臺灣二ＯＯ八年國際科學展覽會

科　　　別：數學

作 品 名 稱：魔術猜牌-由再生訊息延伸推展猜中比值之研究

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作者簡介

我是江盛浩；就讀國立新竹科學工業園區實驗高級中學數理班二年級。

我所研究的題目是以去年參賽的「魔術猜牌」得出的「保證可猜中最多張花色的方法」，再深入的延伸探討。

這個猜牌數學題自國一開始接觸迄今已足足四年有餘，直至去年我才著手將此研究心得撰寫成報告參賽。經由去年評審老師所提出的問題與建議，以及會場上多件作品的觀摩，我發現我的研究內容與方向確有再深入與推廣的空間，於是再經一年的努力，很高興我的研究終於又往前跨出了一大步，得出了比我預期更好的結果。

魔術猜牌－由再生訊息延伸推展猜中比值之研究

中文摘要

本研究是藉由數學手法探討；如何由一疊 36 張四種花色的撲克牌中，尋找出保證可猜中最多張花色的方法。研究過程是以在適當的猜牌時機，以邏輯推理、二進位、分析與歸納……等數學原理與方法，搭配巧妙的策略運用而達到目的。猜牌方法：先約定好猜牌規則，助手將 36 張牌背圖樣相同但非對稱的撲克牌，以旋轉牌背的方向傳達訊息。在本研究中得出「經由巧妙的猜牌方法保證可以猜中不少於 26 張花色」，並得出「當總張數趨近於無窮大時，保證可以猜中不少於 81.07%的牌，並且證出若僅使用獨立的訊息猜牌，無論任何猜牌方法皆無法猜中多於 87.37%的牌」。其中一個猜中多於 80%的例子是：「當總張數等於 23006 張時，保證可以猜中不少於 18405 張牌($\frac{18405}{23006} > \frac{4}{5}$)」。

Magic Guessing ---

Study of the Guessing Ratio Developed from the Derived Information

Abstract

The study is mathematically based with reasonable explanations behind it. We are to correctly guess as many cards as possible from a deck of 36 cards, with random numbers and four different suits. We will apply mathematical methods, such as logic inference, binary system, and analytical reduction, upon right timing. Using careful arrangement of the principles and reasoning, we can reach our ultimate goal. To state guessing: Conference between the guesser and the assistant about the guessing rules, the assistant will have 36 cards with the same exact pattern on the back but not symmetrical. The pattern of the cards will be different when rotated 180º. The only communication between the two is by rotating cards. In this study, we can prove that through mathematical method, we can assure 26 or more cards can be correctly guessed. Furthermore, when the total amount of cards is close to infinity, we can assure 81.07% or more of the cards can be correctly guessed, and prove that if the cards are guessed from independent information, no more than 87.37% of the cards will be correctly guessed by any guessing methods. One of the examples, which 80% of the cards are correctly guessed, is that when the amount of the cards is 23006, 18405 or more of the cards can be correctly guessed. ($\frac{18405}{23006} > \frac{4}{5}$)

本　文

前　言

本文中所提「猜牌、猜對、猜中」等口語，係指運用數學原理與方法深入探討與推測所得之結果，並無機率、運氣之意。而「訊息」代表 0 或 1，「方法」則代表可能情況的量(例如 2 個「訊息」等於 4 種「方法」)。

壹、研究動機

在網路上數學討論區裡，大家熱烈的探討環球城市數學競賽高中組高級卷的題目；「小田宣稱他有魔法，可以從撲克牌的背面透視它的花色，於是大家把包括有黑桃、紅心、方塊及梅花各 9 張的一疊 36 張牌洗好後，交由小方把這疊牌的牌面向下放在小田面前，要他說出這疊牌最上面一張牌的花色。當小田說出答案後即把這張牌翻開來驗證他的答案是否正確，並將這張牌放在一旁不再加入這疊牌中；接著繼續猜測下一張牌，重複以上程序。他試圖使猜測的正確次數愈多愈好。這副牌背面的圖樣完全相同但並非對稱的(即可以分辨出圖案朝前或朝後)。小方雖然知道這疊牌的排列順序，但他不能更動，也不能偷偷告訴小田。但是他可以依照事先與小田約定好的方式擺置撲克牌背面圖案的朝向來暗中協助小田。事實上，小田並無魔法，他只是利用數學方法來作分析判斷。請問在小方的協助下，小田有沒有辦法保證正確地預測到 (a)不少於 19 張牌？ (b)不少於 20 張牌？」。

當我和他人討論之後覺得這個題目很有趣，而且解法應該還有改進的空間，因此我不斷地嘗試並改進已得出的解法；並利用解 36 張的方法，推廣研究至撲克牌總張數不等於 36 張的情況。

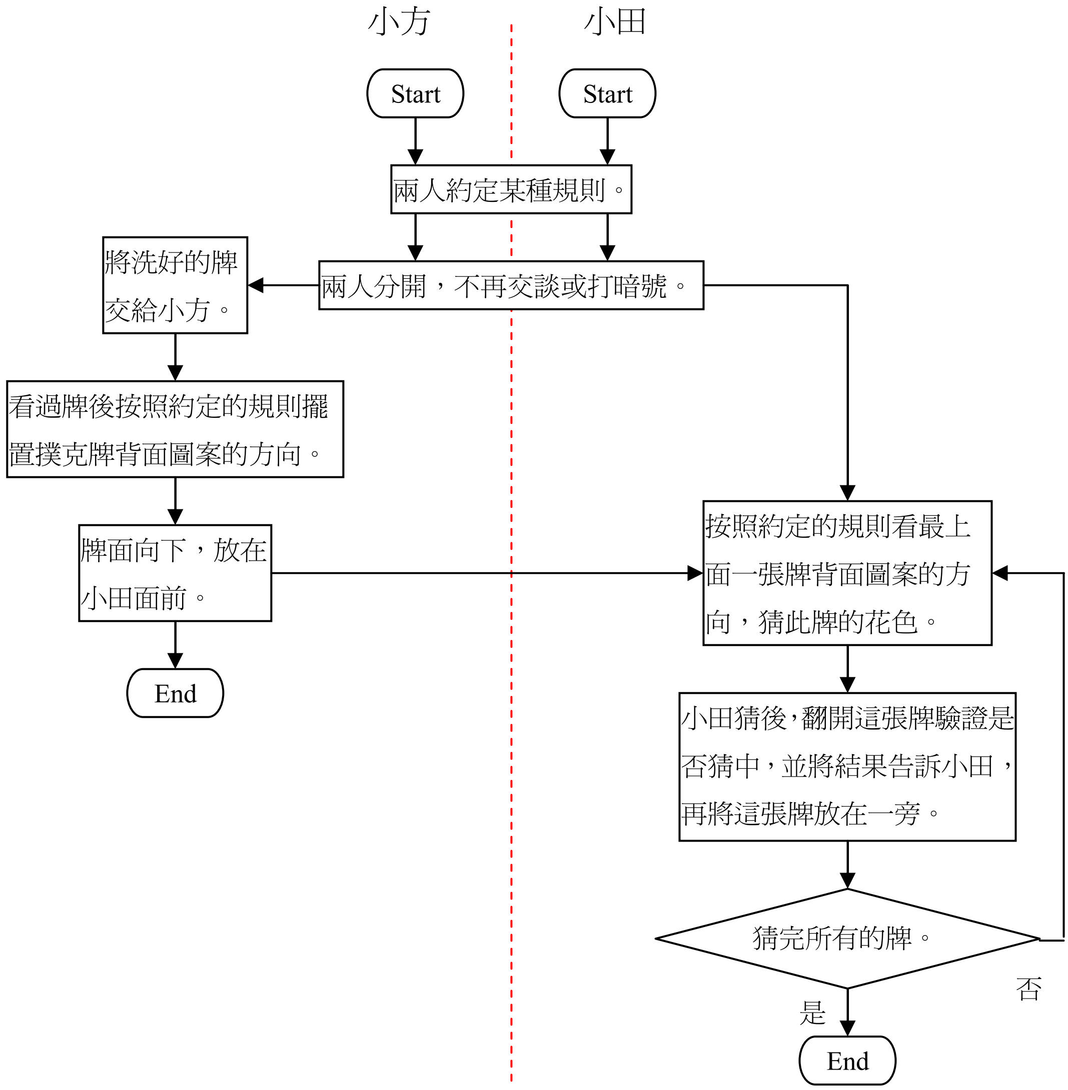
貳、研究目的

一、36 張猜牌問題

(一)遊戲規則：

1. 黑桃、紅心、方塊及梅花各 9 張共 36 張牌，背面的圖樣完全相同但非對稱。
2. 遊戲之前小方和小田可以約定某種規則，但將洗好的牌交給小方後兩人不能再交談。
3. 洗好牌後上下的順序不再更動，但小方可以調整撲克牌背面圖案的方向。
4. 小田只能看到最上面一張牌，下面的牌在猜此牌之前都看不到。

(二)猜牌的流程圖



利用數學原理與方法，尋找出保證可在 36 張牌中猜中最多張花色的猜牌方法。

二、N 張猜牌問題

將總張數改成 N 張且每種花色的數量不固定，其中 N 趨近於無窮大。

(一)找出保證可猜中的張數和總張數的比值 r

(二)計算出此比值的極限 R

參、研究器材

紙、筆、電腦。

肆、研究過程及方法

一、我歸納出以下的猜牌方法：

(一)牌背有朝上、朝下兩種不同的情況，設朝上爲 1、朝下爲 0，等於一個訊息。

(二)兩張牌的牌背有(1，1)、(1，0)、(0，1)、(0，0)四種不同的情況，可猜出某一張牌是黑桃、紅心、方塊或梅花。

(三)第一張牌一定猜不到，因爲猜中一張牌至少要有兩個訊息。

(四)整副牌的最後兩張，只用第 35 張的牌背的訊息就可猜中，只要將四種花色排序：黑桃>紅心>方塊>梅花，第 35 張的花色可能大於、等於或小於第 36 張的花色。

(a)大於時：設第 35 張的牌背爲 1。 (b)小於時：設第 35 張的牌背爲 0。

(c)等於時：不用猜即可知道。

(五)圖示講解（見表 1）

牌序/牌向	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
牌背																																				
牌面																																				
猜對																																				

表 1

說明：

1. 從左到右分別代表第 01 張到第 36 張牌。
2. 在牌背和牌面的方格內，若已塗色表示已使用過。
3. 相關的牌背和牌面會上相同的顏色(例如用 01、02 兩張的牌背猜第 02 張的牌面，用同色表示)。
4. 表 1 的內容爲：用 01、02 兩張牌猜第 02 張牌並猜中；用 35 的牌背代表 35、36 這兩張牌都能猜對。

二、猜牌：

(一)我得出的一種猜法（見表 2）

牌序/牌向	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
牌背																																				
牌面																																				
猜對																																				

表 2

做法如下：

步驟 1：用第 01、02 張的牌背猜第 02、03 張的牌面，兩張都猜同一個花色。

雖然只能猜對 1 張，但小方可以控制猜對前面那張或猜對後面那張兩種，設猜對前面那張為 1、猜對後面那張為 0，等於是多了一個訊息。

步驟 2：第 03、04 張牌的牌背和第 04、05 張牌的牌面也做相同的步驟，一樣只能猜對一張，並產生一個訊息。到此已經猜中 2 張牌，並產生 2 個訊息(表 2 中的橘色)。

步驟 3：用第 05~08 的牌背和前面產生的 2 個訊息猜第 06~09 的牌面，可以猜對 3 張，並再產生 2 個訊息(見詳細講解)。

步驟 4：重複前步驟(步驟 3) 7 次，來猜緊接的下 4 張牌，也就是猜到第 32 張的牌面，共可猜中 21 張牌(表 2 中的藍色)。

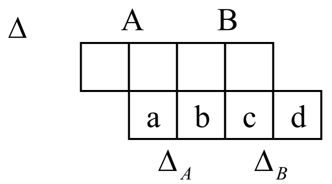
步驟 5：用第 33 張和第 34 張的牌背猜第 34 張牌(表 2 中的粉紅色)。

步驟 6：第 35、36 張也都能猜中(表 2 中的紫色)。

一共是$2+3\times7+1+2=26$張。

(二)詳細講解（此部分講解步驟 3）（見表 3）

設黑桃=3、紅心=2、方塊=1、梅花=0，並設步驟 3 的 4 張牌面的花色分別是 a、b、c、d、$\Delta_A=b-a\pmod 4$、$\Delta_B=d-c\pmod 4$、前面產生的 2 個訊息=Δ、4 張牌背中的前 2 張牌背=A、後 2 張牌背=B (a、b、c、d、Δ_A、Δ_B、Δ、A、B 都是 0、1、2、3，而小方只能控制Δ、A、B)。



$$\Delta_A=b-a\pmod 4$$
$$\Delta_B=d-c\pmod 4$$

表 3

說明：（見表 4）

1. 小方看過 a、b、c、d 這 4 張牌後，可得知Δ_A、Δ_B。
2. 小田猜牌，猜的花色分別是 A、A+Δ(mod4)、B、B+Δ(mod4)。
3. 如果小方要讓小田猜對前面二張，則使$\Delta=\Delta_A$；反之要讓小田猜對後面二張，則使$\Delta=\Delta_B$。
4. 這兩組其中一組可以猜對 2 張，另外一組雖然只能猜對 1 張，但仍然可以選擇猜對前面那張或後面那張，也就是說這 4 張牌有 4 種不同的「方法」可以猜對 3 張牌，而 4 種「方法」等於 2 個訊息。

5. 這 4 張牌先使用了 2 個訊息，猜對了 3 張後又再產生 2 個訊息，以此步驟重複猜下去。

	小方可控制的訊息			實際與猜測花色			
	$\Delta=$	A=	B=	a	b	c	d
方法 1	Δ_A	a	c	a	b	c	c+Δ_A(mod4)
方法 2	Δ_A	a	d-Δ_A(mod4)	a	b	d-Δ_A(mod4)	d
方法 3	Δ_B	a	c	a	a+Δ_B(mod4)	b	d
方法 4	Δ_B	b-Δ_B(mod4)	c	b-Δ_B(mod4)	b	c	d

表 4

特殊情況：

1. 如果第 02、03 的花色恰好一樣時：則兩張牌都會猜中而不會留下訊息，此時繼續用兩張牌背猜兩張牌面，直到產生兩個訊息爲止。猜中的張數只會增加不會減少，第 04、05 張牌也可以使用相同方法。

2. 如果$\Delta_A=\Delta_B$時：當$\Delta_A=\Delta_B$，這 4 張牌都會猜對，這時用兩張牌背猜兩張牌面，直到產生 2 個訊息爲止。由前面得知，最壞的情況是使用 2 次就產生 2 個訊息，這 4 張雖然只猜對 2 張，但是前 4 張全部都猜對，所以這 8 張牌共猜對了 6 張，猜對的張數不會改變（見表 5）。

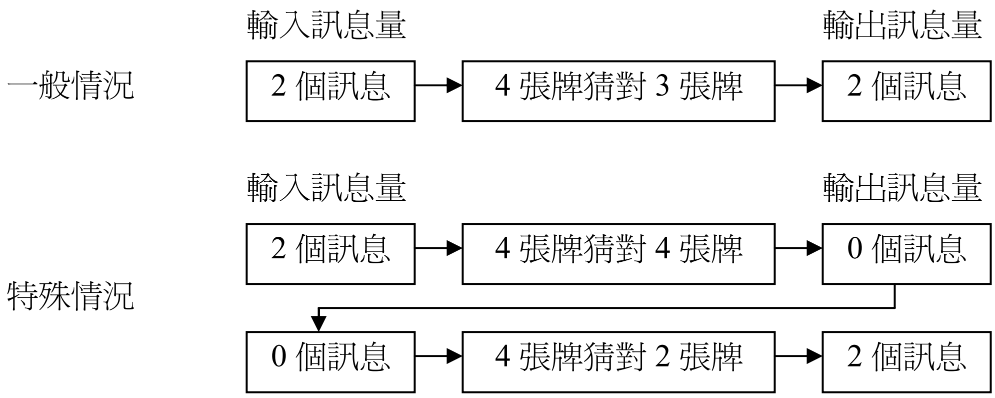


表 5

三、當撲克牌的張數爲 N 張時可猜中張數的比值（N 趨近於無窮大）：

因爲張數較多，可以使用一種特別的猜牌方法，這種猜牌方法要在張數非常多時，才會有較顯著的效果。

(一)猜牌策略：

這 x 張牌中原本可以猜中 y 張，但為了製造出更多的猜牌訊息，故意讓這 x 張牌只猜中 y-z 張，可以控制少掉的牌是第幾張牌、少的原因、……等，雖然少了 z 張，但是相對的卻產生更多訊息，可以在其它地方再多猜出幾張牌。

以前面的猜法為例：

因為 2 張牌背有四種不同的情況，如果要用 2 張牌面猜 1 張牌面，其中一種組合可以猜中 1 張牌面，另外三種組合則不會猜對，也就是說，有三種「方法」不會猜對。

(二) 初步的想法（見表 6）：

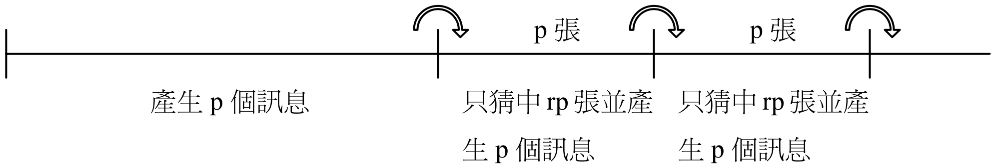


表 6

步驟 1：先使用一些牌製造出 p 個訊息，並與接下來的 p 張牌的牌背猜這 p 張牌面。

步驟 2：使用猜牌策略只猜中 rp 張牌，且產生的訊息量≧p。

步驟 3：使用這 p 個訊息重複步驟 2。

如果總張數 N 趨近於無窮大時，步驟 1 所使用的牌可以忽略，可猜中的張數的比值為 r。

(三) 求出 r 的值：

p 張牌中猜中 rp 張牌，有 C_{rp}^{p} 種組合，要讓一張牌沒有猜對有 3 種「方法」，共有 $(1-r)p$ 張沒有猜對的牌，也就是說每種組合又有 $3^{(1-r)p}$ 種不同的「方法」可以猜對這 rP 張牌，全部共有 $C_{rp}^{p}\cdot 3^{(1-r)p}$ 種「方法」，要讓產生的訊息量≧p，可得出這個不等式：

$$C_{rp}^{p}\cdot 3^{(1-r)p}\geq 2^{p}$$

$$\frac{p!}{(rp)!\left((1-r)p\right)!}\cdot 3^{(1-r)p}\geq 2^{p}$$

這個時候可以使用斯特靈公式：

$$n!\sim\sqrt{2\pi n}\left(\tfrac{n}{e}\right)^{n}$$

將此公式帶入原不等式：

$$\frac{\sqrt{2\pi p}}{\sqrt{2\pi rp}\sqrt{2\pi(1-r)p}}\cdot\frac{\left(\frac{p}{e}\right)^{p}}{\left(\frac{rp}{e}\right)^{rp}\left(\frac{(1-r)p}{e}\right)^{(1-r)p}}\cdot 3^{(1-r)p}\geq 2^{p}$$

化減後可得：

$$\frac{1}{\sqrt{2\pi r(1-r)p}}\cdot\left(\frac{3^{1-r}}{r^{r}\cdot(1-r)^{1-r}}\right)^{p}\geq 2^{p}$$

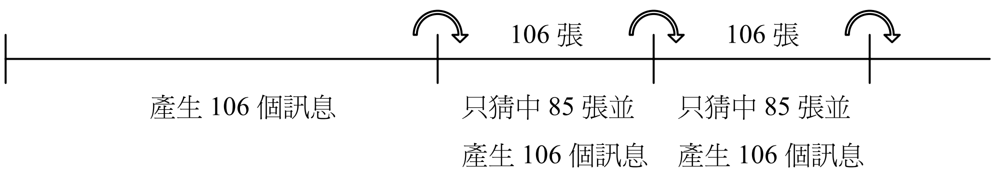
如果 p 趨近於無窮大，則可得出此不等式：

$$\frac{3^{1-r}}{r^{r}\cdot(1-r)^{1-r}}\geq 2$$

使用電腦可算出 r 的值約等於 0.810710375，也就是說；當總張數趨近於無窮大時，保證可以猜中 81.07%的牌。

(四) 猜中多於 80%的一個例子(23006 張牌)：

我先用電腦計算得知 $C_{85}^{106}\cdot 3^{21}>2^{106}$，即 106 個訊息加上 106 張牌可以猜中 85 張牌並產生 106 個訊息，且 $\frac{85}{106}>\frac{4}{5}$ 。



1. 用兩張牌背猜兩張牌面連續使用 106 次；也就是使用 212 張牌背猜 212 張牌面猜對 106 張，並產生 106 個訊息（見表 7）。

牌序／牌向	01	02	03	04	05	06	..	..	..	..	..	205	206	207	208	209	210	211	212	..	..	..	..	..												
牌背							..	..	..	..	..									..	..	..	..	..												
牌面								..	..	..	..	..									..	..	..	..	..											
猜對								..	..	..	..	..									..	..	..	..	..											

表 7

2. 使用產生的 106 個訊息和 106 張牌背可以猜中 85 張牌面並再產生 106 個訊

息，重複此步驟 214 次。也就是使用 106 個訊息和 213~22896 這 22684 張牌背($106\times214=22684$)猜中 18190 張牌面($22684\times\frac{85}{106}=18190$)，並再產生 106 個訊息（見表 8）。

牌序/牌向	..	..	..	..	213	214	215	216	217	218	219	220	221	222	223	224	225	226	..	..	..	..	..	22892	22893	22894	22895	22896	..	..	..	..	..			
牌背	..	..	..	..															..	..	..	..	..						..	..	..	..	..			
牌面	..	..	..	..	..															..	..	..	..	..						..	..	..	..	..		
猜對	..	..	..	..	..															..	..	..	..	..						..	..	..	..	..		

表 8

3. 使用剩餘的 106 個訊息和 22897~23002 這 106 張牌背猜中這 106 張牌面，再使用第 23003 和第 23004 張牌猜第 23004 張牌面，而最後兩張都可以猜對(見表 9）。

牌序/牌向	..	..	..	..	22897	22898	22899	22900	22901	22902	..	..	..	..	..	22986	22987	22988	22989	22990	22991	22992	22993	22994	22995	22996	22997	22998	22999	23000	23001	23002	23003	23004	23005	23006
牌背	..	..	..	..							..	..	..	..	..																					
牌面	..	..	..	..	..							..	..	..	..	..																				
猜對	..	..	..	..	..							..	..	..	..	..																				

表 9

總共猜對了$106+85\times214+106+1+2=18405$張牌，而$\frac{18405}{23006}>\frac{4}{5}$。

以上得出保證可以猜中 81.07%的牌，並證實是可行的。

四、為了找出更好的猜法，我再從另一個角度思考。

設若僅使用獨立的訊息猜牌可猜中 N 張牌中的 RN 張牌。

無論猜法如何都無法改變：$總方法量\geq2^{總訊息量}\geq2^{2RN}$

總方法量即這 N 張牌中，從牌背、牌面或其它地方所有情況的乘積，將這些「方法」經過換算，可以得出總訊息量(雖然$2^{訊息}$=方法，但總方法量不一定是 2 的整數次方)。

而每猜對 1 張牌就要使用 2 個訊息，所以總訊息量一定要大於等於 2RN。

假設這一種絕佳的猜法用盡了 N 張牌中所有的「訊息」和「方法」，也就是：

$總方法量=2^{總訊息量}=2^{2RN}$

N 張牌中保證可以猜中 RN 張牌，但在某些情況下還是可以猜中超過 RN 張牌，我直接讓猜中第 RN 張牌之後的牌通通放棄，這種猜法保證可以猜中恰好 RN 張牌，因為放棄的是多出來的牌，保證猜中的張數沒有減少，一樣是 RN 張牌。

要猜對一張牌，除了猜此花色，沒有其他方法；但不要猜對這張牌，有三種不同的「方法」(猜到另外 3 種花色)。

在這 N 張牌中，保證猜中恰好 RN 張牌，如果事先知道哪些牌會猜中，猜中這 RN 張牌只有$1^{RN}=1$種「方法」；而沒有猜中的$(1-R)N$張牌每張只能產生 3 個「方法」，共可產生$3^{(1-R)N}$種「方法」。

因為事先知道哪些牌會猜中，而這 RN 張牌在 N 張牌中最多有只有C_{RN}^{N}種可能的排列方法，故這 N 張牌的牌面最多只能產生$3^{(1-R)N}C_{RN}^{N}$種「方法」。

牌背最多只有2^N種「方法」，而牌面最多只可產生$3^{(1-R)N}C_{RN}^{N}$種「方法」，由此公式總方法量=2^{2RN} 可以得出下等式：

$$2^N\cdot C_{RN}^{N}\cdot 3^{(1-R)N}=2^{2RN}$$

將斯特靈公式帶入此等式，會得出：

$$\frac{1}{\sqrt{2\pi R(1-R)N}}\cdot\left(\frac{3^{1-R}}{R^R\cdot(1-R)^{1-R}}\right)^N=2^{(2R-1)N}$$

同樣的，因為 N 趨近於無窮大，所以可以化簡此式：

$$\frac{3^{1-R}}{R^R\cdot(1-R)^{1-R}}=2^{2R-1}$$

使用電腦算出 R 的值約等於 0.873697436，也就是說，僅使用獨立的訊息猜牌不可能保證猜中多於 87.37%的牌。

伍、討論及推廣

一、討論

本研究已得出保證 36 張牌中可猜中不少於 26 張花色。當張數為無限大時，已證明保證可猜中超過 81.07%的牌；若僅使用獨立的訊息猜牌保證無法猜中超過 87.37%的牌。至於任何猜法猜中比值的極限值和最大值是多少，目前正持續研究探討中。(我認為極限值不會高於 87.37%，但目前尚未找到證明的方法，會留待日後繼續探討。)

該題目除了我國知名數學網站熱烈討論之外，在網路上發現亦有其它國家數學網站針對如何找出更多張數的猜牌方法詳加討論。其中一位俄羅斯的學生提到他找到了 26 張牌的猜法，但他的猜法經過我仔細驗証發現猜牌過程其實是有問題的，此猜法並無法猜中 26 張牌。

本研究所提出的証明「當總張數趨近於無窮大可以猜中超過 80%」的這一部份搜尋網路數學討論區截至目前尚無人提出。

二、推廣

本研究之過程及所得知之結果，是藉由訊息轉換而成爲訊息加密的方式，讓此訊息除特定的接收者外而不被其他人所辨識。以及在此猜牌過程中會不斷的產生訊息，如果將其運用到電腦傳輸方面，則可增強傳輸之速度。因此；本研究可廣泛應用在密碼與電腦傳輸等方面。

陸、研究結論

一、 此猜牌魔術，小田保證可以從 36 張牌中猜中不少於 26 張花色。

二、 當總張數 N 趨近於無窮大時，保證猜中的張數和總張數的比值 r≒0.810710375。

三、 當總張數等於 23006 張時，保證可以猜中不少於 18405 張牌($\frac{18405}{23006} > \frac{4}{5}$)。

四、 當總張數 N 趨近於無窮大時，僅使用獨立的訊息猜牌保證猜中的張數和總張數的比值的極限值 R≒0.873697436。

五、 0.810710375≦僅使用獨立的訊息猜牌保證猜中的張數和總張數的比值的最大值≦0.873697436。

六、可用此猜法推廣到牌背 X 種角度、牌面 Y 種花色。

柒、參考資料

一、知名數學網站討論區的網址：

http://www.chiuchang.org.tw/modules/newbb/viewtopic.php?topic_id=494&forum=6

二、俄羅斯網站討論區網址：

http://community.livejournal.com/ru_math/114368.html

三、俄羅斯網站討論區網址(英文翻譯)：

http://groups.google.com/group/rec.puzzles/browse_thread/thread/1e3183b9594bf989/a40f7aa688326cb8?lnk=st&q=&rnum=3&hl=en#a40f7aa688326cb8

評語

本作品的數學困難程度超越普通中學生的能力，我們可以看到作者過去數年對於該問題的投入。

My name is Sheng Hao Jiang. I am a senior in National Experimental High School at Hsinchu Science Park math honor class.

My project is based on the method guaranteed to correctly guess the suit of cards as many as possible, which was from my last year project “Magic Guessing”, and I make more research on it.

Since my seventh grade, I have been working on this problem for four years. Not until last year did I write my whole project to participate in the International Science and Engineering Fair. Being questioned and suggested by judges last year, inspecting and learning from others as well, I found that there is indeed more room for my project. Therefore, I make development on the content and research orientation this year. I am so glad that my research has made a great progress and improvement, which is beyond my expectation.

Magician's Prediction of Card Suits

Abstract

The study is mathematically based with reasonable explanations behind it. We are to correctly guess as many cards as possible from a deck of 36 cards, with random numbers and four different suits. We will apply mathematical methods, such as logic inference, binary system, and analytical reduction, upon right timing. Using careful arrangement of the principles and reasoning, we can reach our ultimate goal. To state guessing: Conference between the guesser and the assistant about the guessing rules, the assistant will have 36 cards with the same exact pattern on the back but not symmetrical. The pattern of the cards will be different when rotated 180º. The only communication between the two is by rotating cards. In this study, we can prove that through mathematical method, we can assure 26 or more cards can be correctly guessed. Furthermore, when the total amount of cards is close to infinity, we can assure 81.07% or more of the cards can be correctly guessed, and prove that if the cards are guessed from independent information, no more than 87.37% of the cards will be correctly guessed by any guessing methods. One of the examples, which 80% of the cards are correctly guessed, is that when the amount of the cards is 23006, 18405 or more of the cards can be correctly guessed. ($\frac{18405}{23006} > \frac{4}{5}$)

Text

Introduction

The words”guess, predict, deduce” in this project mean we make the reasoning with mathematical principles and methods, not really guess with luck and probability. “Bit” means 0 or 1. “Information” means the amount of possible situation. For example, 2 bits mean 4 messages.

I、Research motivation

In math discussion website, lots of people are discussing a math problem which is from International Mathematics Tournament of Towns 2004, senior papers, A level.「The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least (a) 19 cards;(b) 20 cards?」。

After discussing with others, I am interested in this question. However, I think there is room for improvement. I keep trying and improving the solutions which have been proposed on the internet. Based on the solution for 36 cards, I extend my research when the card number doesn’t equal 36.

II、Research purpose

A、Guessing suits form 36 cards

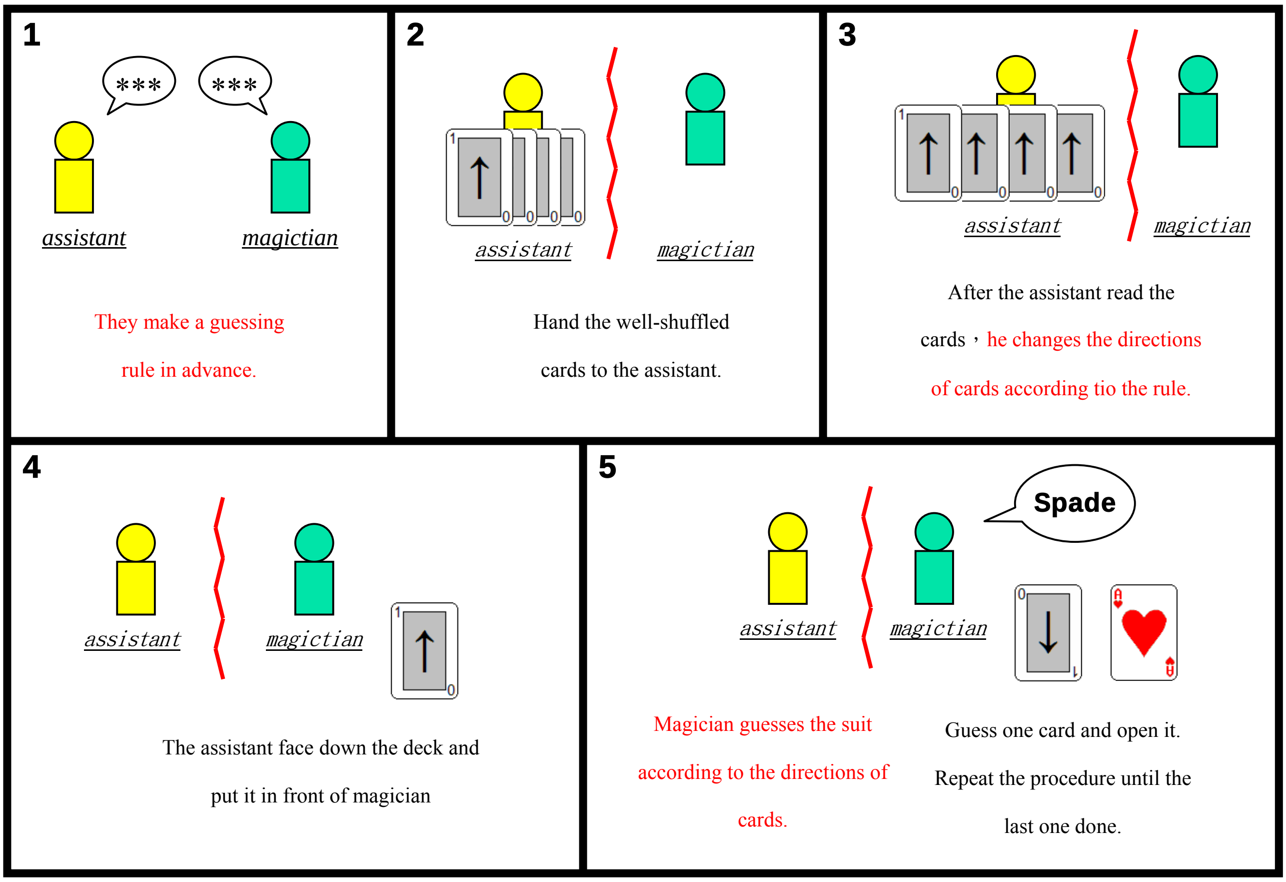
(a) Rules：

1. A 36-card deck contains 9 cards form 4 different suits spade, heart, diamond and club. The patterns on the card back are the same, but not symmetrical.
2. The magician and the assistant have made their guessing protocol in advance before the performance. After the assistant get the well-shuffled cards, they are not allowed to talk any more.
3. The order of the cards can not be changed, but the assistant can adjust the direction of the card back. The assistant can make the arrow head on card backs

point toward or away from the magician.

4. The magician can only see one card back on the top. He guesses the card suit one by one, and flips over.

(b) The flow chart of guessing suits



With mathematical principle and methods, we find out one way guaranteed to guess most card backs correctly among 36 cards.

B、Guessing suits from N cards

N cards, which is not fixed, and N approaches to infinity.

(a) Find the ratio, r, of the guaranteed number of correctly guessed cards to the number of total cards.

(b) Calculate the limit, R, of the ratio.

III、Research tool

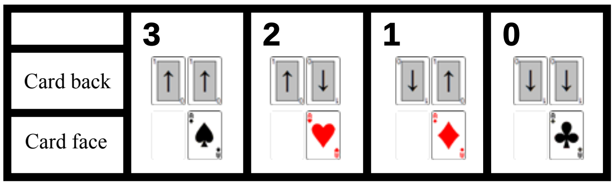
Paper, pen and computer

IV、Research method and process

A、I induct the following guessing rules:

(a) There are two different directions for a card back, go up or down. Assume that going up as 1, and going down as 0, which information is a bit.

(b) There are four different permutations for two card backs: (1，1) 、 (1，0) 、 (0，1) 、 (0 ，0). They represent the suits: spade, heart, diamond and club.



(c) The first card can’t be guessed correctly, because it needs at least two bits to guess one card.

(d) The last two cards can be deduced by the information from 35^{th} card back. Put the suits in order: spade> heart> diamond> club. The suit of 35^{th} card may be greater than, equivalent to, or smaller than that of the 36^{th} card.

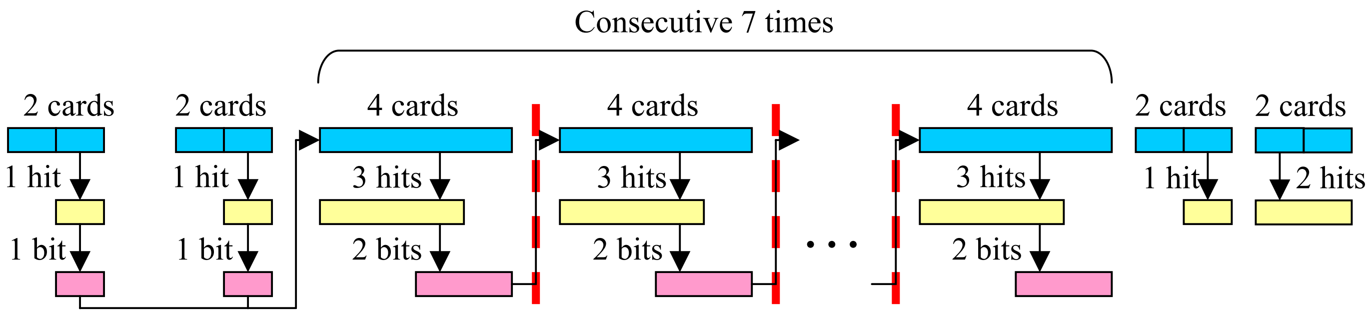
(1) greater: Set the card back of the 35^{th} card as 1

(2) smaller：Set the card back of the 35^{th} card as 0

(3) equivalent：You’ll know it without guessing.

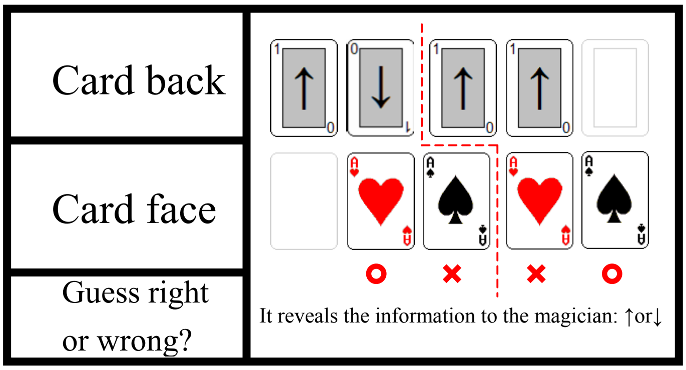
B、Guessing cards：

(a) The method I figure out



Procedure：

Step 1： We can deduce the 2^{nd} and 3^{rd} card faces with the 1^{st} and 2^{nd} card backs. We assume that these two cards have the same suit, but we can only have one correct. The assistant can control which card, the former or the latter, to be correctly guessed by magician. There are two different hints. If the former card is correctly guessed, we code it as “1”. If the latter card is correct, we code it as “0”. Then, we’ll have one bit.



Step 2： Repeat step 1 to deduce the suit of the 4th and 5th card. We can have one correct, which means one bit. So far, we have correctly guessed two cards, and had 2 bits.

Step 3： With the 5th to 8th card backs and the previous 2 bits, we can deduce the suits from the 6th to 9th cards. We can have three right answers. Meanwhile, 2 bits are produced. (See detailed explanation below)

Step 4： From the 5th card, group 4 consecutive cards as a set. Repeat step 3 for 7 times to guess the following 7 sets until the 33rd card face is done. In this process, we can have 21 cards being correctly guessed.

Step 5： We deduce the 34th card face with the 33rd and 34th card back.

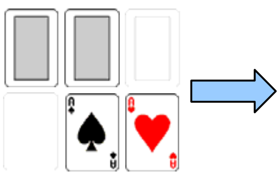
Step 6： The 35th and 36th cards can be guessed correctly.

There are $2+3\times7+1+2=26$ cards correctly guessed.

(b) Detailed explanations (explain step 3；See table 3)

Define the difference of suits：

Assume that: spade=3, heart=2, diamond=1 and club=0. The difference of any two cards is the latter card suit minus the former one. For example: (Δ=3)

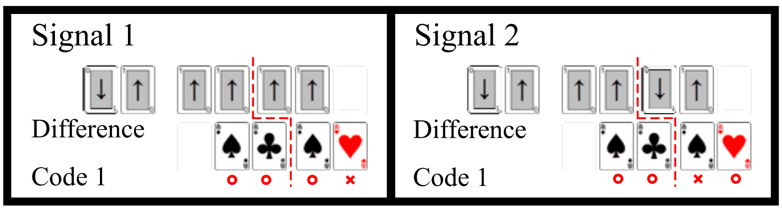
 $2-3(\mathrm{mod}\,4)=3$

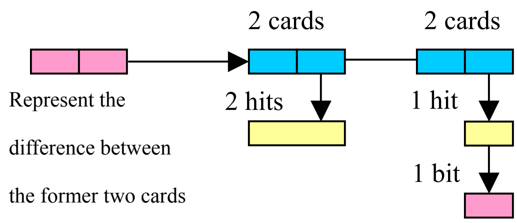
The method as followed:

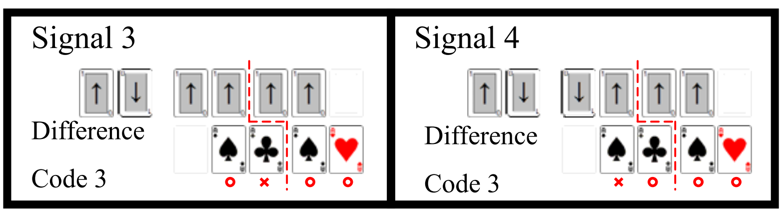
1. In step 3, the assistant divides 4 card backs in two groups. There are two card backs and two faces in each group. Calculate the differences of each group. Use the previous two bits which we get from first two steps to represent one of the differences.
2. After the magician guesses the first card, additionally, he can deduce the second card with the difference shown by previous two bits. After he guesses the third card, he can deduce the fourth card with the previous two bits.

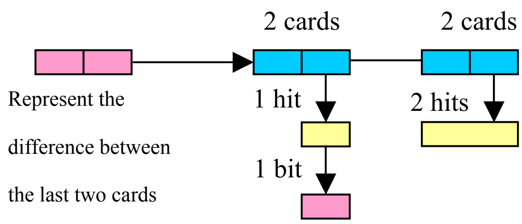
With this method, the assistant has four ways to give hints.

1. Use the previous two bits to show the difference of the first pair of cards. These two card backs can show the suit of the first card face. With the difference shown before, the second card face can be guessed. As for the last two cards, the assistant can control which one be guessed by the magician, either the former one (see signal 1) or the latter one (see signal 2).
2. Use the previous two bits to show the difference of the last pair of cards. Also, the assistant can control which card in first pair be guessed by the magician. Use the last two card backs to show the third card suit, and the fourth card face can be guessed.









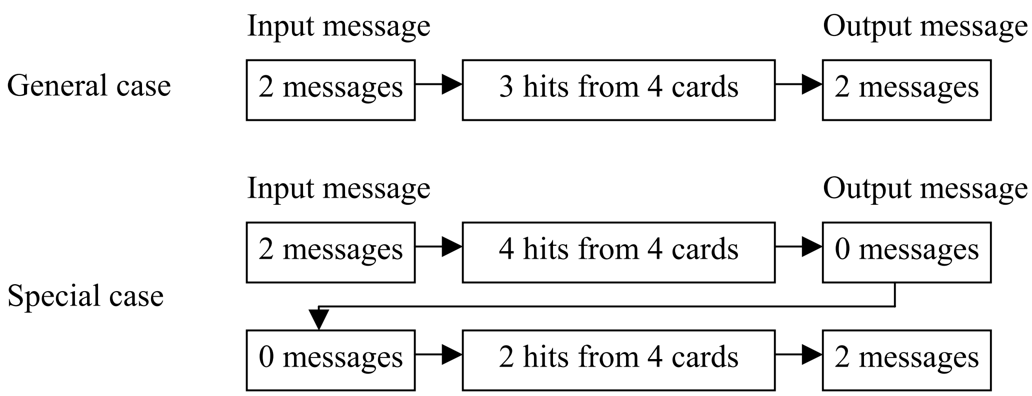
These four signals are two bits.

Special cases：

1. In step 1 and 2, if the 2^{nd} and 3^{rd} suits are the same, these two cards will be guessed without leaving any message. Continue guessing two card faces with

two card backs until two bits are generated. The cards being correctly guessed will just increase. The 4th and 5th cards can use the same way.

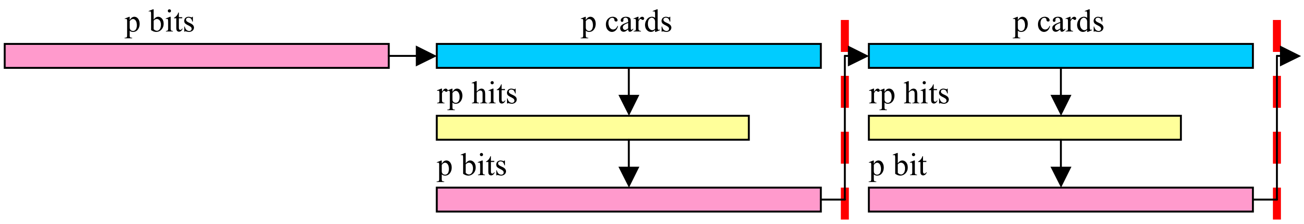
2. Instep 3, if the differences of these two pairs are equivalent, these four cards will be guessed correctly. Then use two card backs to guess two card faces until two bits are generated. The worst situation is that 2 bits come from two guessing, which means 2 hits from 4 cards. But four cards are correct before, there are 6 correct cards from these 8 cards. The total number of correct guessing is not changed.



C、The correct prediction rate of N cards (N is sufficiently large):

We can use a method which is especially for large numbers.

(a) Initial idea：



Step 1： Generate p bits by some cards. With the following p card backs, guess suits of these p card faces.

Step 2： We can only correctly guess rp cards, and the generated messages ≧ 2^p 。

Step 3： Use these p bits messages to repeat step 2.

If N approaches to infinity, the cards used in step 1 can be omitted. The correctly prediction rate is r (r<1).

(b) Strategy：

p bits and p card backs give 4p messages, which means all combination of p cards. So, the assistant can control how many cards being correctly guessed by the magician. But, the purpose to guess only rp cards is to generate p bits; meanwhile, r has to be as great as possible.

(c) Find the value of r：

There are C_{rp}^{p} ways to guess correctly rp cards from p cards, and $(1-r)p$ cards will be guessed wrong. For every card, there are three wrong answers for the suit. Therefore, there are $3^{(1-r)p}$ ways to correctly guess rp cards, and there are $C_{rp}^{p}\cdot 3^{(1-r)p}$ ways in total. In order to produce more than 2^{p} messages, the following inequality equation is obtained: $C_{rp}^{p}\cdot 3^{(1-r)p}\geq 2^{p}$.

$$\frac{p!}{(rp)!\left((1-r)p\right)!}\cdot 3^{(1-r)p}\geq 2^{p}$$

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Now, Stirling Formula can be applied in this case. $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$

Then we will have the following inequality.

$$\frac{\sqrt{2\pi p}}{\sqrt{2\pi rp}\sqrt{2\pi(1-r)p}}\cdot\frac{\left(\frac{p}{e}\right)^{p}}{\left(\frac{rp}{e}\right)^{rp}\left(\frac{(1-r)p}{e}\right)^{(1-r)p}}\cdot 3^{(1-r)p}\geq 2^{p}$$

Simplify, and we have:

$$\frac{1}{\sqrt{2\pi r(1-r)p}}\cdot\left(\frac{3^{1-r}}{r^{r}\cdot(1-r)^{1-r}}\right)^{p}\geq 2^{p}$$

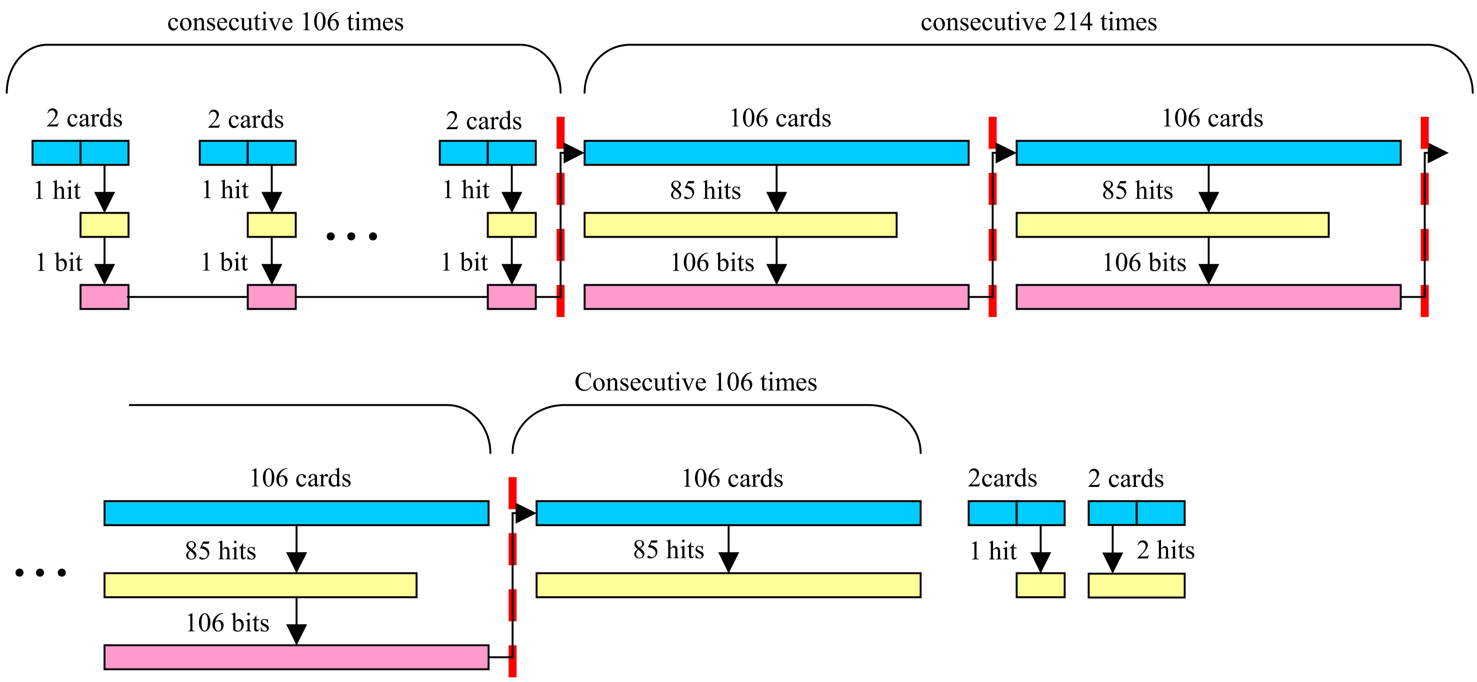
If p approaches to infinity, we will obtain:

$$\frac{3^{1-r}}{r^{r}\cdot(1-r)^{1-r}}\geq 2$$

With computation, r is around 0.810710375. To sum up, it’s guaranteed that we can have 81.07% of the cards correctly predicted, when the number of cards approaches to infinity.

(d) An example of guessing rate more than 80% (23006 cards)：

$C_{85}^{106}\cdot 3^{21} > 2^{106}$. We use 106 bits and 106 cards to correctly predict 85 cards, and generate 106 bits. The prediction rate is $\frac{85}{106} > 80.18\% > 80\%$.



Step 1： Use two card backs to predict two card faces for 106 times. You'll have 106 corrections, and 106 bits.

Step 2：Use the 106bits generated above and 106 card backs to guess correctly 85 cards. Meanwhile, 106 bits are generated. Repeat this step for 214 times, you will have 18,190 correct predictions on card suits and 106 bits.

Step 3：Use the rest of 106 bits and 106 card backs to predict the following 106 card suits.

Step 4： Deduce the last third card suit with the last fourth and the last third card back.

Step 5： The last two cards can be guessed correctly.

Totally speaking, we correctly guess $106+85\times 214+106+1+2=18405$ cards, and the ratio $\frac{18405}{23006} > 80.00087\% > 80\%$. Therefore, it is guaranteed to correctly predict 81.07% of the cards, and it works very well.

V、General discussion and development

A、Discussion

This research has guaranteed that not less than 26 cards can be correctly predicted from 36 cards. When there is large enough number of cards, more than 81.07% of cards are guaranteed to be correctly predicted.

This question is not only discussed in our famous math website, but also in other countries. They focus on how to correctly predict more cards. One Russian student posts his method which can correctly deduce 26 cards. After I carefully verify his method, I find there's something wrong with his method and it can't predict 26 cards.

The proof in my project, "The correct prediction rate is over 80% when there are sufficiently large cards", has not been put forth in math discussion website so far.

B、Development

In the method "not less than 26 cards can be correctly predicted among 36 cards", there are two bits haven't been used. Therefore, we can have the conclusion that"not less than 28 cards can be correctly predicted among 38 cards."

The outcome from my project is that the messages can only be recognized by specific message receivers by transforming and encoding messages. During the guessing process, bits will be constantly generated. If the bits can be applied to computer science, the transporting can be speeded up. As a result, this research can be applied to coding and computer transporting.

VI、Conclusion

A、 In this magic guessing game, the magician can guess not less than 26 card suits among 36 cards.

B、 When there are sufficiently large cards, the guaranteed predict rate, r, is about 0.810710375.

C、 When there are 23006 cards, it is guaranteed that not less than 18405 cards can be correctly predicted. $\left(\frac{18405}{23006} > 80.00087\% > 80\%\right)$。

D、 This method can be expanded to X directions of card backs, and Y suits of card faces.

VII、References material

A、Famous math discussion website:

http://www.chiuchang.org.tw/modules/newbb/viewtopic.php?topic_id=494&forum=6

B、Russian discussion website:：

http://community.livejournal.com/ru_math/114368.html

C、Russian discussion website (English Version)：

http://groups.google.com/group/rec.puzzles/browse_thread/thread/1e3183b9594bf989/a40f7aa688326cb8?lnk=st&q=&rnum=3&hl=en#a40f7aa688326cb8