%	79.87%	45.14%	57.25%	43.77%	44.71%
T1.45.3	62.64%	57.43%	84.26%	54.72%	45.52%	56.35%	43.00%	41.97%
T1.46.1	65.56%	60.42%	60.69%	53.09%	62.16%	74.18%	41.63%	54.81%
T1.46.2	64.97%	59.82%	62.02%	67.86%	45.09%	73.02%	37.98%	53.59%
T1.46.3	64.61%	59.47%	73.14%	55.39%	46.03%	72.37%	35.99%	52.96%
T1.56.1	70.74%	65.68%	58.61%	51.08%	53.45%	46.02%	75.23%	71.77%
T1.56.2	70.28%	65.22%	60.61%	59.40%	43.81%	42.46%	74.28%	70.69%
T1.56.3	70.12%	65.05%	64.92%	54.55%	45.23%	40.52%	73.95%	70.32%
T2.12.1	60.30%	59.47%	48.73%	45.64%	82.88%	76.01%	99.32%	99.23%
T2.12.2	59.55%	58.65%	47.98%	44.83%	85.47%	99.40%	64.04%	82.16%
T2.12.3	59.18%	58.25%	47.61%	44.45%	99.46%	80.69%	65.87%	76.18%
T2.13.1	59.20%	61.91%	47.75%	84.99%	39.14%	67.36%	91.60%	90.50%
T2.13.3	58.39%	60.97%	46.93%	87.22%	38.24%	91.04%	55.76%	73.49%
T2.13.3	57.71%	60.19%	46.27%	99.30%	37.57%	72.54%	58.41%	64.78%
T2.14.1	58.88%	66.48%	47.50%	75.61%	65.26%	36.19%	86.03%	83.81%
T2.14.2	57.43%	64.67%	46.05%	78.80%	83.91%	34.46%	48.06%	59.38%
T2.14.3	57.10%	64.27%	45.74%	91.53%	69.34%	34.13%	48.95%	56.45%
T2.15.1	58.10%	72.60%	46.79%	67.28%	57.13%	82.09%	30.62%	62.43%
T2.15.2	57.37%	71.59%	46.07%	68.31%	80.50%	50.73%	29.76%	54.46%
T2.15.3	56.86%	70.91%	45.59%	87.24%	58.86%	51.46%	29.26%	50.78%
T2.16.1	56.91%	82.08%	45.73%	64.41%	54.21%	64.05%	49.66%	34.44%
T2.16.2	56.41%	81.30%	45.25%	66.73%	65.64%	48.76%	44.31%	33.64%
T2.16.3	56.21%	80.98%	45.05%	73.36%	58.08%	50.40%	41.83%	33.36%
T2.23.1	59.22%	88.65%	47.80%	51.33%	42.24%	57.47%	82.64%	80.83%
T2.23.2	58.30%	90.43%	46.87%	50.19%	41.14%	81.05%	46.65%	63.91%
T2.23.3	57.14%	100.00%	45.76%	48.88%	39.94%	63.00%	49.98%	52.85%
T2.24.1	60.34%	81.02%	48.89%	56.80%	56.98%	40.13%	79.97%	77.38%
T2.24.2	58.68%	83.61%	47.21%	54.58%	75.97%	37.95%	40.60%	52.23%
T2.24.3	57.81%	93.98%	46.39%	53.55%	61.43%	37.06%	42.29%	46.58%
T2.25.1	60.87%	74.14%	49.40%	64.01%	50.23%	77.39%	34.90%	58.32%
T2.25.2	60.02%	75.01%	48.55%	62.76%	74.89%	44.00%	33.79%	49.79%
T2.25.3	58.63%	90.96%	47.23%	60.89%	52.51%	45.43%	32.37%	42.40%

T2.26.1	60.55%	72.12%	49.19%	75.20%	47.97%	60.83%	47.02%	40.28%
T2.26.2	59.85%	74.16%	48.50%	74.01%	61.47%	42.61%	41.11%	39.03%
T2.26.3	59.29%	80.01%	47.96%	73.10%	53.30%	45.95%	36.00%	38.23%
T2.34.1	62.34%	75.61%	50.77%	61.50%	51.67%	44.30%	78.48%	75.60%
T2.34.2	60.03%	78.63%	48.44%	78.77%	48.54%	41.11%	36.73%	45.49%
T2.34.3	59.45%	89.43%	47.90%	65.55%	47.90%	40.52%	37.57%	42.68%
T2.35.1	63.70%	69.36%	52.10%	55.24%	59.44%	76.54%	39.33%	57.42%
T2.35.2	62.04%	70.72%	50.44%	78.19%	57.01%	40.43%	37.07%	43.70%
T2.35.3	61.10%	87.72%	49.55%	57.37%	55.81%	41.16%	36.11%	39.92%
T2.36.1	64.08%	67.62%	52.58%	53.35%	71.92%	60.41%	46.65%	46.42%
T2.36.2	62.74%	70.90%	51.24%	66.33%	69.59%	40.06%	36.93%	43.90%
T2.36.3	62.35%	77.33%	50.88%	58.48%	68.99%	41.79%	34.25%	43.35%
T2.45.1	66.60%	64.61%	54.88%	50.73%	73.46%	58.58%	45.03%	48.09%
T2.45.2	65.72%	65.11%	54.01%	77.56%	42.34%	57.22%	43.80%	42.87%
T2.45.3	64.49%	85.97%	52.83%	51.88%	42.96%	55.57%	42.37%	38.32%
T2.46.1	68.13%	64.10%	56.54%	50.09%	60.66%	74.23%	40.42%	55.05%
T2.46.2	67.44%	65.36%	55.85%	65.92%	42.30%	72.93%	36.59%	53.68%
T2.46.3	66.74%	75.88%	55.17%	52.99%	43.87%	71.73%	33.25%	52.52%
T2.56.1	73.58%	62.24%	62.10%	48.35%	52.33%	44.97%	75.52%	72.23%
T2.56.2	73.00%	64.16%	61.52%	57.88%	41.25%	41.19%	74.36%	70.91%
T2.56.3	72.67%	68.31%	61.19%	52.77%	43.64%	37.90%	73.75%	70.24%
T3.12.1	63.03%	56.45%	50.05%	44.12%	82.72%	75.85%	99.10%	98.97%
T3.12.2	62.25%	55.67%	49.28%	43.34%	85.31%	99.19%	63.89%	81.95%
T3.12.3	61.86%	55.29%	48.90%	42.97%	99.27%	80.52%	65.72%	75.99%
T3.13.1	60.90%	60.62%	86.13%	42.40%	38.20%	61.04%	85.52%	83.85%
T3.13.2	60.01%	59.63%	88.09%	41.50%	37.26%	84.51%	49.89%	67.08%
T3.13.3	59.04%	58.58%	98.66%	40.62%	36.37%	66.40%	52.94%	57.02%
T3.14.1	60.94%	65.51%	77.59%	42.51%	59.78%	35.61%	81.31%	78.69%
T3.14.2	59.34%	63.61%	80.41%	40.93%	78.47%	33.77%	43.05%	54.27%
T3.14.3	58.71%	62.88%	91.70%	40.36%	64.05%	33.20%	44.41%	49.74%
T3.15.1	60.38%	71.86%	69.93%	42.05%	52.34%	78.04%	30.33%	58.65%
T3.15.2	59.57%	70.80%	70.86%	41.26%	76.14%	46.02%	29.41%	50.56%
T3.15.3	58.58%	69.53%	87.91%	40.38%	54.38%	47.17%	28.51%	44.78%
T3.16.1	59.17%	81.29%	67.31%	41.08%	49.62%	60.82%	46.90%	34.24%
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T3.16.3	58.17%	79.79%	75.55%	40.15%	54.27%	46.79%	37.44%	32.80%

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T3.23.2	59.97%	89.83%	54.55%	41.48%	38.48%	80.26%	46.13%	63.08%
T3.23.3	58.77%	99.34%	53.26%	40.40%	37.35%	62.38%	49.43%	52.17%
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T3.25.1	61.67%	73.33%	67.79%	43.19%	49.49%	76.10%	32.13%	56.99%
T3.25.2	60.81%	74.19%	66.62%	42.34%	73.79%	43.27%	31.10%	48.66%
T3.25.3	59.40%	89.97%	64.81%	41.08%	51.74%	44.68%	29.80%	41.44%
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T3.36.2	65.56%	67.61%	69.20%	46.74%	69.24%	37.49%	33.34%	43.71%
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T3.45.3	67.93%	83.50%	55.53%	48.75%	39.72%	55.99%	42.77%	36.42%
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T3.46.3	70.20%	73.73%	56.70%	51.17%	40.59%	72.14%	31.81%	53.06%
T3.56.1	77.74%	58.81%	52.39%	58.76%	51.71%	44.36%	76.78%	73.77%
T3.56.2	76.62%	62.04%	61.63%	57.64%	39.54%	37.93%	74.64%	71.34%
T3.56.3	76.45%	66.55%	56.68%	57.48%	40.53%	36.57%	74.36%	71.03%
T4.12.1	65.40%	55.10%	48.80%	84.67%	37.71%	67.05%	91.14%	89.99%
T4.12.2	64.50%	54.26%	47.96%	86.90%	36.85%	90.63%	55.48%	73.07%
T4.12.3	63.75%	53.57%	47.29%	98.93%	36.20%	72.21%	58.12%	64.41%
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T4.14.3	61.82%	62.34%	85.89%	67.26%	34.86%	32.86%	38.71%	44.55%
T4.15.1	64.04%	71.83%	64.11%	56.47%	37.01%	75.05%	30.53%	55.62%
T4.15.2	62.44%	69.85%	65.53%	78.19%	35.57%	41.43%	28.88%	43.09%
T4.15.3	61.94%	69.25%	83.34%	58.33%	35.17%	41.85%	28.47%	40.94%
T4.16.1	62.79%	81.30%	61.66%	53.97%	36.19%	58.44%	44.80%	34.70%
T4.16.2	61.62%	79.63%	65.02%	65.63%	35.15%	40.48%	36.06%	33.02%
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T4.24.1	65.46%	74.24%	61.03%	60.16%	38.07%	37.45%	75.98%	72.76%
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T4.25.1	65.69%	67.80%	68.13%	53.73%	38.32%	73.85%	32.56%	54.61%
T4.25.2	63.98%	69.13%	65.96%	76.05%	36.75%	39.02%	30.70%	41.56%
T4.25.3	63.01%	85.75%	64.80%	55.80%	35.98%	39.71%	29.90%	37.96%
T4.26.1	64.85%	65.71%	78.41%	51.51%	37.85%	57.73%	44.24%	37.42%
T4.26.2	63.49%	68.90%	76.42%	64.04%	36.63%	38.28%	35.01%	35.39%
T4.26.3	63.10%	75.15%	75.87%	56.46%	36.31%	39.94%	32.47%	34.94%
T4.34.1	67.30%	70.32%	63.84%	57.02%	39.53%	40.47%	75.07%	71.72%
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T4.34.3	63.98%	84.38%	67.47%	53.13%	36.44%	36.76%	33.27%	38.45%
T4.35.1	68.28%	64.41%	57.92%	64.74%	40.40%	73.46%	35.74%	54.22%
T4.35.2	65.78%	66.04%	78.78%	61.53%	38.12%	36.60%	32.97%	37.79%
T4.35.3	65.30%	83.36%	59.85%	60.97%	37.75%	36.88%	32.57%	36.33%
T4.36.1	67.97%	62.60%	56.05%	76.14%	40.42%	57.69%	44.21%	41.73%
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T4.45.2	73.53%	57.32%	78.29%	44.04%	44.71%	57.94%	44.53%	37.00%
T4.45.3	72.77%	81.93%	51.27%	44.27%	44.07%	57.03%	43.75%	34.95%
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T4.56.3	81.83%	65.51%	52.53%	45.39%	53.25%	35.71%	76.02%	73.09%
T5.12.1	69.79%	54.52%	48.26%	75.16%	64.83%	34.74%	85.33%	83.04%
T5.12.2	68.08%	53.03%	46.79%	78.32%	83.35%	33.07%	47.67%	58.83%
T5.12.3	67.69%	52.70%	46.47%	90.98%	68.88%	32.76%	48.55%	55.92%
T5.13.1	69.10%	56.60%	77.27%	43.27%	59.49%	34.36%	80.82%	78.15%
T5.13.2	67.29%	54.95%	80.08%	41.65%	78.08%	32.59%	42.79%	53.90%
T5.13.3	66.57%	54.32%	91.32%	41.08%	63.74%	32.03%	44.15%	49.41%
T5.14.1	69.13%	60.12%	70.95%	63.28%	38.00%	34.52%	77.43%	74.27%
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T5.14.3	66.34%	57.44%	85.69%	67.09%	35.55%	31.91%	38.58%	44.38%
T5.15.1	68.45%	71.16%	58.48%	51.02%	70.94%	34.17%	30.20%	45.03%
T5.15.2	67.59%	70.18%	58.99%	75.75%	42.52%	33.42%	29.43%	40.47%
T5.15.3	66.92%	69.43%	80.36%	51.90%	42.87%	32.90%	28.90%	37.98%
T5.16.1	67.29%	80.78%	57.24%	49.72%	57.70%	33.67%	37.56%	34.72%
T5.16.2	66.68%	79.97%	58.50%	63.56%	41.85%	33.14%	34.27%	33.95%
T5.16.3	66.31%	79.50%	68.98%	51.88%	42.72%	32.85%	32.47%	33.55%
T5.23.1	69.09%	79.68%	51.96%	44.86%	55.75%	34.42%	77.77%	74.92%
T5.23.2	67.20%	82.23%	50.18%	43.11%	74.33%	32.55%	39.48%	50.57%
T5.23.3	66.20%	92.43%	49.31%	42.29%	60.10%	31.79%	41.13%	45.10%
T5.24.1	69.82%	73.91%	55.81%	59.84%	39.80%	35.05%	75.43%	72.17%
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T5.24.3	66.59%	87.42%	52.65%	63.78%	36.90%	32.06%	36.11%	40.74%
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T5.26.2	68.96%	62.65%	77.18%	62.53%	39.83%	34.88%	33.64%	36.64%
T5.26.3	68.24%	72.74%	76.24%	50.27%	41.30%	34.31%	30.57%	35.85%
T5.34.1	71.28%	69.83%	63.36%	51.67%	42.66%	36.15%	74.21%	70.77%
T5.34.2	68.07%	73.02%	78.72%	48.42%	39.57%	33.06%	32.58%	38.99%
T5.34.3	67.76%	83.79%	66.96%	48.15%	39.32%	32.84%	32.89%	37.94%
T5.35.1	73.40%	59.33%	52.94%	64.54%	69.73%	37.94%	35.66%	44.08%
T5.35.2	71.57%	60.12%	77.26%	62.40%	38.09%	36.32%	33.90%	35.98%
T5.35.3	70.92%	81.41%	53.99%	61.69%	38.33%	35.82%	33.38%	34.26%

T5.36.1	73.18%	58.61%	52.16%	76.10%	57.17%	38.30%	37.22%	42.07%
T5.36.2	71.79%	60.63%	66.74%	74.18%	38.25%	37.07%	31.19%	40.18%
T5.36.3	71.43%	71.54%	54.83%	73.71%	38.86%	36.78%	29.91%	39.78%
T5.45.1	76.36%	56.22%	49.94%	73.88%	59.50%	40.22%	39.57%	39.05%
T5.45.2	75.41%	56.53%	77.09%	43.28%	58.42%	39.40%	38.64%	35.76%
T5.45.3	74.64%	80.79%	50.49%	43.51%	57.60%	38.78%	37.96%	33.78%
T5.46.1	77.14%	56.36%	50.02%	62.67%	72.54%	41.57%	33.77%	47.80%
T5.46.2	76.42%	57.18%	66.83%	43.74%	71.56%	40.93%	31.26%	46.80%
T5.46.3	75.92%	71.22%	51.45%	44.35%	70.91%	40.51%	29.75%	46.18%
T5.56.1	89.74%	55.14%	48.85%	50.83%	41.81%	54.18%	79.42%	77.40%
T5.56.2	89.30%	56.85%	55.35%	43.46%	39.30%	53.75%	78.70%	76.59%
T5.56.3	89.14%	60.01%	51.88%	44.75%	37.79%	53.60%	78.46%	76.32%
T6.12.1	75.62%	53.44%	47.25%	66.67%	56.57%	81.21%	29.25%	61.57%
T6.12.2	74.66%	52.70%	46.52%	67.70%	79.71%	50.18%	28.43%	53.72%
T6.12.3	74.01%	52.20%	46.03%	86.45%	58.28%	50.91%	27.95%	50.08%
T6.13.1	75.17%	55.72%	69.44%	42.51%	51.91%	77.33%	29.14%	57.98%
T6.13.2	74.16%	54.90%	70.36%	41.72%	75.51%	45.61%	28.25%	49.98%
T6.13.3	72.94%	53.92%	87.29%	40.82%	53.93%	46.74%	27.39%	44.27%
T6.14.1	75.45%	59.40%	63.76%	56.15%	37.48%	74.54%	29.52%	55.15%
T6.14.2	73.56%	57.77%	65.18%	77.74%	36.01%	41.14%	27.93%	42.72%
T6.14.3	72.97%	57.27%	82.89%	58.00%	35.61%	41.56%	27.53%	40.58%
T6.15.1	75.16%	63.87%	58.29%	50.84%	70.68%	34.67%	29.42%	44.81%
T6.15.2	74.21%	62.99%	58.80%	75.50%	42.36%	33.91%	28.67%	40.27%
T6.15.3	73.48%	62.32%	80.11%	51.72%	42.72%	33.38%	28.16%	37.80%
T6.16.1	73.90%	80.39%	54.78%	47.31%	48.90%	41.41%	29.80%	35.60%
T6.16.2	73.42%	79.83%	56.64%	55.00%	40.09%	38.20%	29.42%	35.07%
T6.16.3	73.25%	79.63%	60.68%	50.51%	41.38%	36.46%	29.29%	34.89%
T6.23.1	75.43%	72.34%	51.32%	44.27%	48.64%	74.65%	29.38%	55.60%
T6.23.2	74.38%	73.20%	50.43%	43.40%	72.51%	42.44%	28.44%	47.47%
T6.23.3	72.65%	88.76%	49.06%	42.11%	50.85%	43.83%	27.24%	40.43%
T6.24.1	76.44%	67.10%	55.31%	53.08%	39.42%	72.78%	30.19%	53.61%
T6.24.2	74.45%	68.42%	53.54%	75.13%	37.80%	38.45%	28.46%	40.79%
T6.24.3	73.32%	84.87%	52.60%	55.13%	37.01%	39.14%	27.72%	37.26%
T6.25.1	76.84%	62.02%	60.11%	48.25%	69.42%	36.84%	30.50%	43.82%
T6.25.2	75.83%	62.50%	59.15%	73.76%	40.00%	35.99%	29.67%	39.06%
T6.25.3	74.41%	82.53%	57.86%	49.34%	40.59%	34.95%	28.70%	34.91%

T6.26.1	76.58%	59.05%	77.91%	45.20%	48.40%	40.98%	31.67%	38.61%
T6.26.2	75.98%	60.87%	77.18%	54.11%	38.15%	37.53%	31.18%	37.91%
T6.26.3	75.64%	64.80%	76.77%	49.34%	40.36%	34.53%	30.93%	37.55%
T6.34.1	78.26%	63.39%	56.93%	51.37%	42.42%	71.78%	31.41%	52.61%
T6.34.2	75.40%	64.99%	77.42%	48.82%	40.03%	35.77%	28.98%	36.66%
T6.34.3	74.85%	82.03%	58.83%	48.38%	39.63%	36.04%	28.63%	35.25%
T6.35.1	79.44%	58.78%	52.41%	56.52%	68.89%	40.16%	32.23%	43.31%
T6.35.2	77.45%	59.56%	76.48%	54.64%	37.63%	38.45%	30.63%	35.36%
T6.35.3	76.75%	80.65%	53.44%	54.02%	37.86%	37.92%	30.16%	33.67%
T6.36.1	80.52%	56.32%	49.90%	76.08%	48.46%	41.05%	34.53%	43.31%
T6.36.2	79.35%	59.40%	58.70%	74.63%	37.06%	35.10%	33.57%	41.88%
T6.36.3	79.18%	63.72%	53.98%	74.42%	37.99%	33.84%	33.44%	41.70%
T6.45.1	81.82%	55.51%	49.25%	72.76%	51.22%	44.01%	33.76%	38.06%
T6.45.2	80.80%	55.81%	76.02%	42.62%	50.30%	43.11%	32.96%	34.86%
T6.45.3	79.97%	79.77%	49.79%	42.85%	49.58%	42.43%	32.38%	32.93%
T6.46.1	85.08%	54.76%	48.43%	54.26%	73.01%	37.71%	38.07%	49.61%
T6.46.2	84.49%	56.04%	58.78%	42.61%	72.28%	35.20%	37.58%	48.86%
T6.46.3	84.17%	63.45%	50.67%	43.55%	71.88%	33.69%	37.32%	48.47%
T6.56.1	91.35%	54.16%	47.87%	49.65%	40.68%	77.52%	43.68%	60.61%
T6.56.2	90.90%	55.84%	54.24%	42.46%	38.24%	76.91%	43.29%	59.97%
T6.56.3	90.74%	58.94%	50.83%	43.72%	36.77%	76.69%	43.15%	59.76%
T7.12.1	84.41%	51.99%	45.89%	63.58%	53.45%	63.07%	48.82%	32.71%
T7.12.2	83.67%	51.50%	45.41%	65.87%	64.72%	48.01%	43.56%	31.94%
T7.12.3	83.38%	51.30%	45.22%	72.42%	57.27%	49.62%	41.12%	31.68%
T7.13.1	83.95%	54.24%	66.58%	41.25%	48.98%	59.97%	46.18%	32.71%
T7.13.2	83.09%	53.63%	68.69%	40.67%	61.23%	43.57%	40.75%	31.81%
T7.13.3	82.53%	53.24%	74.73%	40.32%	53.58%	46.13%	36.87%	31.33%
T7.14.1	84.31%	57.86%	61.07%	53.41%	36.36%	57.73%	44.20%	33.38%
T7.14.2	82.73%	56.67%	64.40%	64.96%	35.32%	39.99%	35.58%	31.76%
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T7.15.1	84.20%	62.39%	56.81%	49.32%	57.20%	33.86%	37.17%	33.66%
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T7.16.1	84.21%	69.18%	54.52%	47.07%	48.64%	41.17%	30.02%	34.85%
T7.16.2	83.67%	68.70%	56.38%	54.73%	39.87%	37.98%	29.64%	34.33%
T7.16.3	83.47%	68.52%	60.40%	50.26%	41.16%	36.25%	29.51%	34.15%

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T7.24.3	83.14%	73.76%	52.07%	55.25%	36.72%	38.92%	31.55%	32.00%
T7.25.1	86.06%	60.54%	58.64%	46.70%	56.05%	35.98%	36.33%	35.03%
T7.25.2	85.19%	61.73%	57.92%	61.47%	39.09%	35.35%	32.88%	34.16%
T7.25.3	84.30%	71.67%	57.22%	49.41%	40.54%	34.77%	29.88%	33.42%
T7.26.1	86.72%	58.51%	65.86%	44.72%	47.85%	40.46%	32.21%	36.77%
T7.26.2	86.04%	60.31%	65.25%	53.54%	37.71%	37.06%	31.72%	36.10%
T7.26.3	85.65%	64.21%	64.90%	48.81%	39.90%	34.10%	31.46%	35.76%
T7.34.1	87.50%	60.82%	54.33%	49.94%	41.21%	55.37%	42.21%	35.88%
T7.34.2	84.98%	64.71%	65.98%	48.00%	39.40%	35.11%	30.93%	33.27%
T7.34.3	84.73%	71.18%	58.93%	47.82%	39.24%	35.76%	29.93%	33.07%
T7.35.1	88.95%	57.26%	50.87%	55.08%	55.49%	39.21%	35.88%	37.26%
T7.35.2	87.27%	59.24%	65.09%	53.70%	37.13%	37.95%	30.07%	35.58%
T7.35.3	86.83%	69.90%	53.47%	53.35%	37.72%	37.65%	28.83%	35.23%
T7.36.1	90.43%	55.51%	49.13%	62.88%	47.57%	40.22%	35.57%	39.77%
T7.36.2	89.12%	58.55%	57.79%	61.68%	36.38%	34.39%	34.58%	38.47%
T7.36.3	88.93%	62.81%	53.15%	61.51%	37.29%	33.16%	34.45%	38.30%
T7.45.1	91.83%	54.60%	48.33%	60.36%	50.11%	43.17%	32.04%	39.47%
T7.45.2	90.98%	55.39%	64.56%	42.13%	49.44%	42.51%	29.66%	38.64%
T7.45.3	90.38%	68.99%	49.71%	42.71%	48.98%	42.08%	28.23%	38.13%
T7.46.1	94.54%	53.67%	47.39%	52.98%	58.59%	36.62%	39.95%	43.31%
T7.46.2	93.88%	54.92%	57.52%	41.61%	58.00%	34.19%	39.43%	42.65%
T7.46.3	93.52%	62.18%	49.58%	42.53%	57.68%	32.73%	39.16%	42.32%
T7.56.1	100.00%	52.80%	46.56%	48.16%	39.32%	59.74%	47.15%	49.06%
T7.56.2	99.51%	54.44%	52.75%	41.18%	36.96%	59.26%	46.72%	48.55%
T7.56.3	99.33%	57.46%	49.44%	42.40%	35.53%	59.10%	46.58%	48.38%

Analysis of Single-elimination Tournaments

Research Paper

Project ID: MA014
Finalist: Chi-Hua Wang

2010 Intel International Science and Engineering Fair

Analysis of Single-elimination Tournaments

Abstract

Single-elimination tournaments studied in this project are the knockout tournaments which satisfy the property that each round has a winner and a loser; a loser of a round is not involved in any further round. Let a, b denote the strength of players A, B. Then the winning probability of player A when playing against player B is expressed by the quantity a / (a + b). Along the same line, in order to formulate the winning probability of player A when reaching the nth round based on scheme T, we introduce the following concepts: "Imaginary Opponent", "Threshold of Threat", "Formula of Winning Probability", " Probability-to-Strength Ratio (PSR) ", "Rate of Scheme T ". These concepts enable us to study the influence of the variations in player's strength on winning probability (a) based on the traditional scheme and (b) based on the winning probability when the strength of the player is kept fixed. Results of this project clarify some myths in the single-elimination tournaments. We find it surprising that the conventional wisdom "Once the strength of our opponent increases, our winning probability will decrease" actually is false! This project may be of importance in explaining the fairness of single-elimination tournaments, as well as in providing a better understanding of single-elimination tournaments.

Introduction

Single-elimination tournaments studied in this project are the knockout tournaments which satisfy the property that each game has a winner and a loser; a loser of a game is not involved in any further games. Single-elimination tournaments are widely adopted in the sports such as tennis and World Cup Soccer Final. About single-elimination tournaments, we have many common beliefs.

Here are the conventional wisdoms we want to demonstrate:

(1) Once the strength of our opponent increases, our winning probability will decrease.

(2) Once the strength of us increases, our winning probability should increase.

(3) The stronger the player is, the higher winning probability he will have.

In the second part, we introduce "Probability-to-Strength Ratio (PSR)" to measure how favorable the scheme is to the player for explaining the fairness of single-elimination tournament.

Notations

1. Let a, b, c, d denote the strength of players A, B, C, D. [1-5]
2. Let P_{AB} denote the winning probability of player A when competing against player B: $P_{AB} = a / (a + b)$. [6]
3. Let $P(n,A,T)$ denote the winning probability of player A when reaching the nth round based on scheme T.
4. Let $PSR(n,A,T)$ denote the ratio of winning probability to the strength of player A when reaching the *nth* round based on scheme *T*.

Results

I. The Influence of the Variation in Player's Strength on Winning Probability

(I) A knockout Tournament Plan among 4 Players

1. Imaginary Opponent X

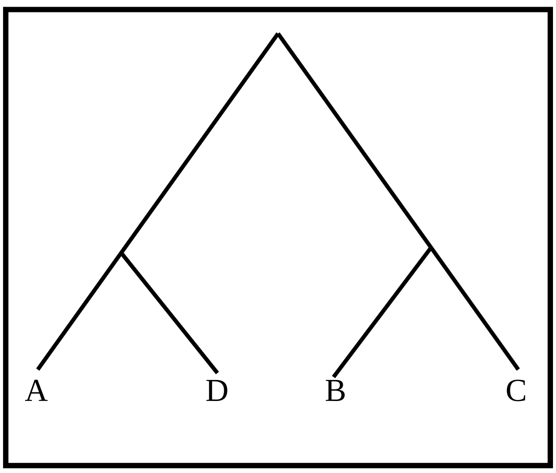

Fig.1

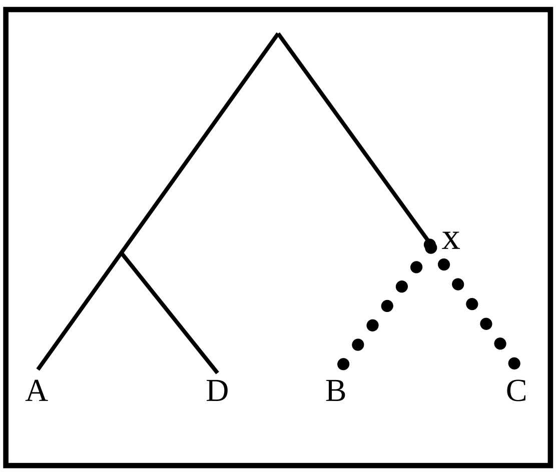

Fig.2

There is a knockout tournament plan among 4 players as Fig.1 .As for player A, the opponent of the first round must be player D .We can judge the threat of player D by comparing the strength between player A and player D.

However, the opponent of the second round may be player B or player C. Before any round proceeds, how much threat player B or player C will impose on player A? We attempt to build an Imaginary opponent X .By substituting Imaginary opponent X for player B and player C, we can get a simplified tournament plan which is a ladder tree for player A as Fig. 1.2 .Thus we have a impressive theorem:

Theorem 1. Let x denote the strength of Imaginary opponent X for player A , then:

$$x = \frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)}$$

Proof. We can calculate the winning probability of player A when reaching the 2nd round based on Fig.1 and Fig.2 as (1) and (2) :

$$\text{P (2,A,T)} = \text{P}_{\text{AD}}(\text{P}_{\text{BC}}\text{P}_{\text{AB}} + \text{P}_{\text{CB}}\text{P}_{\text{AC}}) = \frac{a}{a+d}\left(\frac{b}{b+c}\frac{a}{a+b} + \frac{c}{b+c}\frac{a}{a+c}\right) \ldots\ldots\ldots\ldots (1)$$

$$\text{P (2,A,T)} = \text{P}_{\text{AD}}\text{P}_{\text{AX}} = \frac{a}{a+d}\frac{a}{a+x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Fig.1 and Fig.2 indicate the same tournament plan, thus the winning probability of player A when reaching 2nd round based on Fig.1 and Fig.2 should be equal. By equaling (1) and (2) , we obtain:

$$x = \frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)}$$

We can use Imaginary opponent X to judge the threat caused by any opponent that player A may compete against at the second round before any round be proceeds.

2. Existence of Threshold at the Second Round

From theorem 1, we can judge the threat caused by any opponent that the player A may compete against at the second round. At Fig. 1.2, player A will compete against the Imaginary opponent X. According to the convention wisdom ," Once the strength of player B increase, the strength of the Imaginary opponent X will also increase and the winning probability of the player A will decrease." Thus, one would guess **the x is an increasing function of b.** To our surprise, after we have proved the guess, actually that is not. We have an amazing theorem:

Theorem 2. The winning probability of the player A is (a) an increasing function of b over the interval [0, b0]. (b) a decreasing function of b over the interval [b0,+∞). Where:

$$\text{b}_0 = \frac{c(\sqrt{2a(a+c)}-a)}{a+2c}$$

Proof. Define x as function f(b). Examine whether f(b) is an increasing function of b or not.

$$f(b) = \frac{b^2(a+c)+c^2(a+b)}{b(a+c)+c(a+b)} = \frac{(a+c)b^2+c^2b+c^2a}{(a+2c)b+ac}$$

Take the first derivative of f(b).

$$\frac{df(b)}{db} = \frac{(a+c)(a+2c)b^2+2ac(a+c)b-(ac^2(a+c))}{((a+2c)b+ac)^2}$$

The first derivative of f(b) is a fraction. The denominator is positive and the numerator is a quadratic of b. Define the numerator as g(b).

$$g(b) = (a+c)(a+2c)\mathrm{b}^2 + 2ac(a+c)b - (ac^2(a+c))$$

From the first term and the discriminant are positive, we can confirm the picture of g(b) as Fig. 1.3. Let g(b) = 0 and use the quadratic formula, we can solve the function and get a positive root and a negative root.

$$b = \frac{-2ac(a+c) \pm \sqrt{4a^2c^2(a+c)^2 + 4(a+c)^2(a+2c)ac^2}}{2(a+c)(a+2c)} = \frac{c\,(\,-a \pm \sqrt{2a(a+c)}\,)}{a+2c}$$

Name the positive root as b_0, and then we have

$$\mathrm{b}_0 = \frac{c(\sqrt{2a(a+c)} - a)}{a+2c}$$

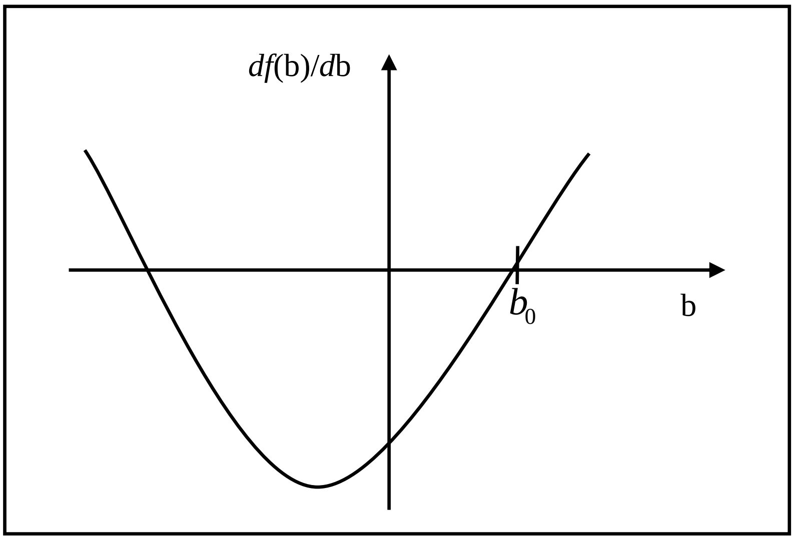

Fig.3

From these observations, we can conclude that:

For b>b_0, the derivative is positive and f(b) is an increasing function of b, then once b increase, the winning probability of the player A will decrease. For b<b_0, the derivative is negative and f(b) is a decreasing function of b, then once b increase, the winning probability of the player A will also increase!

Theorem 2 is a very interesting result. It reveals that there exist a Threshold of Threat b_0, if the strength b of player B satisfied b>b_0 ,then once the strength of player B increase, the winning probability of the player A will decrease; if the strength b of player B satisfied b<b_0 ,then once the strength of player B increase, the winning probability of the player A will increase.

The result prove that the convention wisdom," Once the strength of our opponent increases, our winning probability will decrease", actually is false.

Here is an example. At Fig. 1.1, let c = a and d = a/10. Under such circumstances, b_0 is a/3. Let b increase from a/5 to a. Observe the strength b of the player B and the winning probability of player A at fallowing form.

The strength b of the player B	a/5	a/4	a/3	a/2	a
The winning probability P(2,A,T) of player A	50.51%	50.91%	51.14%	50.51%	45.45%

We can find that when the strength of player B is greater than b_0, than the increase of b will decrease the winning probability of the player A; when the strength of player B is less than b_0, than the increase of b will also increase the winning probability of the player A.

3. Threshold of Threat of the Seed

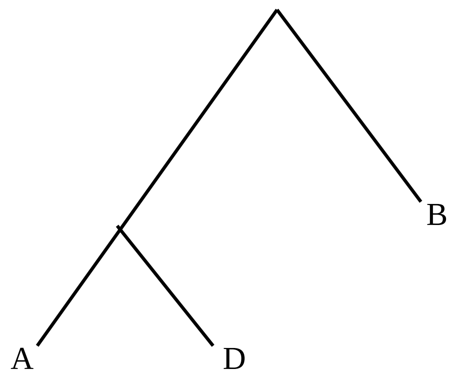

Fig. 4

Furthermore, a player would be arranged as the Seed as the player B at Fig. 1.4. Then we have:

Proposition 1. Threshold b_0 of the Seed at the second round is equal to 0.

Proof. The player B is the Seed and we can see the situation as the strength of the player C is equal to 0 (the player B must reache to the second round). Substitute c = 0 into b_0,we have:

$$b_0 = \frac{c(\sqrt{2a(a+c)} - a)}{a+2c}$$

$$c = 0$$

$$b_0 = 0$$

Surely, Threshold b_0 of the Seed at the second round is equal to 0.

4. Influence of the Increase in Player's Strength on Winning Probability

According to the conventional wisdom,” Once our strength increases, our winning probability should increase.” At a knockout tournament plan among 4 players as Fig. 1.1, we examine whether the winning probability of player A is an increasing function of the strength of player A or not and we have:

Lemma 1. At a knockout tournament plan among 4 players, the winning probability is an increasing function of the strength of the player.

Proof. From the Fig. 1.1,we can confirm the winning probability of player A.

$$P(n,A,T) = \frac{a}{a+d}\left(\frac{b}{b+c}\frac{a}{a+b} + \frac{c}{b+c}\frac{a}{a+c}\right)$$

According to the conventional wisdom, one would guess P(A) is an increasing function of the strength of player A. We examine the first derivative is positive or not. Define the former term as G(a),we can easily confirm that G(a) is the strictly increasing function of the strength a of player A.

Define the latter term as H(a) and do the first derivative, we have:

$$\mathrm{H}(a) = \frac{b}{b+c}\frac{a}{a+b} + \frac{c}{b+c}\frac{a}{a+c} = \frac{(b+c)a^2 + 2bca}{(b+c)a^2 + (b+c)^2 a + bc(b+c)}$$

$$\frac{d\mathrm{H}(a)}{da} = \frac{bc(b+c)^2 a + 2b^2c^2(b+c)}{((b+c)a^2 + (b+c)^2 a + bc(b+c))^2}$$

From the first derivative is positive we can prove that at a knockout tournament plan among 4 players, the winning probability is an increasing function of the strength of the player..

(II) A knockout Tournament Plan among 2^n Players

1. Formula of Winning Probability (1)

During the tournament, we always focus on the winning probability of the player who we support. For this reason, we want to find a formula of winning probability to assist us calculate .Thus, we have:

Theorem 3. Let P (n,Q1,T) denote the winning probability of player Q1 when reaching the nth round based on scheme T, then:

$$P(n,Q_1,T) = \prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} (P(j-1,Q_i,T)P_{Q_1Q_i})$$

Proof. Let the scheme be given by Fig.5 which is a knockout tournament plan among 2^n players. Name the player from left to right as $Q_1,Q_2,Q_3,\ldots\ldots$ as Fig.6. Once the player wins a game, he will proceed to the next round.

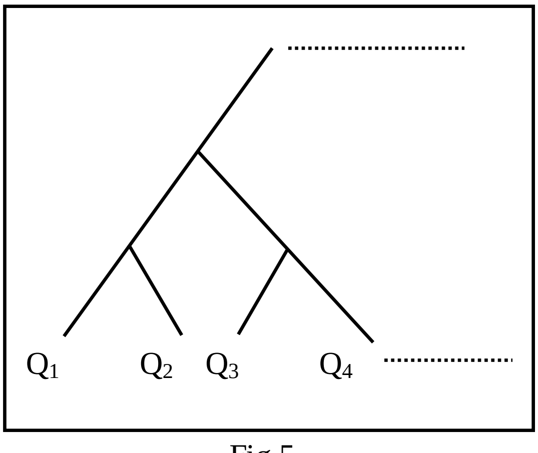

Fig.5

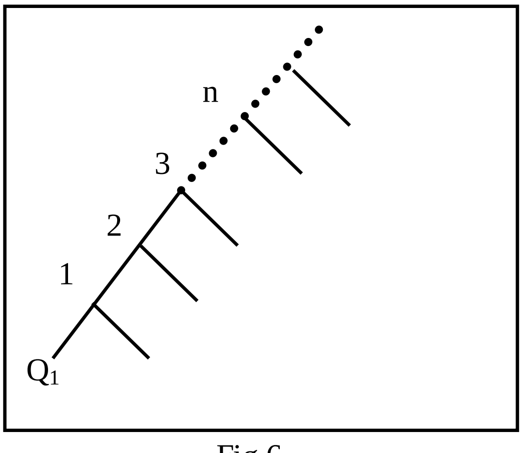

Fig.6

Let P (n,Q_1,T) denote the winning probability of player Q_1 when reaching the nth round based on scheme T, then:

$$P(1,Q_1,T) = P_{Q1Q2}$$

$$P(2,Q_1,T) = P(1,Q_1,T)(P_{Q_3Q_4}P_{Q_1Q_3} + P_{Q_4Q_3}P_{Q_1Q_4}) = P(1,Q_1,T)(P(1,Q_3,T)P_{Q_1Q_3} + P(1,Q_4,T)P_{Q_1Q_4})$$

$$P(3,Q_1,T)=P(2,Q_1,T)(P(2,Q_5,T)P_{Q_1Q_5}+P(2,Q_6,T)P_{Q_1Q_6}+P(2,Q_7,T)P_{Q_1Q_7}+P(2,Q_8,T)P_{Q_1Q_8})$$

We can find a recursive algorithm as :

$$P(n,Q_1,T)=P(n\text{-}1,Q_1,T)\sum_{i=2^{n-1}+1}^{2^n}(P(n\text{-}1,Q_i,T)P_{Q_1Q_i})$$

Furthermore, we can find the general formula of P (n,Q_1,T) as follows.

$$P\ (1,Q_1,T)=P(0,Q_1,T)\sum_{i=2^0+1}^{2}(P\ (0,Q_i,T)P_{Q_1Q_i})$$

$$P\ (2,Q_1,T)=P(1,Q_1,T)\sum_{i=2^0+1}^{2}(P\ (1,Q_i,T)P_{Q_1Q_i})$$

$$P\ (3,Q_1,T)=P(2,Q_1,T)\sum_{i=2^0+1}^{2}(P\ (2,Q_i,T)P_{Q_1Q_i})$$

.

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×) $$P\ (n,Q_1,T)=P(n-1,Q_1,T)\sum_{i=2^0+1}^{2}(P\ (n-1,Q_i,T)P_{Q_1Q_i})$$

$$P(n,Q_1,T)=P(0,Q_1,T)\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P(j\text{-}1,Q_i,T)P_{Q_1Q_i})$$

Among the formula, P(0,Q_1,T) means the probability of player Q_1 when reaching the zero round. Before any round proceeds, every player is locating at the zero round. Thus , P(0,Q_1,T) = 1 and the winning probability of player Q_1 when reaching the nth round based on scheme T is:

$$P(n,Q_1,T)=\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P(j\text{-}1,Q_i,T)P_{Q_1Q_i})$$

From Theorem 3 and condition probability, we can confirm an Proposition as follows.

Proposition 2. Let P((n,m),Q_1,T) denote the winning probability of player Q_1 when mth round from nth round, then:

$$P((n,m),Q_1,T)=\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})$$

Proof. Define Event A as the player Q_1 reaching the nth round, Event B as the player Q_1 reaching the mth round and Event C as the player Q_1 reaching the mth from nth round. Because of Event B lies in Event A, by condition probability, we have:

$$P(C) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$$

$$\mathrm{P}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T}) = \frac{\mathrm{P}(\mathrm{m},\mathrm{Q}_1,\mathrm{T})}{\mathrm{P}(\mathrm{n},\mathrm{Q}_1,\mathrm{T})} = \frac{\prod_{j=1}^{m}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(j-1,\mathrm{Q}_i,\mathrm{T})\mathrm{P}_{\mathrm{Q}_1\mathrm{Q}_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(j-1,\mathrm{Q}_i,\mathrm{T})\mathrm{P}_{\mathrm{Q}_1\mathrm{Q}_i})} = \prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(j-1,\mathrm{Q}_i,\mathrm{T})\mathrm{P}_{\mathrm{Q}_1\mathrm{Q}_i})$$

At Lemma 1, we confirm that "at a knockout tournament plan among 4 players, once the strength of player increase, the winning probability of player increase." Then, is it still a truth at a knockout tournament plan among $\mathbf{2^n}$ players?

Theorem 4. At a knockout tournament plan among $\mathbf{2^n}$ players, the winning probability is an increasing function of the strength of the player.

Proof. From Theorem 3, we obtain the winning probability of player Q_1 when reaching the nth round. According to the conventional wisdom, one would guess P (n,Q_1,T) is an increasing function of the strength of the player Q_1. We examine whether P (n,Q_1,T) is an increasing function of the q_1 or not.

Take the first derivative of P (n,Q_1,T).

$$\mathrm{P}(\mathrm{n},\mathrm{Q}_1,\mathrm{T}) = \prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(j-1,\mathrm{Q}_i,\mathrm{T})\mathrm{P}_{\mathrm{Q}_1\mathrm{Q}_i})$$

$$\frac{d\mathrm{P}(\mathrm{n},\mathrm{Q}_1,\mathrm{T})}{d\mathrm{q}_i} = \sum_{k=1}^{n}\Big(\sum_{i=2^{k-1}+1}^{2^k}\Big(\frac{\mathrm{q}_i}{(\mathrm{q}_i+\mathrm{q}_1)^2}\mathrm{P}(\mathrm{k}-1,\mathrm{Q}_1,\mathrm{T})\Big)\prod_{j=1;\,j\neq k}^{n}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(j-1,\mathrm{Q}_1,\mathrm{T})\mathrm{P}_{\mathrm{Q}_1\mathrm{Q}_i})\Big)$$

Since all terms of the first derivative of P (n,Q_1,T) is positive, we see that P (n,Q_1,T) is an increasing function of the strength of player Q_1. We confirm that "at a knockout tournament plan among $\mathbf{2^n}$ players, the winning probability is an increasing function of the strength of the player."

2. Imaginary Opponent $\mathbf{X_n}$

At Theorem 1, we use Imaginary opponent X to judge the threat caused by any opponent that player A may compete against at the second round. After we obtain "Formula of winning probability," we can generalize Imaginary opponent X to Imaginary Opponent $\mathbf{X_n}$ as follows.

Theorem 5. Let x_n denote the strength of Imaginary opponent $\mathbf{X_n}$ for player Q_1 , then:

$$x_n = \frac{1 - \sum_{i=2^{n-1}+1}^{2^n} (P(n-1, Q_i, T) P_{Q_1 Q_i})}{\sum_{i=2^{n-1}+1}^{2^n} (P(n-1, Q_i, T) P_{Q_1 Q_i})} q_1$$

Proof.

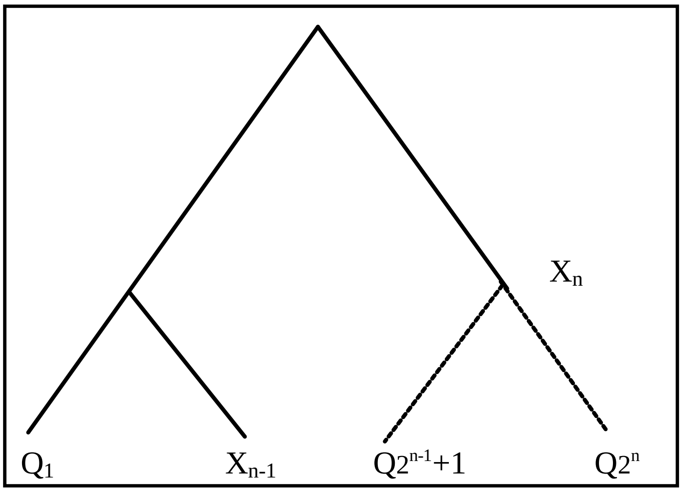

Fig.7

By substituting Imaginary opponent $\mathbf{X_n}$ for player $Q2^{n-1}+1$, $Q2^{n-1}+2$,..., $Q2^n$, we can get a simplified tournament plan which is a ladder tree for player A as Fig.7. Then we have:

$$P(n,Q_1,T) = P(n-1,Q_1,T)\frac{q_1}{q_1 + x_n}$$

$$x_n P(n,Q_1,T) + q_1 P(n,Q_1,T) = q_1 P(n-1,Q_1,T)$$

$$x_n = \frac{P(n\text{-}1,Q_1,T) - P(n,Q_1,T)}{P(n,Q_1,T)} q_1 == \frac{1\text{-} \sum_{i=2^{n-1}+1}^{2^n} (P(n\text{-}1,Q_i,T) P_{Q_1 Q_i})}{\sum_{i=2^{n-1}+1}^{2^n} (P(n\text{-}1,Q_i,T) P_{Q_1 Q_i})} q_1$$

Therefore, we can use Imaginary opponent $\mathbf{X_n}$ to judge the threat caused by any opponent that the player Q_1 may compete against at the nth round before any round be proceeds.

3. The Threshold of Threat at the 3rd Round

At Theorem 2, we know there exists a Threshold of Threat at the 2nd round. Is there still exists a Threshold of Threat at the 3rd round?

Theorem 6. If the strength a,f,g,h of the player A,F,G,H satisfy:

$$\frac{(g-h)^2 - \sqrt{(g^2+h^2)^2 + 4g^2h^2}}{2(g+h)} < f < \frac{(g-h)^2 + \sqrt{(g^2+h^2)^2 + 4g^2h^2}}{2(g+h)}, f > \sqrt{gh}$$

Then there exists a Threshold of Threat e0, if the strength e of player E satisfied e>e0 ,then once the strength e of player E increase, the winning probability of player A will decrease; if the strength e of player E satisfied e<e0 ,then once the strength e of player E increase, the winning

probability of player A will increase.

Proof.

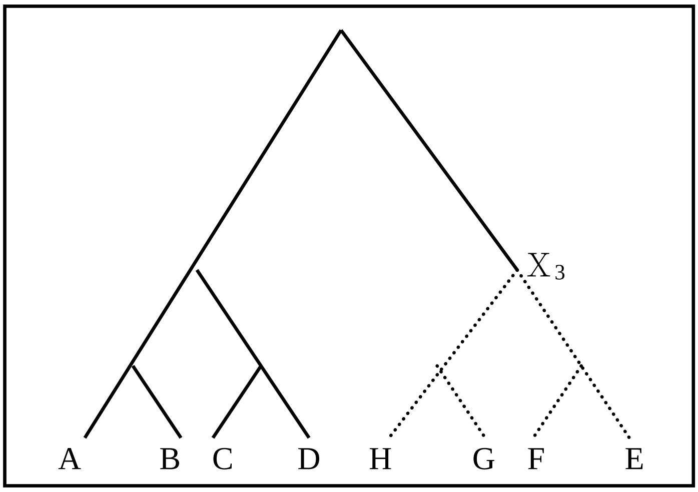

Fig.8

From Theorem 5, we can know the strength of Imaginary opponent $\mathbf{X_3}$ for player Q_1 as follows:

$$x_3 = \frac{1-(P_2(E)P_{AE} + P_2(F)P_{AF} + P_2(G)P_{AG} + P_2(H)P_{AH})}{(P_2(E)P_{AE} + P_2(F)P_{AF} + P_2(G)P_{AG} + P_2(H)P_{AH})} a$$

Next ,we examine whether x_3 is the strictly increasing function of the strength e of player E or not. It is too enormous to expand, we use other function of e to indicate x_3 as follows.

$$x_3 = \frac{1-(P_2(E)P_{AE} + P_2(F)P_{AF} + P_2(G)P_{AG} + P_2(H)P_{AH})}{(P_2(E)P_{AE} + P_2(F)P_{AF} + P_2(G)P_{AG} + P_2(H)P_{AH})} a = \frac{f(e)}{g(e)}$$

Take the first derivative of x_3

$$\frac{dx_3}{de} = \frac{\frac{df(e)}{de} g(x) - f(x)\frac{dg(e)}{de}}{(g(e))^2} = \frac{h(e)}{(g(e))^2}$$

We solve h(e) and confirm the root of h(e) is same as m(e).

$$\mathrm{m(e)} = \mathrm{a}_6\mathrm{e}^6 + \mathrm{a}_5\mathrm{e}^5 + \mathrm{a}_4\mathrm{e}^4 + \mathrm{a}_3\mathrm{e}^3 + \mathrm{a}_2\mathrm{e}^2 + \mathrm{a}_1\mathrm{e}^1 + \mathrm{a}_0$$

The first term is positive. If the constant term is negative, the positive roots of the equation will more than one, and the Threshold of Threat at the 3rd round exists.

$$a_0 = a^2 g^2 h^2 f^2((-hf^2 - gf^2 - 2fgh + fg^2 + fh^2 + gh^2 + g^2 h)a + (-2ghf^2 + 2g^2h^2))$$

If a_0 is negative, it should satisfy:

$$-hf^2 - gf^2 - 2fgh + fg^2 + fh^2 + gh^2 + g^2 h < 0 \ldots(1)$$
$$-2ghf^2 + 2g^2h^2 < 0 \ldots(2)$$

First, we can find the interval which satisfies the inequality (1) as follows.

$$-hf^2-gf^2-2fgh+fg^2+fh^2+gh^2+g^2h<0$$
$$-(g+h)f^2+(g-h)^2f+(g+h)gh<0$$
$$\frac{(g-h)^2-\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}<f<\frac{(g-h)^2+\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}$$

Next, we find the interval which satisfies the inequality (2) as follows.

$$-2ghf^2+2g^2h^2<0$$
$$-2gh(f^2-gh)<0$$
$$f>\sqrt{gh}\ ,\ f<-\sqrt{gh}\ (Unsuited,\ f>0)$$

Combine (1) and (2), we can conclude that "If the strength a,f,g,h of the player A,F,G,H satisfy:

$$\frac{(g-h)^2-\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}<f<\frac{(g-h)^2+\sqrt{(g^2+h^2)^2+4g^2h^2}}{2(g+h)}\ ,\ f>\sqrt{gh}$$

Then there exists a Threshold of Threat e_0, if the strength e of player E satisfied $e>e_0$,then once the strength b of player E increase, the winning probability of the player A will decrease; if the strength e of player B satisfied $e<e_0$,then once the strength b of player B increase, the winning probability of the player A will increase."

4. The Threshold of Threat at The nth Round

At a knockout tournament plan among $\mathbf{2^n}$ players as Fig. 1.7, the strength of Imaginary opponent $\mathbf{X_n}$ is too enormous to expand and we have hard time to do the first derivative of x_n. However, we can reveal the conditions that should be satisfied if there exists a Threshold of Threat at The round of $\mathbf{2^n}$. Then we have:

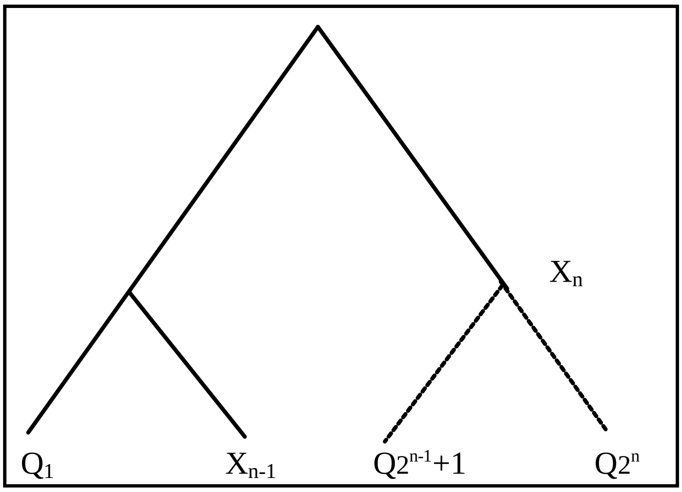

Fig.9

Theorem 7. If the strength $q_{2^{n-1}+2}, q_{2^{n-1}+3}, \ldots, q_{2^n}>0$ of the player $Q_{2^{n-1}+2}, Q_{2^{n-1}+3}, \ldots, Q_{2^n}$ satisfy:

$$\frac{dx_n}{dq_{2^n-1}}<0$$

Then there exists a Threshold $q_{(2^{n-1}+1)0}$, if the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ satisfied $q_{(2^{n-1}+1)}>q_{(2^{n-1}+1)0}$,then once the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ increase, the winning probability of player Q_1 will decrease; if the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ satisfied $q_{(2^{n-1}+1)} < q_{(2^{n-1}+1)0}$,then once the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ increase, the winning probability of player Q_1 will increase.

Proof.

The strength of Imaginary opponent $\mathbf{X_n}$ for player Q_1:

$$x_n = \frac{1-\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})}{\sum_{i=2^{n-1}+1}^{2^n}(P_{i-1}(Q_i)P_{Q_1Q_i})} q_1$$

If the strength $q_{2^{n-1}+2}, q_{2^{n-1}+3}, \ldots , q_{2^n} > 0$ of the player $Q_{2^{n-1}+2}, Q_{2^{n-1}+3}, \ldots , Q_{2^n}$ satisfy:

$$\frac{dx_n}{dq_{2^n-1}} < 0$$

Then there exists a Threshold of Threat $q_{(2^{n-1}+1)0}$, if the strength e of player E satisfied $q_{(2^{n-1}+1)} > q_{(2^{n-1}+1)0}$,then once the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ increase, the winning probability of the player Q_1 will decrease; if the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ satisfied $q_{(2^{n-1}+1)} < q_{(2^{n-1}+1)0}$,then once the strength $q_{(2^{n-1}+1)}$ of player $Q_{(2^{n-1}+1)}$ increase, the winning probability of the player Q_1 will increase.

5. Formula of Winning Probability (2)

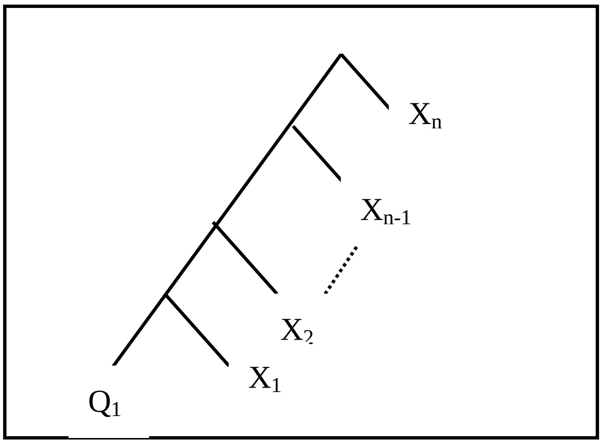

Fig.10

By using the conception of " Imaginary Opponent," we can simplify the tournament plan as Fig. 10. Then we have:

Theorem 8. The roots of an equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ are $x_1, x_2, x_3, \ldots, x_n$. Let P (n,Q_1,T) denote the winning probability of player Q_1 when reaching the nth round based on scheme T, then:

$$P(n,Q_1,T) = \frac{q_1^n}{\sum_{k=0}^{n}(-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}$$

Proof.

By observing Fig.10, we can obtain the winning probability of player Q_1 when reaching the nth round based on scheme T as follows.

$$P(n, Q_1, T) = \prod_{k=1}^{n} P_{Q_1 X_K}$$

The roots of an equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ are x_1, x_2, x_3, ... , x_n. From the relation of roots and coefficients, we can find:

$$-\frac{a_{n-1}}{a_n} = x_1 + x_2 + \ldots + x_n$$

$$\frac{a_{n-2}}{a_n} = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n$$

.
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$$(-1)^k \frac{a_{n-k}}{a_n} = \text{the sum of the the product of multiplication of choosing k roots from n roots}$$

.
.
.

$$(-1)^n \frac{a_0}{a_n} = x_1 x_2 x_3 \ldots x_{n-1} x_n$$

Return to the probability of the player Q_1 reached to N round, we have:

$$P(n, Q_1, T) = \prod_{k=1}^{n} P_{Q_1 X_K}$$

$$= \frac{q_1^n}{(q_1 + x_1)(q_1 + x_2)\ldots(q_1 + x_n)}$$

$$= \frac{q_1^n}{q_1^n + (x_1 + x_2 + \ldots + x_n) q_1^{n-1} + (x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n) q_1^{n-2} + \ldots + x_1 x_2 x_3 \ldots x_{n-1} x_n}$$

$$= \frac{q_1^n}{\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}$$

From Theorem 8 and condition probability, we can confirm a proposition as follows.

Proposition 3. The roots of an equation $a_{m-n}x^{m-n}+a_{m-n-1}x^{m-n-1}+\ldots+a_1x+a_0=0$ are x_n, x_{n+1}, $x_{n+2}, \ldots, x_m$. Let $P((n,m),Q_1,T)$ denote the winning probability of player Q_1 when reaching the ***mth*** round from ***nth*** round, then:

$$P((n,m),Q_1,T)=\frac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\frac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

Proof. Define Event A as the player Q_1 reaching the nth round, Event B as the player Q_1 reaching the mth round and Event C as the player Q_1 reaching the mth from nth round. Because of Event B lies in Event A, by condition probability, we have:

$$P(C)=P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{P(B)}{P(A)}$$

$$P((n,m),Q_1,T)=\frac{P(m,Q_1,T)}{P(n,Q_1,T)}=\frac{\dfrac{q_1^m}{\sum_{k=0}^{m}(-1)^k\frac{a_{m-k}}{a_m}q_1^{m-k}}}{\dfrac{q_1^n}{\sum_{k=0}^{n}(-1)^k\frac{a_{n-k}}{a_n}q_1^{n-k}}}=\frac{\prod_{k=1}^{m}P_{Q_1X_K}}{\prod_{k=1}^{n}P_{Q_1X_K}}=\prod_{k=n}^{n}P_{Q_1X_K}$$

$$=\frac{q_1^{m-n}}{(q_1+x_n)(q_1+x_{n+1})\ldots(q_1+x_m)}=\frac{q_1^{m-n}}{\sum_{k=0}^{(m-n)}(-1)^k\frac{a_{(m-n)-k}}{a_{(m-n)}}q_1^{n-k}}$$

6. Formula of Winning Probability (3)

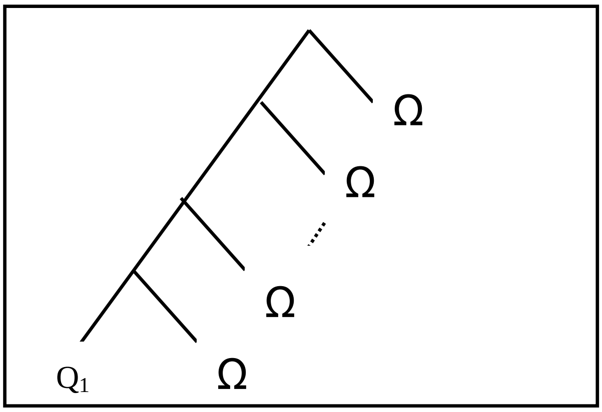

Fig.11

Besides, we see the tournament as facing the player Ω and winning n times continuously, how about the strength of the player Ω? For this reason, we have:

Theorem 9. Let $P(n,Q_1,T)$ denote the winning probability of player Q_1 when reaching the nth round based on scheme T, then:

$$P(n, Q_1, T) = \frac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k} \Omega^k} \qquad \Omega = \sqrt[n]{\prod_{k=1}^{n} (q_1 + x_k)} - q_1$$

Proof. By observing Fig.10 and Fig.11, we can obtain the winning probability of player Q_1 when reaching the nth round based on scheme T as follows.

$$P(n, Q_1, T) = \prod_{k=1}^{n} P_{Q_1 X_K} = (P_{Q_1 \Omega})^n$$

$$\frac{q_1^n}{\prod_{k=1}^{n} (q_1 + x_k)} = \frac{q_1^n}{(q_1 + \Omega)^n}$$

$$\sqrt[n]{\prod_{k=1}^{n} (q_1 + x_k)} = \Omega + q_1$$

$$\Omega = \sqrt[n]{\prod_{k=1}^{n} (q_1 + x_k)} - q_1$$

We obtain the strength of the player Ω and find the formula of winning probability as follows.

$$P_n(Q_1) = (P_{Q_1 \Omega})^n = \frac{q_1^n}{(q_1 + \Omega)^n} = \frac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k} \Omega^k}$$

From Theorem 9 and condition probability, we can confirm a proposition as follows.

Proposition 4. Let P((n,m),Q_1,T) denote the winning probability of player Q_1 when reaching the mth round from nth round, then:

$$P((n,m), Q_1, T) = \frac{q_1^{m-n}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k} \quad , \quad \Omega = \sqrt[m-n]{\prod_{k=n}^{m} (q_1 + x_k)} - q_1$$

Proof. Define Event A as the player Q_1 reaching the nth round, Event B as the player Q_1 reaching the mth round and Event C as the player Q_1 reaching the mth from nth round. Because of Event B lies in Event A, by condition probability, we have:

$$P(C) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$$

$$P((n,m),Q_1,T)=\frac{P(m,Q_1,T)}{P(n,Q_1,T)}=\frac{\prod_{k=1}^{m}P_{Q_1X_K}}{\prod_{k=1}^{n}P_{Q_1X_K}}=\prod_{k=n}^{m}P_{Q_1X_K}=(P_{Q_1\Omega})^{m-n}$$

$$=\frac{q_1^{m-n}}{(q_1+\Omega)^{m-n}}=\frac{q_1^{m-n}}{\sum_{k=0}^{m-n}C_k^{m-n}q_1^{m-n-k}\Omega^k}$$

The strength of the player Ω is as follows.

$$\frac{q_1^{m-n}}{\prod_{k=n}^{m}(q_1+x_k)}=\frac{q_1^{m-n}}{(q_1+\Omega)^{m-n}}$$

$$\sqrt[m-n]{\prod_{k=n}^{m}(q_1+x_k)}=\Omega+q_1$$

$$\Omega=\sqrt[m-n]{\prod_{k=n}^{m}(q_1+x_k)}-q_1$$

7. Formula of Winning Probability (4)

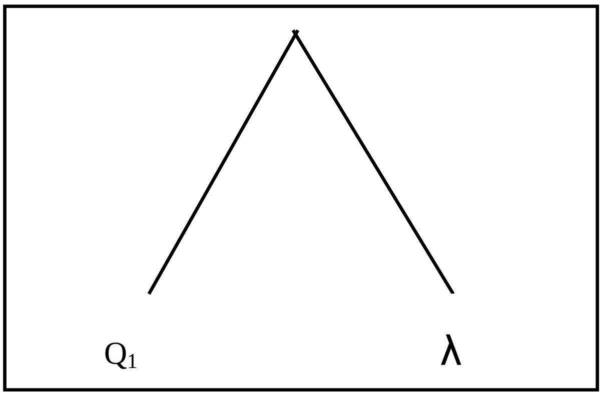

Fig.12

Furthermore, we see the tournament as facing just one player λ . How about the strength of the player λ? For this reason, we have:

Theorem 10. Let P (n,Q_1,T) denote the winning probability of player Q_1 when reaching the nth round based on scheme T, then::

$$P(n,Q_1,T)=\frac{q_1}{q_1+\lambda}\qquad \lambda=\frac{1-\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})}{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})}q_1$$

Proof. By observing Fig.12, we can obtain the winning probability of player Q_1 when reaching the nth round based on scheme T as follows.

$$\mathrm{P}(\mathrm{n},\mathrm{Q}_1,\mathrm{T}) = \prod_{\mathrm{j}=1}^{\mathrm{n}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (P_{\mathrm{j}-1}(Q_{\mathrm{i}}) P_{\mathrm{Q_1Q_i}}) = \mathrm{P}_{\mathrm{Q_1}\lambda}$$

$$\prod_{\mathrm{j}=1}^{\mathrm{n}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}}) = \frac{\mathrm{q}_1}{\mathrm{q}_1 + \lambda}$$

$$\lambda = \frac{1 - \prod_{\mathrm{j}=1}^{\mathrm{n}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j}=1}^{\mathrm{n}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})} \mathrm{q}_1$$

From Theorem 10 and condition probability, we can confirm a proposition as follows.

Proposition 5. Let P((n,m),Q_1,T) denote the winning probability of player Q_1 when reaching the mth round from nth round, then:

$$\mathrm{P}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T}) = \frac{\mathrm{q}_1}{\mathrm{q}_1 + \lambda} \quad , \quad \lambda = \frac{1 - \prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})} \mathrm{q}_1$$

Proof. Define Event A as the player Q_1 reaching the nth round, Event B as the player Q_1 reaching the mth round and Event C as the player Q_1 reaching the mth from nth round. Because of Event B lies in Event A, by condition probability, we have:

$$P(\mathrm{C}) = \mathrm{P}(\mathrm{B}|\mathrm{A}) = \frac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{\mathrm{P}(\mathrm{A})} = \frac{\mathrm{P}(\mathrm{B})}{\mathrm{P}(\mathrm{A})}$$

$$\mathrm{P}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T}) = \frac{\mathrm{P}(\mathrm{m},\mathrm{Q}_1,\mathrm{T})}{\mathrm{P}(\mathrm{n},\mathrm{Q}_1,\mathrm{T})} = \frac{\prod_{\mathrm{j}=1}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j}=1}^{\mathrm{n}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})} = \prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}}) = \mathrm{P}_{\mathrm{Q_1}\lambda}$$

$$\prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}}) = \frac{\mathrm{q}_1}{\mathrm{q}_1 + \lambda}$$

$$\lambda = \frac{1 - \prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})}{\prod_{\mathrm{j}=\mathrm{n}}^{\mathrm{m}} \sum_{i=2^{\mathrm{j}-1}+1}^{2^{\mathrm{j}}} (\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_{\mathrm{i}},\mathrm{T}) P_{\mathrm{Q_1Q_i}})} \mathrm{q}_1$$

II. The Influence of the Scheme of Players on Winning Probability

(I) Probability-to-Strength Ratio (PSR)

1. Probability-to-Strength Ratio (PSR) (1)

It is believed that" the stronger the player is, the higher winning probability he will have." However, if the scheme is fair to every player, we consider the ratio of winning probability to player's strength of every player should be equal. For this reason, we define "Probability-to-Strength Ratio" as the ratio of winning probability to player's strength, and we have a Theorem as follows.

Theorem 11.Let PSR(n,Q_1,T) denote PSR of player Q_1 when reaching the nth round based on scheme T, then:

$$\mathrm{PSR}(n,Q_1,T)=q_1^{\,n-1}\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}\frac{P(j-1,Q_i,T)}{q_1+q_i}$$

Proof. From the definition of PSR, we can confirm :

$$\mathrm{PSR}(n,Q_1,T)=\frac{P(n,Q_1,T)}{q_1}=\frac{\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})}{q_1}=q_1^{\,n-1}\prod_{j=1}^{n}\sum_{i=2^{j-1}+1}^{2^j}\frac{P(j-1,Q_i,T)}{q_1+q_i}$$

From Theorem 3 and Proposition 2, we can confirm a proposition as follows.

Proposition 6. Let PSR((n,m),Q_1,T) denote PSR of player Q_1 when reaching the mth round from the nth round, then:

$$\mathrm{PSR}((n,m),Q_1,T)=q_1^{\,m-n-1}\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})$$

Proof. From Proposition 2, we obtain the winning probability of the player Q_1 when reaching the mth round from nth round. Moreover, from Proposition 2 and the definition of "PSR," we can confirm:

$$\mathrm{PSR}(n,Q_1,T)=\frac{P(n,Q_1,T)}{q_1}=\frac{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})}{q_1}=q_1^{\,m-n-1}\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(P(j-1,Q_i,T)P_{Q_1Q_i})$$

2. Probability-to-Strength Ratio (PSR) (2)

In the same way, from the Formula of Winning Probability (2), we can confirm Probability-to-Strength Ratio (PSR) (2) as follows.

Theorem 12. The roots of an equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ are $x_1, x_2, x_3, \ldots,$ x_n. Let PSR(n,Q_1,T) denote PSR of player Q_1 when reaching the nth round based on scheme T, then:

$$\mathrm{PSR}(n, Q_1, T) = \frac{q_1^{n-1}}{\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}$$

Proof. From the definition of "Ratio of Winning Probability to Player's Strength," we can confirm :

$$\mathrm{PSR}(n, Q_1, T) = \frac{P(n, Q_1, T)}{q_1} = \frac{\dfrac{q_1^{n}}{\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}}{q_1} = \frac{q_1^{n-1}}{\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{a_n} q_1^{n-k}}$$

From Theorem 8 and Proposition 3, we can confirm a proposition as follows

Proposition 7. The roots of an equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ are $x_1, x_2, x_3, \ldots, x_n$. Let PSR((n,m),$Q_1$,T)denote PSR of player Q_1 when reaching the mth round from nth round, then:

$$\mathrm{PSR}((n, m), Q_1, T) = \frac{q_1^{m-n-1}}{\sum_{k=0}^{(m-n)} (-1)^k \frac{a_{(m-n)-k}}{a_{(m-n)}} q_1^{n-k}}$$

Proof. From Proposition 3, we obtain the winning probability of the player Q_1 when reaching the mth round from nth round. Moreover, from Proposition 2 and the definition of "PSR," we can confirm:

$$F_{n,m}(Q_1) = \frac{P_{n,m}(Q_1)}{q_1} = \frac{\dfrac{q_1^{m-n}}{\sum_{k=0}^{(m-n)} (-1)^k \frac{a_{(m-n)-k}}{a_{(m-n)}} q_1^{n-k}}}{q_1} = \frac{q_1^{m-n-1}}{\sum_{k=0}^{(m-n)} (-1)^k \frac{a_{(m-n)-k}}{a_{(m-n)}} q_1^{n-k}}$$

3. Ratio of Winning Probability to Player's Strength (3)

In the same way, from the Formula of Winning Probability (3), we can confirm Ratio of Winning Probability to Player's Strength (3) as follows.

Theorem 13. Let $F_n(Q_1)$ denote the ratio of winning probability to player's strength of the player Q_1 reached to N round, then:

$$F_n(Q_1) = \frac{q_1^{n-1}}{\sum_{k=0}^{n} C_k^n q_1^{n-k} \Omega^k} \qquad \Omega = \sqrt[n]{\prod_{k=1}^{n} (q_1 + x_k)} - q_1$$

Proof. From the definition of PSR, we can confirm :

$$PSR(n, Q_1, T) = \frac{P(n, Q_1, T)}{q_1} = \frac{\dfrac{q_1^n}{\sum_{k=0}^{n} C_k^n q_1^{n-k} \Omega^k}}{q_1} = \frac{q_1^{n-1}}{\sum_{k=0}^{n} C_k^n q_1^{n-k} \Omega^k}$$

From Theorem 9 and Proposition 4, we can confirm a proposition as follows.

Proposition 8. Let PSR((n,m),Q_1,T) denote PSR of player Q_1 when reaching the mth round from the nth round, then:

$$PSR((n,m), Q_1, T) = \frac{q_1^{m-n-1}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k} \qquad \Omega = \sqrt[m-n]{\prod_{k=n}^{m} (q_1 + x_k)} - q_1$$

Proof. From Proposition 4, we obtain the winning probability of the player Q_1 when reaching the mth round from nth round. Moreover, from Proposition 2 and the definition of "PSR," we can confirm:

$$PSR(n, Q_1, T) = \frac{P(n, Q_1, T)}{q_1} = \frac{\dfrac{q_1^{m-n}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k}}{q_1} = \frac{q_1^{m-n-1}}{\sum_{k=0}^{m-n} C_k^{m-n} q_1^{m-n-k} \Omega^k}$$

4. Probability-to-Strength Ratio (PSR) (4)

In the same way, from the Formula of Winning Probability (4), we can confirm Probability-to-Strength Ratio (PSR) (4) as follows.

Theorem 14. Let PSR(n,Q_1,T) denote PSR of player Q_1 when reaching the nth round based on scheme T, then:

$$PSR(n, Q_1, T) = \frac{1}{q_1 + \lambda} \qquad \lambda = \frac{1 - \prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} (P(j-1, Q_i, T) P_{Q_1 Q_i})}{\prod_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} (P(j-1, Q_i, T) P_{Q_1 Q_i})} q_1$$

Proof. From the definition of "Ratio of Winning Probability to Player's Strength," we can confirm :

$$PSR(n, Q_1, T) = \frac{P(n, Q_1, T)}{q_1} = \frac{\dfrac{q_1}{q_1 + \lambda}}{q_1} = \frac{1}{q_1 + \lambda}$$

From Theorem 10 and Proposition 5, we can confirm a proposition as follows.

Proposition 9. Let PSR((n,m),Q_1,T)denote PSR of player Q_1 when reaching the mth round from nth round, then:

$$\mathrm{PSR}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T}) = \frac{1}{\mathrm{q}_1+\lambda} \qquad \lambda = \frac{1-\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_i,\mathrm{T})P_{\mathrm{Q}_1\mathrm{Q}_i})}{\prod_{j=n}^{m}\sum_{i=2^{j-1}+1}^{2^j}(\mathrm{P}(\mathrm{j}\text{-}1,\mathrm{Q}_i,\mathrm{T})P_{\mathrm{Q}_1\mathrm{Q}_i})}\mathrm{q}_1$$

Proof. From Proposition 5, we obtain the winning probability of the player Q_1 when reaching the mth round from nth round. Moreover, from Proposition 2 and the definition of "PSR," we can confirm:

$$\mathrm{PSR}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T}) = \frac{\mathrm{P}((\mathrm{n},\mathrm{m}),\mathrm{Q}_1,\mathrm{T})}{\mathrm{q}_1} = \frac{\frac{\mathrm{q}_1}{\mathrm{q}_1+\lambda}}{\mathrm{q}_1} = \frac{1}{\mathrm{q}_1+\lambda}$$

(II) The Deviation from the Optimum Scheme

1. Influence of the Scheme of Players on Winning Probability

If the scheme is fair to every player, we consider the PSR of every player should be equal. We calculate the PSR for each player based on all possible scheme and examine whether the PSR of each player is equal or not.

There are 3 possible schemes in total of a knockout tournament plan among 4 players as Fig.13, Fig.14, and Fig.15.

Name Fig.13, Fig.14, and Fig.15 as T_1,T_2 and T_3 .Without loss of generality, we assume the strength a,b,c and d of player A,B,C and D satisfy $a \geqq b \geqq c \geqq d$. By calculating "PSR," we can find the optimum scheme for each player.

Let PSR(n,Q_1,T) denote PSR of player Q_1 when reaching the nth round based on scheme T. We can confirm some lemma from the player A,B,C and D separately as follows.

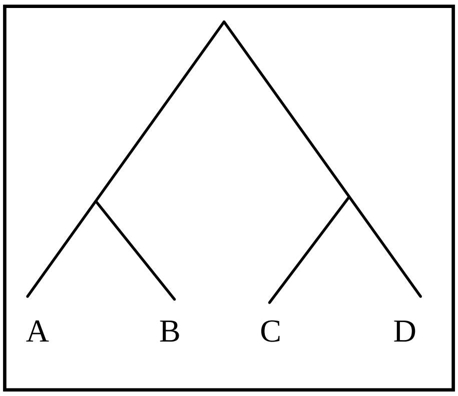

Fig.13 Scheme *T_1*

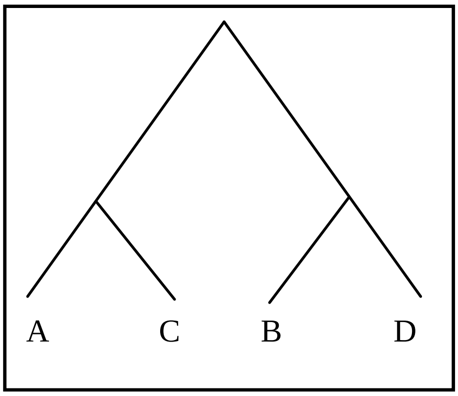

Fig.14 Scheme *T_2*

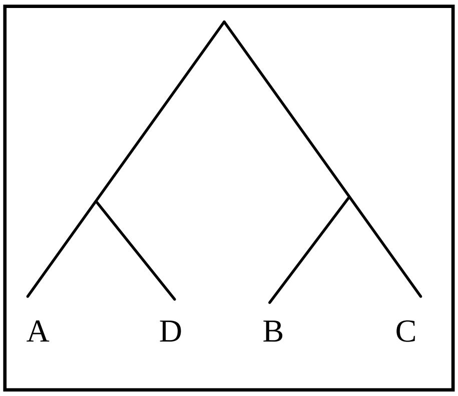

Fig.15 Scheme *T_3*

Lemma2. The PSR of the player A when reaching the 2nd round satisfy:

$$\mathrm{PSR}(2,\mathrm{A},\mathrm{T}_3) > \mathrm{PSR}(2,\mathrm{A},\mathrm{T}_2) > \mathrm{PSR}(2,\mathrm{A},\mathrm{T}_1)$$

Proof. Enumerate the PSR of player A when reaching the 2nd round based on the scheme T_1 ,T_2 and T_3.

$$PSR(2,A,T_1)=a\frac{P_0(B)}{a+b}(\frac{P_1(C)}{a+c}+\frac{P_1(D)}{a+d})$$

$$PSR(2,A,T_2)=a\frac{P_0(C)}{a+c}(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d})$$

$$PSR(2,A,T_3)=a\frac{P_0(D)}{a+d}(\frac{P_1(B)}{a+b}+\frac{P_1(C)}{a+c})$$

First, compare the value of $PSR(2,A,T_3)$ with the value of $PSR(2,A,T_2)$.

$$PSR(2,A,T_3) - PSR(2,A,T_2)$$

$$=a\frac{P_0(D)}{a+d}(\frac{P_1(B)}{a+b}+\frac{P_1(C)}{a+c}) - a\frac{P_0(C)}{a+c}(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d})$$

$$=a((\frac{P_0(D)}{a+d}-\frac{P_0(C)}{a+c})\frac{P_1(B)}{a+b}+\frac{P_0(D)P_1(C)-P_0(C)P_1(D)}{(a+c)(a+d)})$$

$$=a((\frac{1}{a+d}-\frac{1}{a+c})\frac{P_1(B)}{a+b}+\frac{P_1(C)-P_1(D)}{(a+c)(a+d)})>0$$

$$=> PSR(2,A,T_3) > PSR(2,A,T_2) \;......\; (1)$$

Second, compare the value of $PSR(2,A,T_2)$ with the value of $PSR(2,A,T_1)$.

$$PSR(2,A,T_2) - PSR(2,A,T_1)$$

$$=a\frac{P_0(C)}{a+c}(\frac{P_1(B)}{a+b}+\frac{P_1(D)}{a+d}) - a\frac{P_0(B)}{a+b}(\frac{P_1(C)}{a+c}+\frac{P_1(D)}{a+d})$$

$$=a((\frac{P_0(C)}{a+c}-\frac{P_0(B)}{a+b})\frac{P_1(D)}{a+d}+\frac{P_0(C)P_1(B)-P_0(B)P_1(C)}{(a+b)(a+c)})$$

$$=a((\frac{1}{a+c}-\frac{1}{a+b})\frac{P_1(D)}{a+d}+\frac{P_1(B)-P_1(C)}{(a+b)(a+c)})>0$$

$$=> PSR(2,A,T_2) > PSR(2,A,T_1) \;......\; (2)$$

Combine (1) and (2),we can conclude that:

$$PSR(2,A,T_3) > PSR(2,A,T_2) > PSR(2,A,T_1)$$

Lemma 3. The PSR of the player B when reaching the 2nd round satisfy:

$$PSR(2,B,T_2) > PSR(2,B,T_3) > PSR(2,B,T_1)$$

Proof. Enumerate the PSR of player B when reaching the 2nd round based on the scheme T_1 ,T_2 and T_3.

$$PSR(2,B,T_1)=b\frac{P_0(A)}{a+b}(\frac{P_1(C)}{b+c}+\frac{P_1(D)}{b+d})$$

$$PSR(2,B,T_2)=b\frac{P_0(D)}{b+d}(\frac{P_1(A)}{a+b}+\frac{P_1(C)}{b+c})$$

$$PSR(2,B,T_3)=b\frac{P_0(C)}{b+c}(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d})$$

First, compare the value of $PSR(2,B,T_2)$ with the value of $PSR(2,B,T_3)$.

$$PSR(2,B,T_2)-PSR(2,B,T_3)$$

$$=b\frac{P_0(D)}{b+d}(\frac{P_1(A)}{a+b}+\frac{P_1(C)}{b+c}) - b\frac{P_0(C)}{b+c}(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d})$$

$$=b((\frac{P_0(D)}{b+d}-\frac{P_0(C)}{b+c})\frac{P_1(A)}{a+b}+\frac{P_0(D)P_1(C)-P_0(C)P_1(D)}{(b+c)(b+d)})$$

$$=b((\frac{1}{b+d}-\frac{1}{b+c})\frac{P_1(A)}{a+b}+\frac{P_1(C)-P_1(D)}{(b+c)(b+d)})>0$$

=> PSR(2,B,T_2) > PSR(2,B,T_3) ...... (1)

Second, compare the value of PSR(2,B,T_3) with the value of PSR(2,B,T_1).

PSR(2,B,T_3) >PSR(2,B,T_1)

$$=b\frac{P_0(C)}{b+c}(\frac{P_1(A)}{a+b}+\frac{P_1(D)}{b+d})-b\frac{P_0(A)}{a+b}(\frac{P_1(C)}{b+c}+\frac{P_1(D)}{b+d})$$

$$=b((\frac{P_0(C)}{b+c}-\frac{P_0(A)}{a+b})\frac{P_1(D)}{b+d}+\frac{P_0(C)P_1(A)-P_0(A)P_1(C)}{(a+b)(b+c)})$$

$$=b((\frac{1}{b+c}-\frac{1}{a+b})\frac{P_1(D)}{b+d}+\frac{P_1(A)-P_1(C)}{(a+b)(b+c)})>0$$

=> PSR(2,A,T_2) > PSR(2,A,T_1) ...... (2)

Combine (1) and (2),we can conclude that:

PSR(2,B,T_2) > PSR(2,B,T_3) > PSR(2,B,T_1)

Lemma 4. The PSR of the player C when reaching the 2nd round satisfy:

PSR(2,C,T_1) > PSR(2,C,T_3) > PSR(2,C,T_2)

Proof. Enumerate the PSR of player A when reaching the 2nd round based on the scheme T_1 ,T_2 and T_3.

$$PSR(2,C,T_1)=c\frac{P_0(D)}{c+d}(\frac{P_1(A)}{a+c}+\frac{P_1(B)}{b+c})$$

$$PSR(2,C,T_2)=c\frac{P_0(A)}{a+c}(\frac{P_1(B)}{b+c}+\frac{P_1(D)}{c+d})$$

$$PSR(2,C,T_3)=c\frac{P_0(B)}{b+c}(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d})$$

First, compare the value of PSR(2,C,T_1) with the value of PSR(2,C,T_3).

PSR(2,C,T_1) - PSR(2,C,T_3)

$$=c\frac{P_0(D)}{c+d}(\frac{P_1(A)}{a+c}+\frac{P_1(B)}{b+c})-c\frac{P_0(B)}{b+c}(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d})$$

$$=c((\frac{P_0(D)}{c+d}-\frac{P_0(B)}{b+c})\frac{P_1(A)}{a+c}+\frac{P_0(D)P_1(B)-P_0(B)P_1(D)}{(b+c)(c+d)})$$

$$=c((\frac{1}{c+d}-\frac{1}{b+c})\frac{P_1(A)}{a+c}+\frac{P_1(B)-P_1(D)}{(b+c)(c+d)})>0$$

=> PSR(2,C,T_1) > PSR(2,C,T_3) ...... (1)

Second, compare the value of PSR(2,C,T_3) with the value of PSR(2,C,T_2).

PSR(2,C,T_3) - PSR(2,C,T_2)

$$=c\frac{P_0(B)}{b+c}(\frac{P_1(A)}{a+c}+\frac{P_1(D)}{c+d})-c\frac{P_0(A)}{a+c}(\frac{P_1(B)}{b+c}+\frac{P_1(D)}{c+d})$$

$$= c((\frac{P_0(B)}{b+c} - \frac{P_0(A)}{a+c})\frac{P_1(D)}{c+d} + \frac{P_0(B)P_1(A) - P_0(A)P_1(B)}{(a+c)(b+c)})$$

$$= (c(\frac{1}{b+c} - \frac{1}{a+c})\frac{P_1(D)}{c+d} + \frac{P_1(A) - P_1(B)}{(a+c)(b+c)}) > 0$$

=> PSR(2,C,T_3) > PSR(2,C,T_2) ...... (2)

Combine (1) and (2),we can conclude that:

PSR(2,C,T_1) > PSR(2,C,T_3) > PSR(2,C,T_2)

Lemma 5. The PSR of the player D when reaching the 2nd round satisfy:

PSR(2,D,T_1) > PSR(2,D,T_2) > PSR(2,D,T_3)

Proof. Enumerate the PSR of player A when reaching the 2nd round based on the scheme T_1 ,T_2 and T_3.

$$PSR(2, D, T_1) = d\frac{P_0(C)}{c+d}(\frac{P_1(A)}{a+d} + \frac{P_1(B)}{b+d})$$

$$PSR(2, D, T_2) = d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d} + \frac{P_1(C)}{c+d})$$

$$PSR(2, D, T_3) = d\frac{P_0(A)}{a+d}(\frac{P_1(B)}{b+d} + \frac{P_1(C)}{c+d})$$

First, compare the value of PSR(2,D,T_1) with the value of PSR(2,D,T_2).

PSR(2,D,T_1)- PSR(2,D,T_2)

$$= d\frac{P_0(C)}{c+d}(\frac{P_1(A)}{a+d} + \frac{P_1(B)}{b+d}) - d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d} + \frac{P_1(C)}{c+d})$$

$$= d((\frac{P_0(C)}{c+d} - \frac{P_0(B)}{b+d})\frac{P_1(A)}{a+d} + \frac{P_0(C)P_1(B) - P_0(B)P_1(C)}{(b+d)(c+d)})$$

$$= d((\frac{1}{c+d} - \frac{1}{b+d})\frac{P_1(A)}{a+d} + \frac{P_1(B) - P_1(C)}{(b+d)(c+d)}) > 0$$

=> PSR(2,D,T_1) > PSR(2,D,T_2) ...... (1)

Second, compare the value of PSR(2,D,T_2) with the value of PSR(2,D,T_3).

PSR(2,D,T_2) - PSR(2,D,T_3)

$$= d\frac{P_0(B)}{b+d}(\frac{P_1(A)}{a+d} + \frac{P_1(C)}{c+d}) - d\frac{P_0(A)}{a+d}(\frac{P_1(B)}{b+d} + \frac{P_1(C)}{c+d})$$

$$= d((\frac{P_0(B)}{b+d} - \frac{P_0(A)}{a+d})\frac{P_1(C)}{c+d} + \frac{P_0(B)P_1(A) - P_0(A)P_1(B)}{(a+d)(b+d)})$$

$$= d((\frac{1}{b+d} - \frac{1}{a+d})\frac{P_1(C)}{c+d} + \frac{P_1(A) - P_1(B)}{(a+d)(b+d)}) > 0$$

=> PSR(2,D,T_2) > PSR(2,D,T_3) ...... (2)

Combine (1) and (2),we can conclude that:

PSR(2,D,T_1) > PSR(2,D,T_2) > PSR(2,D,T_3)

Combine lemma 2~5, we can conclude a significant theorem of a knockout tournament among 4 players as follows.

Theorem 15. At a tournament plan among 4 players, the scheme is optimum for a player if and only if the player competes against the weakest player (except of himself) at the first round.

Proof. The following forms reveal the consequence of lemma 2~5.

	A	B	C	D
best	T_3	T_2	T_1	T_1
medium	T_2	T_3	T_3	T_2
worst	T_1	T_1	T_2	T_3

	A	B	C	D
T_1	worst	worst	best	best
T_2	medium	best	worst	medium
T_3	best	medium	medium	worst

By observing the best scheme of the player, we can find that at a knockout tournament among 4 players, the best scheme of the player satisfy that competing against the weakest player except himself at the first round. According the theorem, we can find the optimum scheme of the player for every player at a knockout tournament among 4 players.

2. The Deviation from Optimum Scheme

From Theorem 15, we realize the winning probability would vary from scheme to scheme. Therefore, we want to mesure how much the scheme deviate from the optimum scheme. For this reason, we take the ratio of the PSR based on scheme T to the PSR based on the optimum scheme.

Thus, we have a theorem as follows

Theorem 16. The *PSR Ratio* of player Q_1 when reaching the 2nd round based on scheme *T* is:

$$\frac{PSR(2, Q_1, T)}{PSR(2, Q_1, T_{Optimum})} = \frac{q_1 + \dfrac{2q_y q_z}{q_y + q_z}}{q_1 + \dfrac{2q_x q_y}{q_x + q_y}}$$

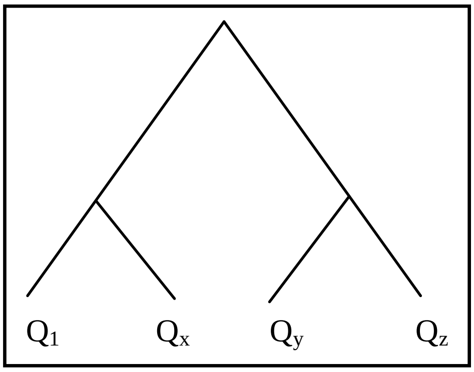

Fig.16 Scheme T

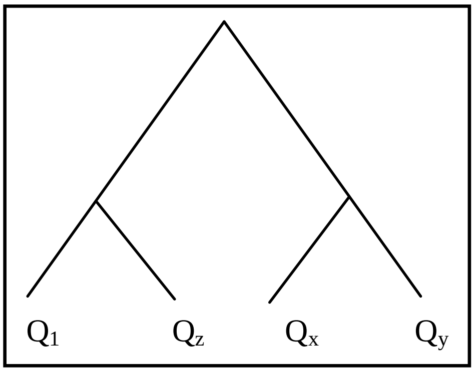

Fig. 17 Scheme $T_{Qptimum}$

Proof. Name Fig. 16 as the Scheme T_j of the player Q_1 and Fig. 2.5 as the Optimum Scheme $T_{Qptimum}$ of the player Q_1. Enumerate the Ratio of Winning Probability to Player's Strength at the scheme as follows and we have:

$$\mathrm{PSR}(2,Q_1,T)=q_i\frac{P_0(Q_x)}{q_i+q_x}\left(\frac{P_1(Q_y)}{q_i+q_y}+\frac{P_1(Q_z)}{q_i+q_z}\right)=q_i\frac{q_i(P_1(Q_y)+P_1(Q_z))+P_1(Q_y)q_z+P_1(Q_z)q_y}{(q_i+q_x)(q_i+q_y)(q_i+q_z)}$$

$$=\frac{q_i\left(q_i+\dfrac{2q_yq_z}{q_y+q_z}\right)}{(q_i+q_x)(q_i+q_y)(q_i+q_z)}$$

$$\mathrm{PSR}(2,Q_1,T_{\mathrm{Optimum}})=q_i\frac{P_0(Q_z)}{q_i+q_z}\left(\frac{P_1(Q_x)}{q_i+q_x}+\frac{P_1(Q_y)}{q_i+q_y}\right)=q_i\frac{q_i(P_1(Q_x)+P_1(Q_y))+P_1(Q_x)q_y+P_1(Q_y)q_x}{(q_i+q_x)(q_i+q_y)(q_i+q_z)}$$

$$=\frac{q_i\left(q_i+\dfrac{2q_xq_y}{q_x+q_y}\right)}{(q_i+q_x)(q_i+q_y)(q_i+q_z)}$$

$$\frac{\mathrm{PSR}(2,Q_1,T)}{\mathrm{PSR}(2,Q_1,T_{\mathrm{Optimum}})}=\frac{q_i+\dfrac{2q_yq_z}{q_y+q_z}}{q_i+\dfrac{2q_xq_y}{q_x+q_y}}$$

From Theorem 16, we can confirm the PSR ratio of each player at the knockout tournament among 4 players as Proposition 10 ~ 13.

Proposition 10. The PSR ratio of player A:

$$\frac{\mathrm{PSR}(2,A,T_1)}{\mathrm{PSR}(2,A,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2cd}{c+d}}{a+\dfrac{2bc}{b+c}},\quad \frac{\mathrm{PSR}(2,A,T_2)}{\mathrm{PSR}(2,A,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2bd}{b+d}}{a+\dfrac{2bc}{b+c}},\quad \frac{\mathrm{PSR}(2,A,T_3)}{\mathrm{PSR}(2,A,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2bc}{b+c}}{a+\dfrac{2bc}{b+c}}$$

Proof.

$$\mathrm{PSR}(2,A,T_1)=\frac{a\left(a+\dfrac{2cd}{c+d}\right)}{(a+b)(a+c)(a+d)}\qquad \mathrm{PSR}(2,A,T_2)=\frac{a\left(a+\dfrac{2bd}{b+d}\right)}{(a+b)(a+c)(a+d)}$$

$$\mathrm{PSR}(2,A,T_3)=\frac{a\left(a+\dfrac{2bc}{b+c}\right)}{(a+b)(a+c)(a+d)}$$

$$\frac{\mathrm{PSR}(2,Q_1,T_1)}{\mathrm{PSR}(2,Q_1,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2cd}{c+d}}{a+\dfrac{2bc}{b+c}}\quad \frac{\mathrm{PSR}(2,Q_1,T_2)}{\mathrm{PSR}(2,Q_1,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2bd}{b+d}}{a+\dfrac{2bc}{b+c}}\quad \frac{\mathrm{PSR}(2,Q_1,T_3)}{\mathrm{PSR}(2,Q_1,T_{\mathrm{Optimum}})}=\frac{a+\dfrac{2bc}{b+c}}{a+\dfrac{2bc}{b+c}}$$

Proposition 11. The PSR ratio of player B:

$$\frac{\mathrm{PSR}(2,B,T_1)}{\mathrm{PSR}(2,B,T_{\mathrm{Optimum}})}=\frac{b+\dfrac{2cd}{c+d}}{b+\dfrac{2ac}{a+c}}\text{、}\frac{\mathrm{PSR}(2,B,T_2)}{\mathrm{PSR}(2,B,T_{\mathrm{Optimum}})}=\frac{b+\dfrac{2ac}{a+c}}{b+\dfrac{2ac}{a+c}}\text{、}\frac{\mathrm{PSR}(2,B,T_3)}{\mathrm{PSR}(2,B,T_{\mathrm{Optimum}})}=\frac{b+\dfrac{2ad}{a+d}}{b+\dfrac{2ac}{a+c}}$$

Proof.

$$PSR(2,B,T_1)=\frac{b(b+\frac{2cd}{c+d})}{(a+b)(b+c)(b+d)} \qquad PSR(2,B,T_2)=\frac{b(b+\frac{2ac}{a+c})}{(a+b)(b+c)(b+d)}$$

$$PSR(2,B,T_3)=\frac{b(b+\frac{2ad}{a+d})}{(a+b)(b+c)(b+d)}$$

$$\frac{PSR(2,B,T_1)}{PSR(2,B,T_{Optimum})}=\frac{b+\frac{2cd}{c+d}}{b+\frac{2ac}{a+c}} \quad \frac{PSR(2,B,T_2)}{PSR(2,B,T_{Optimum})}=\frac{b+\frac{2ac}{a+c}}{b+\frac{2ac}{a+c}} \quad \frac{PSR(2,B,T_3)}{PSR(2,B,T_{Optimum})}=\frac{b+\frac{2ad}{a+d}}{b+\frac{2ac}{a+c}}$$

Proposition 12. The PSR ratio of player C:

$$\frac{PSR(2,C,T_1)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2ab}{a+b}}{c+\frac{2ab}{a+b}} 、 \frac{PSR(2,C,T_2)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2bd}{b+d}}{c+\frac{2ab}{a+b}} 、 \frac{PSR(2,C,T_3)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2ad}{a+d}}{c+\frac{2ab}{a+b}}$$

Proof.

$$PSR(2,C,T_1)=\frac{c(c+\frac{2ab}{a+b})}{(a+c)(b+c)(c+d)} \qquad PSR(2,C,T_1)=\frac{c(c+\frac{2bd}{b+d})}{(a+c)(b+c)(c+d)}$$

$$PSR(2,C,T_1)=\frac{c(c+\frac{2ad}{a+d})}{(a+c)(b+c)(c+d)}$$

$$\frac{PSR(2,C,T_1)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2ab}{a+b}}{c+\frac{2ab}{a+b}} \quad \frac{PSR(2,C,T_2)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2bd}{b+d}}{c+\frac{2ab}{a+b}} \quad \frac{PSR(2,C,T_3)}{PSR(2,C,T_{Optimum})}=\frac{c+\frac{2ad}{a+d}}{c+\frac{2ab}{a+b}}$$

Proposition 13. The PSR ratio of player D:

$$\frac{PSR(2,D,T_1)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2ab}{a+b}}{d+\frac{2ab}{a+b}} 、 \frac{PSR(2,D,T_2)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2ac}{a+c}}{d+\frac{2ab}{a+b}} 、 \frac{PSR(2,D,T_3)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2bc}{b+c}}{d+\frac{2ab}{a+b}}$$

Proof.

$$PSR(2,D,T_1)=\frac{d(d+\frac{2ab}{a+b})}{(a+d)(b+d)(c+d)} \qquad PSR(2,D,T_2)=\frac{d(d+\frac{2ac}{a+c})}{(a+d)(b+d)(c+d)}$$

$$PSR(2,D,T_3)=\frac{d(d+\frac{2bc}{b+c})}{(a+d)(b+d)(c+d)}$$

$$\frac{PSR(2,D,T_1)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2ab}{a+b}}{d+\frac{2ab}{a+b}} \quad \frac{PSR(2,D,T_2)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2ac}{a+c}}{d+\frac{2ab}{a+b}} \quad \frac{PSR(2,D,T_3)}{PSR(2,D,T_{Optimum})}=\frac{d+\frac{2bc}{b+c}}{d+\frac{2ab}{a+b}}$$

After all, how much the influence of the scheme caused on the winning probability of the player? We make some examples. One is that the strength of the player satisfy the arithmetic progression, another is that the strength of the player satisfy geometric progression. Let's see the consequence as follows.

(1) The Strength of the Player Satisfy the Arithmetic Progression

The strength of the player A,B,C and D satisfy that a = 4d、b = 3d、c = 2d、d = d. We show The PSR ratio of every player as follows form.

Arithmetic progression (%)	A	B	C	D
T_1	83	76	100	100
T_2	86	100	64	82
T_3	100	81	66	77

(2)The Strength of the Player Satisfy Geometric Progression

The strength of the player A,B,C and D satisfy that a = 8d、b = 4d、c = 2d、d = d. We show The PSR ratio of every player as follows form.

Geometric progression(%)	A	B	C	D
T_1	70	77	100	100
T_2	72	100	49	66
T_3	100	83	52	58

In the international game, the adopted scheme is T_3. From The PSR ratio, we can realize that the scheme is beneficial the strongest player and it will decrease the winning probability of other player.

III. Inequality of Winning Probability

1. The Maximum Value of Winning Probability from the First round to the Second round

When we know all the strength of the player, how about the maximum value of Winning Probability reached from the first round to the second round?

Lemma 6. If the player Q_1 may compete against the player Q_3 and the player Q_4 at the second round, then the maximum of winning probability of the player Q_1 when reaching the 2nd round from the 1st round is:

$$P((1,2),Q_1,T) \le \frac{P^2_{Q_1Q_3} + P^2_{Q_1Q_4}}{P_{Q_1Q_3} + P_{Q_1Q_4}}$$

Proof. We can know the winning probability of the player Q_1 when reaching the 2nd round from the 1st round is:

$$P((1,2),Q_1,T) = P(1,Q_3,T)P_{Q_1Q_3} + P(1,Q_4,T)P_{Q_1Q_4}$$

From Cauchy's inequality, we can know:

$$(P^2(1,Q_3,T) + P^2(1,Q_4,T))(P_{Q_1Q_3}{}^2 + P_{Q_1Q_4}{}^2)$$
$$\ge (P(1,Q_3,T)P_{Q_1Q_3} + P(1,Q_4,T)P_{Q_1Q_4})^2 = P^2((1,2),Q_1,T)$$

If (1) be satisfied, the equality holds.

$$\frac{P(1,Q_3,T)}{P_{Q_1Q_3}} = \frac{P(1,Q_4,T)}{P_{Q_1Q_4}} = t \quad ......(1)$$

Then P ((1,2),Q_1,T) have the maximum. Besides, from (1), we have:

$$P_1(Q_3) = tP_{Q_1Q_3} \quad , \quad P_1(Q_4) = tP_{Q_1Q_4}$$

The sum of the winning probability of the player Q_3 and the player Q_4 equals 1.

$$P(1,Q_3,T) + P(1,Q_4,T) = 1$$

$$t(P_{Q_1Q_3} + P_{Q_1Q_4}) = 1, \; t = \frac{1}{P_{Q_1Q_3} + P_{Q_1Q_4}}$$

We have:

$$P(1,Q_3,T) = \frac{P_{Q_1Q_3}}{P_{Q_1Q_3} + P_{Q_1Q_4}}, \quad P(1,Q_4,T) = \frac{P_{Q_1Q_4}}{P_{Q_1Q_3} + P_{Q_1Q_4}}$$

Therefore, we can confirm the maximum of the winning probability of the player Q_1 when reaching the 2nd round from the 1st round as follows.

$$P((1,2),Q_1,T) = P(1,Q_3,T)P_{Q_1Q_3} + P(1,Q_3,T)P_{Q_1Q_4} \le \frac{P^2_{Q_1Q_3} + P^2_{Q_1Q_4}}{P_{Q_1Q_3} + P_{Q_1Q_4}}$$

2. The Maximum Value of Winning Probability reached from the n-1st round to the nth round

Theorem 17. If the player Q_1 may compete against the player $Q2^{n-1}+1$, $Q2^{n-1}+2$,..., $Q2^{n}$ at the nth round, then the maximum value of winning probability of player Q_1 when reaching the nth round from the n-1th is:

$$P((n-1,n),Q_1,T) \le \frac{\sum_{i=2^{n-1}+1}^{2^n} P^2_{Q_1Q_i}}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$$

Proof. We can know the winning probability of the player Q_1 when reaching the nth round from the n-1th is:

$$P((n-1,n),Q_1,T) = \sum_{i=2^{n-1}+1}^{2^n} P(n-1,Q_i,T)P_{Q_1Q_i})$$

From Cauchy's inequality, we can know:

$$\sum_{i=2^{n-1}+1}^{2^n} P^2(n-1,Q_i,T) \sum_{i=2^{n-1}+1}^{2^n} P^2{}_{Q_1Q_i} \ge (\sum_{i=2^{n-1}+1}^{2^n}(P(n-1,Q_i,T)P_{Q_1Q_i}))^2 = P^2((n-1,n),Q_1,T)$$

If (1) be satisfied, the equality holds.

$$\frac{P(n-1,Q_i,T)}{P_{Q_1Qi}} = t \ \ , i = 2^{n-1}+1,\ldots,2^n \ \ldots\ldots(1)$$

Then P $((n-1,n),Q_1,T)$ have the maximum. Besides, from (1), we have:

$$P(n-1,Q_i,T) = tP_{Q_1Q_i}, i = 2^{n-1}+1 \sim 2^n$$

The sum of the winning probability of the player $Q2^{n-1}+1$, $Q2^{n-1}+2$,..., $Q2^n$ equals 1.

$$\sum_{i=2^{n-1}+1}^{2^n} P_{n-1}(Q_i) = 1 \quad , \quad \sum_{i=2^{n-1}+1}^{2^n} t\,P_{Q_1Q_i} \ \ = 1$$

We have:

$$t = \frac{1}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}} \quad , \quad P_{n-1}(Q_i) = \frac{P_{Q_1Q_i}}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$$

Therefore, we can confirm the maximum value of the winning probability of the player Q_1 when reaching the nth round from the n-1th as follows.

$$P_{n-1,n}(Q_1) = \sum_{i=2^{n-1}+1}^{2^n}(P_{n-1}(Q_i)P_{Q_1Q_i}) \le \frac{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}{}^2}{\sum_{i=2^{n-1}+1}^{2^n} P_{Q_1Q_i}}$$

3. Inequality of Winning Probability

Theorem 18. The Winning Probability of player Q_1 satisfy the inequality:

$$P(n, Q_1, T) \le \prod_{j=1}^{n} \frac{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}^2}{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}}$$

The equality holds if and only if all players have same strength.

Proof. Combine the Formula of Winning Probability and the maximum value of the winning probability of the player, we can conclude a theorem as follows.

$$P(n, Q_1, T) = \prod_{j=1}^{n} \sum_{i=2^{n-1}+1}^{2^n} (P(n-1, Q_i, T) P_{Q_1Q_i}) \le \prod_{j=1}^{n} \frac{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}^2}{\sum_{i=2^{j-1}+1}^{2^j} P_{Q_1Q_i}}$$

Theorem 18 reveal an interesting truth. After decided the player we may compete against, our winning probability is be decreased. And after sure the scheme, our winning probability will be decreased again .

Conclusions

Results of this project clarify common beliefs in the single-elimination tournaments. We find it surprising that the conventional wisdom "Once the strength of our opponent increases, our winning probability will decrease" actually is false! This project may be of importance in explaining the fairness of single-elimination tournaments, as well as in providing a better understanding of single-elimination tournaments.

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