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作者簡介

我是林子瑜，在家排行老二，目前就讀台南女中數理資優班二年級。從小對數學就極具濃厚興趣，平時喜歡動腦筋，解一些老師出的難題或哥哥的自創題。不過由於高中之前就讀的學校位在郊區，因此並未參加過科展。這次是我第一次參加，對過去只曾參加數學競賽的我來說的確是跨出了一大步。

在我的求學生涯中，曾擔任過班長及數學小老師等不同職務的幹部，這是一項能爲他人服務和對自我能力的考驗，感謝班上同學及老師給我這個機會。除學科外，我也喜歡聽音樂，或自行吹奏。我發覺吹吹直笛對我理清思路有很大的幫助。

高中進入了數理資優班，讓我遇到許多同樣喜歡科學的朋友們。我很高興能夠與指導老師一起探究數學奧妙的地方，感謝老師的指導，使我獲益匪淺。

M&m Sequences 之研究

中文摘要

本專題的目的是研究以任意實數 a_1 、 a_2 、 a_3 爲起始的 *M&m Sequences* 之穩定性質。我們主要關心的問題是：

(1) 是否任給定三數 a_1 、 a_2 、 a_3 爲起始的 *M&m* 數列皆會穩定？

(2) 若上述的 *M&m* 數列穩定，則其穩定的長度與 a_1 、 a_2 、 a_3 的關係爲何？

(3) 其穩定的值與 a_1 、 a_2 、 a_3 的關係爲何？

我們研究的主要步驟及結果如下：

1. 當 $a_1 < a_2 = a_3$ 或 $a_1 = a_2 < a_3$ ，此 M&m Sequences 穩定，穩定值爲 a_2，穩定長度爲 5。

2. 當 $a_1 < a_2 < a_3$ 及三數不成等差(三數成等差的結果是明顯的)時，存在一線性變換 $f(x) = \alpha x + \beta$ 使得以 a_1 、 a_2 、 a_3 爲起始的 *M&m* 數列可轉換成以 $-x, 1, x\ (x > 1)$ 爲起始的 *M&m* 數列。

3. 我們證明了下列性質：

(1) 若 *M&m* 數列中前 n 項($a_1, a_2, a_3, \cdots, a_n$)所成數列的中位數爲 m_n ，則下式成立：

$$a_{k+1} = m_k + k(m_k - m_{k-1}) \quad k \geq 4$$

(2) 當存在 $k > 4$，$k \in N$ ，使得 $m_{k-1} = m_{k-2}$ 成立時，則此數列穩定，且穩定長度 p 滿足：
$p = \min\{k \mid k > 4 \text{且 } m_{k-1} = m_{k-2}\}$ ，其中 p 必爲奇數。

(3) $\{m_n\}$ 爲單調遞增且 $a_n \geq m_{n-1}$ ，$n \geq 5$

4. 如果 $x \geq 41.625$，則 $\{-x, 1, x\}$ 爲起始的 M&m 數列，其對應的數列 $a_1(x), a_2(x), a_3(x), \cdots, a_n(x), \cdots$ 有相同的大小次序且此 *M&m* 數列會穩定，穩定值爲 41.625，且穩定長度爲 73。

5. 我們觀察發現：如果 $x < 41.625$，則 $\{-x, 1, x\}$ 爲起始的 M&m 數列也會穩定但沒有一致的規則，其穩定長度變化極大，且穩定長度與 x 值的小數位數隱含著一個數量關係。事實上我們更進一步得到了下列的結果：

(1)若 x 等於節點，其穩定值必爲節點 x ，同時節點穩定長度爲中位數節點位置+2；

(2)越靠近 41.625 的分枝，除了邊緣地帶以外幾乎是常數值，越靠近 1 的分枝，其穩定長度分布越混亂；

(3)在節點附近多一點(K= 3, 5, 7, …, 67, 69)的 x，即節點的鄰近區域，其數列都會收斂且其穩定長度爲節點的穩定長度(K+2)加上新的 b 數列的穩定長度減 3。(如表 6,7 及性質一)

The Study of M&m Sequences

Abstract

This research studies the property of stability of an M&m sequence which starts with any three real numbers a_1, a_2, a_3. The questions we concern are as follows:

(1) Is every M&m sequence starting with any three real numbers a_1, a_2, a_3 stable?

(2) If the M&m sequence is stable, what is the relationship between the stable length and the initial values a_1, a_2, a_3?

(3) What is the relationship between the stable values and the initial values a_1, a_2, a_3?

The approaches and the results are as follows:

1. If $a_1 < a_2 = a_3$ or $a_1 = a_2 < a_3$, the M&m sequences will be stable, the stable value is a_2 and the stable length is 5.
2. When $a_1 < a_2 < a_3$ and they are not equal difference sequences (it is trivial for the equal difference case), there is a linear transformation $f(x) = \alpha x + \beta$ such that the M&m Sequence beginning with a_1, a_2, a_3 can be transformed into another M&m Sequence beginning with –x , 1 , x (x>1).
3. We prove the following properties:

   (1) If the median of the former n numbers $(a_1, a_2, a_3, \cdots, a_n)$ of the M&m sequence is m_n, we obtain $a_{k+1} = m_k + k(m_k - m_{k-1}) \quad k \geq 4$

   (2) There exist $k > 4$, $k \in N$ such that $m_{k-1} = m_{k-2}$, then the sequence is stable and the stable length $p = \min\{k \mid k > 4 \ and \ m_{k-1} = m_{k-2}\}$, where p must be an odd number.

   (3) $\{m_n\}$ is monotone increasing and $a_n \geq m_{n-1}$, $n \geq 5$.
4. Suppose $x \geq 41.625$, then the all M&m Sequences beginning with –x , 1 , x are the same, and the sequences will be stable, the stable value is 41.625 and the stable length is 73.
5. By the computer experiments, we observe that if x is any positive real number less than 41.625, the M&m Sequence starting with –x, 1, x, will be also stable but does not appear to follow any clearly discernible pattern of behavior. However, the stable lengths are much variant and exist some unknown relation with point format of x. Moreover, we have the following properties:

   (1)If x is a node, then the stable value is x and the stable length equals to the index of median of the node + 2；

   (2)Near the branch of 41.625, the stable length is almost a constant except at the edge area，the stable length of (-x,1,x) as x around branch 1 is chaos；

   (3)If x near the node (K= 3, 5, 7, …, 67, 69), then the stable length is $l(K) + K - 1$ where the positive integral $l(K)$ is determined by Prop1 (see Table 6 and 7).

目次

壹、研究動機

在高中《數列與級數》的單元中，老師介紹了一些特殊的數列，其中有一種數列稱作「M&m Sequences」引起我們極大的興趣，於是我們便想進一步研究，希望能提出一種方法來解決文章[1]中所提出未能證明的猜想。

貳、研究目的及步驟

本專題的目的是研究任取 3 個實數 a_1 、 a_2 、 a_3 為起始的 M&m Sequences 之穩定性質。研究此數列是否都會穩定的步驟如下：

一、當 $a_1=a_2=a_3$ 或 $a_1<a_2=a_3$ 或 $a_1=a_2<a_3$ ，此 M&m Sequences 是否穩定?

二、當 $a_1<a_2<a_3$ 時，再分成三數成等差及三數不成等差的狀況下討論。不成等差的情況再經過線性變換 $f(x)=\alpha x+\beta$ 轉換成以 $(-x,1,x)$ 起始的數列。

三、證明經過線性變換 $f(x)=\alpha x+\beta$ 轉換成以 $(-x,1,x)$ 起始的數列依然滿足 M&m Sequences 的定義。

四、透過 MATLAB 程式試驗及觀察。

五、證明我們得到的觀察結果：如果 x 是任何大於等於 41.625 的實數，則由 $(-x,1,x)$ 為起始所形成的 M&m 序列，其收斂的特性有一定規則可尋。

六、如果 x 是任何小於 41.625 的實數，則由 $(-x,1,x)$ 為起始所形成的 M&m 序列，其收斂的特性相當複雜，是否能找出隱藏其中的穩定規則？

七、文章[1]中所提出未能證明的猜想是正確的嗎？

參、研究設備及器材

研究設備及器材：紙、筆、電腦、運算軟體 (Excel、MATLAB)

肆、研究過程

M&m Sequences 是由 Shultz 及 Shiflett 在[1]中所提出的一種特殊數列。M&m Sequences 中的 M 與 m 分別代表平均數及中位數，此數列剛開始先取三個實數 a、b、c，第四個數 d 必須滿足其與前三個數的平均數（即$\frac{a+b+c+d}{4}$）要等於前三個數所成數列（即 a、b、c）的中位數，第五個數則滿足其與前四個數的平均數要等於前四個數所成數列（即 a、b、c、d）的中位數，依次類推，所形成之數列即為 M&m Sequences。

例如：先取三個數 11、25、33，令第四個數為 x_4 ，則(11+25+33+ x_4)/4=25， x_4=31；第五個數為 x_5 ，(11+25+33+31+ x_5)/5=(25+31)/2， x_5=40，……當我們把這個數列全寫出來 (Excel 程式詳見[http://members.cox.net/mathematics/mean~median.xls , 1], 附件四)：

11,25,33,31,40,36,38,40,55.5,60.5,49,51,40,40,40,……

我們發現最後”40” 一直重複出現。再舉一個例子：先給定 53、90、76，接下來的數列依序為：

53,90,76,85,98.5,107.5,102.5,107.5,128.25,136.75,120.5,124.5,135,140,107.5,107.5,……

最後”107.5”依然一直重複出現。

在前述例子中，當 M&m 數列最後會一直重複出現同一數字時，則我們稱這個數列是穩定(stable)的，最後一直重複出現的數字稱為穩定值(stable value)，在上兩例中的 40 及 107.5 即是，穩定長度(stable length)則定義為重複出現的第一數字序列編號，在上兩例中分別為 13 及 15。

Shultz 及 Shiflett 在[1]中猜測：任取三實數起始產生的 M&m Sequences 最後都會是穩定的。果真如此的話，我們想透過另一種研究方法進一步知道第幾個數字開始會重複出現同一個數字，即 M&m Sequences 的穩定長度，並同時研究穩定時之穩定值為何。

一開始我們先任取 3 個數 a_1 、 a_2 、 a_3 ，假設 $a_1 \le a_2 \le a_3$ ，則

$$a_4 = 4a_2 - (a_1 + a_2 + a_3) = 3a_2 - (a_1 + a_3)$$

定義 $S_n \equiv \sum_{i=1}^{n} a_i$ ，且 m_k 表示前 k(≥ 3)個數所成數列$\{a_k\}_{k\in N}$之中位數， M_k 表示前 k(≥ 3)個數所成數列$\{a_k\}_{k\in N}$之平均數，則 $m_k = M_{k+1} \equiv \frac{S_k + a_{k+1}}{k+1}$ 。

一、 考慮$a_1=a_2=a_3=a$的情形

假若$a_1=a_2=a_3=a$，則

$$a_4=4a_2-(a_1+a_2+a_3)=4a-3a=a$$

$$a_5=5(\frac{a_2+a_3}{2})-(a_1+a_2+a_3+a_4)=5a-4a=a$$

依此類推可得 $a_1=a_2=a_3=a_4=a_5=\cdots\cdots=a_k=a$

此數列穩定。

二、 考慮a_1、a_2、a_3中有兩項相同的情形

若a_1、a_2、a_3中有兩項相同(考慮$a_1<a_2=a_3$或$a_1=a_2<a_3$)，當$a_1<a_2=a_3$時，令$a_2=a_3=a$，

$$a_4=3a_2-(a_1+a_3)=3a-a-a_1=2a-a_1>a$$

$$\Rightarrow a_1, a, a, 2a-a_1$$

$$a_5=5a-(2a-a_1+a+a+a_1)=a$$

$$a_6=6a-(2a-a_1+a+a+a+a_1)=a$$

$$\Rightarrow a_5=a_6=\ldots\ldots=a_k=a$$

此數列穩定。

當$a_1=a_2<a_3$，令$a_1=a_2=a$

$$a_4=3a_2-(a_1+a_3)=3a-a-a_3=2a-a_3<a$$

$$a_5=5a-(2a-a_3+a+a+a_3)=a$$

$$a_6=6a-(2a-a_3+a+a+a+a_3)=a$$

$$\Rightarrow a_5=a_6=\ldots\ldots=a_k=a$$

此數列穩定。

三、 考慮$a_1 < a_2 < a_3$的情形

接著考慮$a_1 < a_2 < a_3$的情形。因爲若a_1、a_2、a_3三數成等差數列，平均數即爲中位數，明顯會穩定，所以我們只考慮a_1、a_2、a_3三數不成等差數列的情形。在本作品中我們提出一種研究方法，即利用線性函數將a_1、a_2、a_3轉換成以 –x , 1 , x (x > 1)爲起始的數列。

考慮函數 $f(t) = \alpha t + \beta$ 使得 $f(a_1) = -x$，$f(a_2) = 1$，$f(a_3) = x$。 首先，由 $f(a_1) + f(a_3) = 0$ 知$\alpha \cdot a_1 + \beta + \alpha \cdot a_3 + \beta = 0$，即

$$\alpha(a_1 + a_3) + 2\beta = 0 \quad \Rightarrow \beta = \frac{-\alpha(a_1 + a_3)}{2}$$

再將結果代回 $f(a_2) = 1$即$\alpha \cdot a_2 + \beta = 1$ 得到

$$\alpha = \frac{-2}{a_1 + a_3 - 2a_2} \ ; \ \beta = \frac{a_1 + a_3}{a_1 + a_3 - 2a_2}$$

故原函數 $f(x)$ 可表示成：

$$f(t) = \frac{2t - (a_1 + a_3)}{2a_2 - (a_1 + a_3)}$$

因爲a_1, a_2, a_3不爲等差，所以$a_1 + a_3 - 2a_2$不爲零。由此，若a_1、a_2、a_3三數不成等差數列，必然可以找到 $f(t) = \alpha t + \beta$ 使得：

(1)當$a_3 - a_2 < a_2 - a_1$，$f(a_1) = -x$，$f(a_2) = 1$，$f(a_3) = x = \dfrac{a_3 - a_1}{2a_2 - (a_1 + a_3)} > 1$；

(2)當$a_3 - a_2 > a_2 - a_1$，$f(a_1) = -y$，$f(a_2) = 1$，$f(a_3) = y = \dfrac{a_3 - a_1}{2a_2 - (a_1 + a_3)} < -1$，

經過重新排序改成 $f(a_3) = y = -x$， $f(a_2) = 1$，$f(a_1) = -y = x$，其中$x > 1$依然成立。

四、 M&m 數列經線性變換之討論

接著我們要說明，若$\{a_k\}_{k\in N}$是一 M&m 數列，則$\{a_k\}_{k\in N}$經過線性變換 $f(x)=\alpha x+\beta$ 轉換成另一數列$\{a'_k\}_{k\in N}$ ，其中$a'_k = f(a_k)$ ，$k=1,2,3,\ldots$ ，則新數列$\{a'_k\}_{k\in N}$也是由a'_1, a'_2, a'_3起始的 M&m 數列。

因為$\{a_k\}_{k\in N}$為 M&m Sequences 即$m_k = M_{k+1}(=\frac{S_k+a_{k+1}}{k+1})$且$m'_k=\{a'_1,a'_2,a'_3\ldots a'_k\}$的中位數，而 $a'_k = f(a_k)=\alpha a_k+\beta$ ，所以$a'_1, a'_2, a'_3, \ldots a'_k$與$a_1, a_2, a_3, \ldots a_k$的順序相同，故

$$m'_k = f(m_k) = f(M_{k+1}) = \alpha M_{k+1} + \beta = \alpha(\frac{a_1+\ldots+a_{k+1}}{k+1}) + \beta = \frac{1}{k+1}\sum_{i=1}^{k} f(a_i) = M'_{k+1}$$

因此新數列$\{a'_k\}_{k\in N}$也是 M&m Sequences。這意味著我們只需考慮以–x , 1 , x 為起始的 M&m Sequences 即可。

例如取 3 個數 (0, 67, 78)，經由線性變換 $f(x)=\alpha x+\beta$ 轉成另一數列 (-1.3929, 1.0000, 1.3929)，兩數列的收斂情形相同，如圖 1 所示 (MATLAB 程式如附件一)。

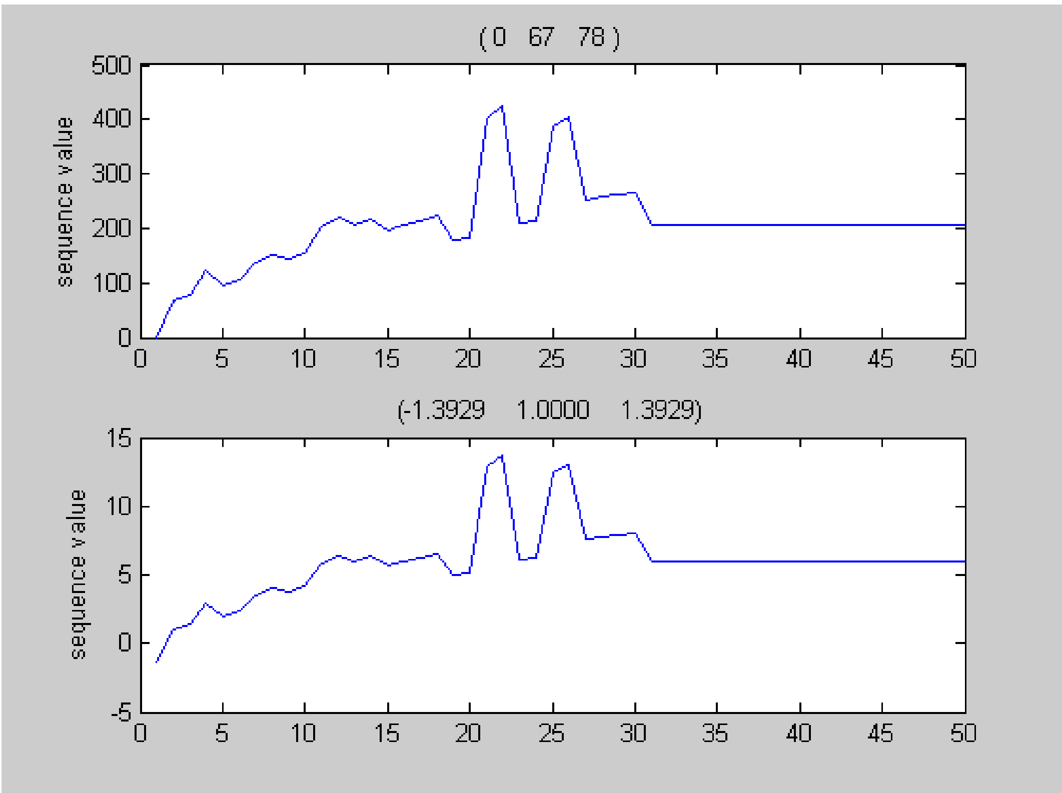


圖 1

透過多次的試驗及觀察，我們得到底下的定理。

定理 1：如果 x 是任何大於或等於 41.625 的實數，則由(–x , 1 , x)起始所形成的 M&m 數列，一定會收斂在 41.625 且穩定長度爲 73。

證明：定理 1 的証明在 p12。

爲了說明定理 1 是正確的，我們改變 x 的值從 1~50，以觀察數列的收斂情形。如圖 2 及 3 所示，當 x 大於等於 41.625 時，我們觀察到穩定值都是 41.625 且穩定長度都是 73。由圖 2 我們也可知道，當 x 小於 41.625 時，我們暫時看不出數列的規則性。

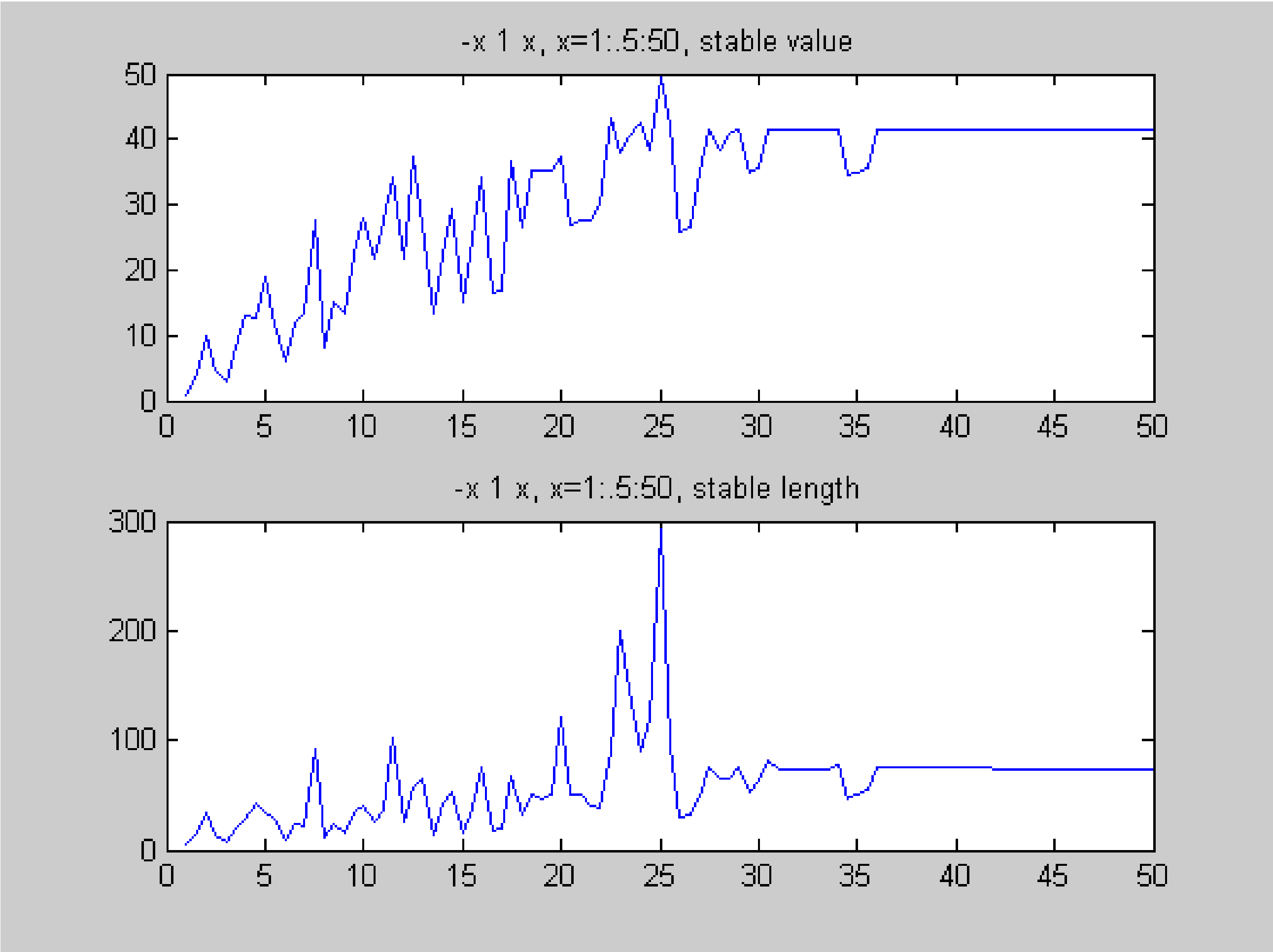


圖 2

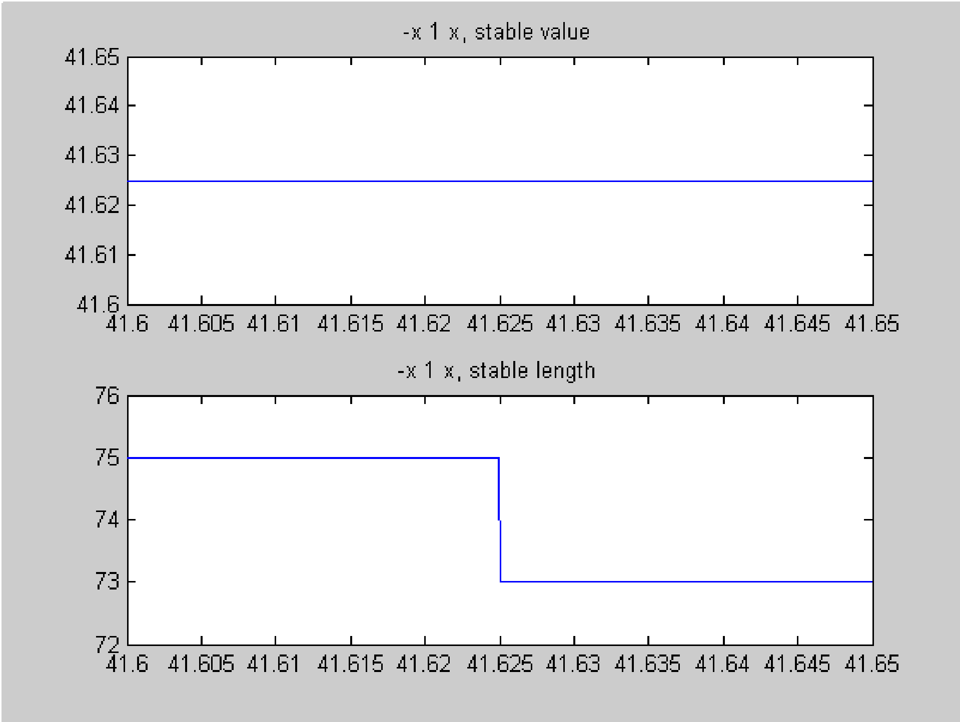


圖 3

爲了再一次說明定理一是正確的，我們選擇 x=41.625 及 x=999，分別得到序列如表 1 及圖 4 所示。由表 1 可知，當 x 大於或等於 41.625，所有 M&m 序列第四項以後都完全相同。

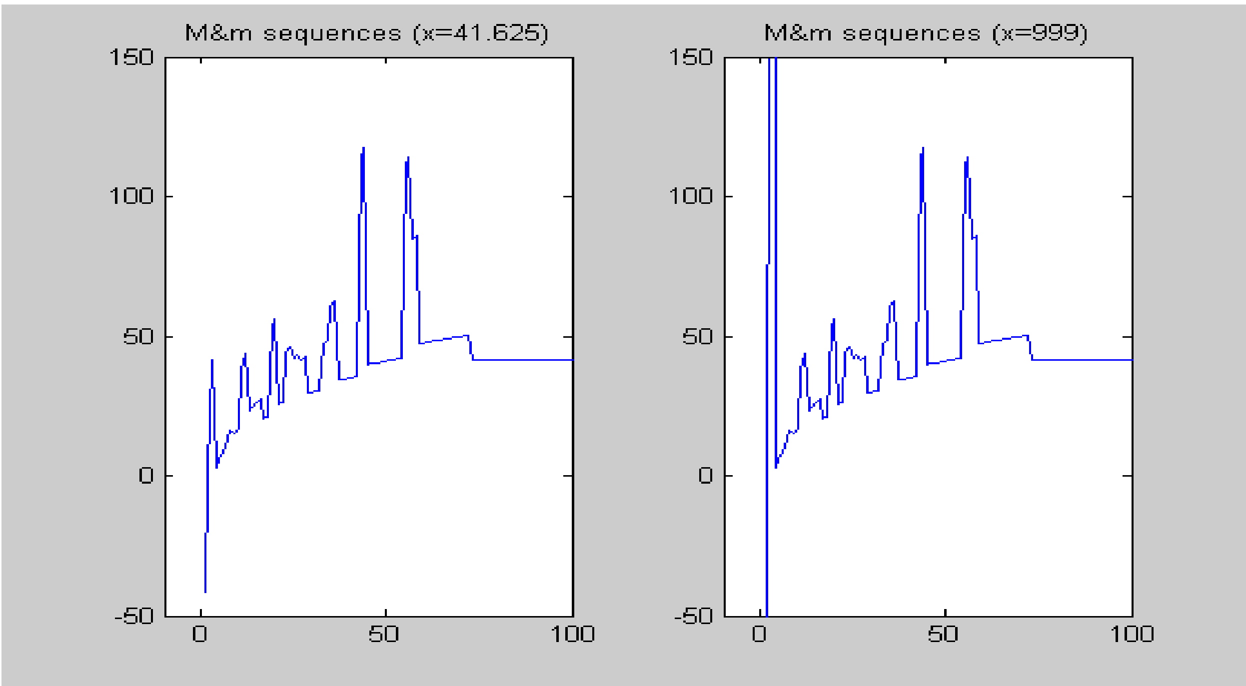


圖 4(左圖表示 x=41.625；右圖表示 x=999)

表 1

Columns 1 through 11

-41.6250	1.0000	41.6250	3.0000	6.0000	8.0000	13.5000	16.5000	15.0000	17.0000	38.2500
-999.0000	1.0000	999.0000	3.0000	6.0000	8.0000	13.5000	16.5000	15.0000	17.0000	38.2500

Columns 12 through 22

43.7500	23.2500	24.7500	26.2500	27.7500	20.7500	21.2500	52.6250	56.3750	26.0000	26.5000
43.7500	23.2500	24.7500	26.2500	27.7500	20.7500	21.2500	52.6250	56.3750	26.0000	26.5000

Columns 23 through 33

44.2500	46.2500	42.0000	43.5000	41.6250	42.8750	29.6250	29.8750	30.1250	30.3750	47.1250
44.2500	46.2500	42.0000	43.5000	41.6250	42.8750	29.6250	29.8750	30.1250	30.3750	47.1250

Columns 34 through 44

48.3750	60.5625	62.4375	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	113.6875	117.5625
48.3750	60.5625	62.4375	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	113.6875	117.5625

Columns 45 through 55

39.8750	40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250	41.8750	42.1250	111.1250
39.8750	40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250	41.8750	42.1250	111.1250

Columns 56 through 66

113.8750	84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500	48.5000	48.7500	49.0000
113.8750	84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500	48.5000	48.7500	49.0000

Columns 67 through 77

49.2500	49.5000	49.7500	50.0000	50.2500	50.5000	41.6250	41.6250	41.6250	41.6250	41.6250
49.2500	49.5000	49.7500	50.0000	50.2500	50.5000	41.6250	41.6250	41.6250	41.6250	41.6250

五、 定理 1 的証明

接著我們打算証明定理 1。令$\{a_k\}_{k\in N}$是一 M&m 數列，而且$a_1 = -x$ 、 $a_2 =1$、 $a_3 = x$ 。 定義 $S_n \equiv \sum_{i=1}^{n} a_i$ ，且 m_k 表示前 $k(\geq 3)$個數$\{a_k\}_{k\in N}$所成數列之中位數，則 $m_k = \dfrac{S_{k+1}}{k+1}$ ， $k \geq 3$ ，即 $S_{k+1} = m_k(k+1)$ ， $k \geq 3$ 。因此，我們有

$$a_{k+1} = S_{k+1} - S_k = (k+1)m_k - km_{k-1} = m_k + k(m_k - m_{k-1}) \quad k \geq 4 \qquad \textbf{(1)}$$

引理 1：當存在 $k > 4$, $k \in N$ ，使得 $m_{k-1} = m_{k-2}$ 成立，則此數列穩定，且穩定長度(stable length) p 滿足： $p = \min\{k \mid k > 4 且 m_{k-1} = m_{k-2}\}$ ，其中 p 必為奇數。

証明： 令 $A = \{k \mid k > 4 且 m_{k-1} = m_{k-2}\}$， $p = \min\{k \mid k > 4 且 m_{k-1} = m_{k-2}\}$，則由(1)及 p 的定義我們可得 $a_p = m_{p-1} + (p-1)(m_{p-1} - m_{p-2}) = m_{p-1}$ 。

更進一步，我們可得：

$$\begin{cases} m_p = \dfrac{a_p + m_{p-1}}{2} = m_{p-1} & p\text{爲偶數} \\ m_p = m_{p-1} & p\text{爲奇數} \end{cases}$$

同理可證 $m_p = m_{p+1} = m_{p+2} = \ldots$，所以 $m_{k-1} = m_{k-2}$ 成立即 $a_k = a_{k+1} = a_{k+2} = \ldots$ 成立，所以數列穩定，且穩定長度爲 p。

最後我們利用反證法證明 p 爲奇數。假設 p 爲偶數，已知 $m_{p-1} = m_{p-2} \neq m_{p-3}$，由於 p-2 爲偶數，故 $m_{p-2} = \frac{1}{2}(m_{p-1} + m_{p-3}) \Rightarrow m_{p-2} = m_{p-3}$ 與已知矛盾，所以 p 必爲奇數。

以 a_1=-8、a_2=1、a_3=8 爲例，M&m 數列及中位數數列分別如表 2 所示，在此例題中 $a_{11} = m_{10} = m_9 = 8$ 且穩定長度 $p = 11$ 爲奇數。再舉兩個例子：(1) $x = 41.3750$，$a_{71} = m_{60} = m_{69} = 41.3750$ 且穩定長度 $p = 71$ 爲奇數；(2) $x = 35$，$a_{51} = m_{50} = m_{49} = 35$ 且穩定長度 $p = 51$ 爲奇數。

表 2

序號	1	2	3	4	5	6	7	8	9	10	**11**	12
M&m 數列	-8.0	1.0	8.0	3.0	6.0	8.0	13.5	16.5	15.0	17.0	**8.0**	8.0
中位數數列			1.0	2.0	3.0	4.5	6.0	7.0	**8.0**	**8.0**	8.0	8.0

引理 2：{ m_n }爲單調遞增且未達穩定值之前 $a_n > m_{n-1}$ 。

証明： 使用數學歸納法，

1. 先証明 $m_4 > m_3$

因爲初始三數爲 –x , 1 , x (x>1)，所以 m_3=1，a_4=3。當 $x \geq 3$ 時，所以 m_4=2> m_3；當 $x < 3$，$m_4 = \dfrac{1+x}{2} > 1 = m_3$ 。

2. 如果 $m_{n-1} > m_{n-2}$ (n>5)成立，由(1)得知

$$a_n = S_n - S_{n-1} = m_{n-1} + (n-1)(m_{n-1} - m_{n-2}) > m_{n-1}$$

由於 $a_n > m_{n-1}$，所以 $m_n > m_{n-1}$。

3. 故由數學歸納法得知中位數數列 $\{m_n\}$ 爲單調遞增且未達穩定值之前 $a_n > m_{n-1}$。

爲了說明引理 2 的概念，我們以初始三數 –25 , 1 , 25 爲例，畫出 M&m sequence 及對應的中位數如圖 5 (MATLAB 程式如附件 3)，由圖中可清楚的看出中位數數列 $\{m_n\}$ 爲單調遞增且未達穩定之前 $a_n > m_{n-1}$。另外由表 2 也可得到相同的結果。

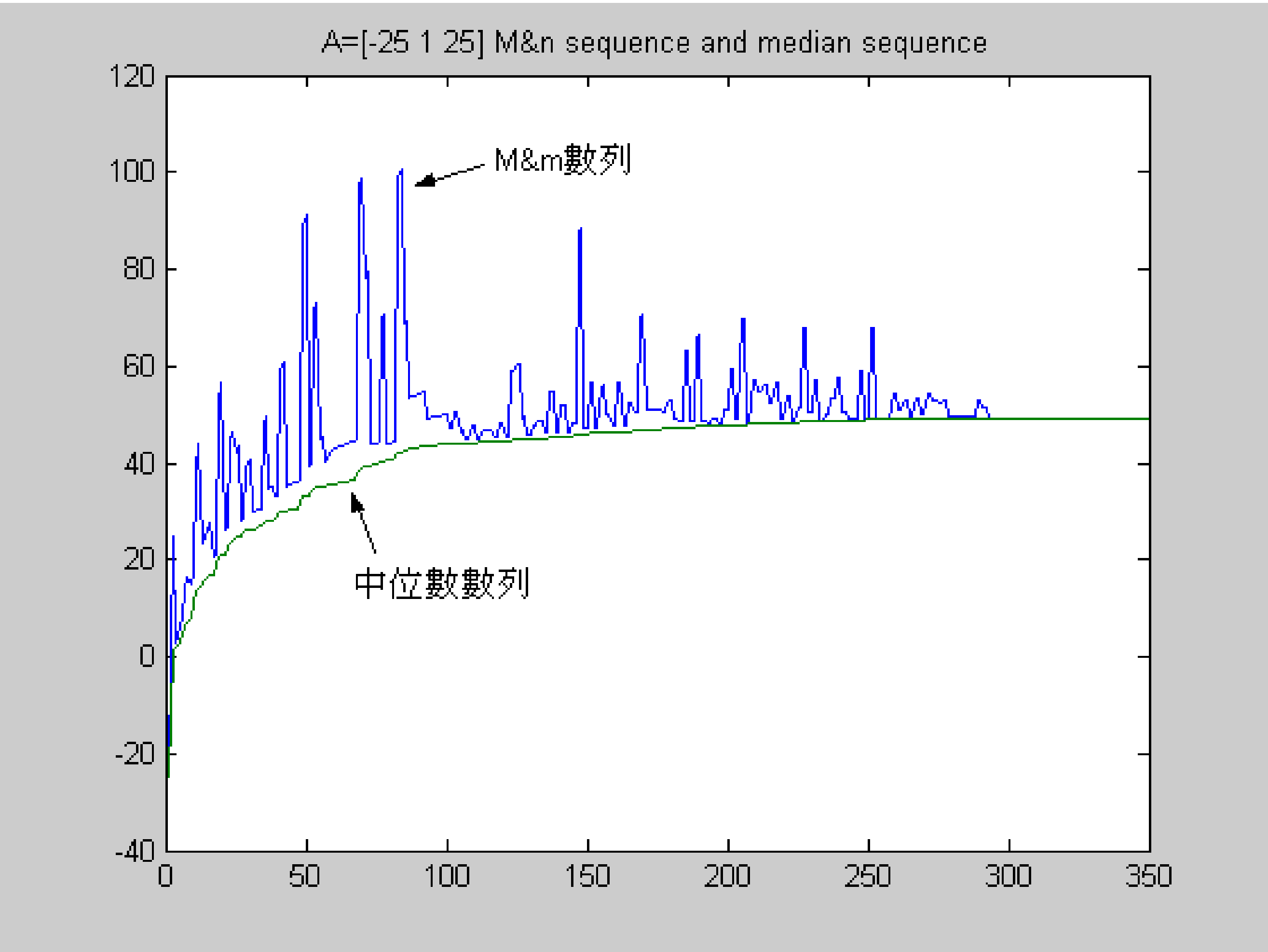


圖 5

引理 3：若$\{a_k\}_{k\in N}$是一 M&m 數列，且$\{a_k\}_{k\in N}$有界，則此數列穩定。

証明：由反證法，設$\{a_k\}_{k\in N}$不是穩定的，由於$\{a_k\}_{k\in N}$不可能以週期性的振盪出現，因此不穩定的$\{a_k\}_{k\in N}$必然出現$a_{k1}=\infty$或$a_{k1}=-\infty$，此與$\{a_k\}_{k\in N}$有界的敘述矛盾，引理得証。

引理 4：若$\{a_k\}_{k\in N}$是一 M&m 數列，$a_1=-x$，$a_2=1$，$a_3=x$，且存在兩自然數r及q，其中$r>q>3$使得$a_r=a_q$，則此數列穩定，且穩定值爲a_q。

証明：由反證法，設$\{a_k\}_{k\in N}$不是穩定的，則由引理 3 可知$\{a_k\}_{k\in N}$是無界的，更進一步此數列的中位數所成的數列$\{m_n\}$也是無界的，再由引理 2，$\{m_n\}$單調遞增至$+\infty$。因此存在自然數k'使得$m_{k'}=a_q$。設$A=\{k\mid k\in N 且 m_k=a_q\}$，故由前述討論知$A\neq\phi$。

再令$k_1=\min\{A\}$，則$m_{k_1+1}=\dfrac{m_{k_1}+a_q}{2}=\dfrac{a_r+a_q}{2}=a_q$，所以由引理 3 得知：$\{m_n\}$穩定且穩定值爲$a_q$，和所設矛盾，故得證。

以a_1=-2、a_2=1、a_3=2 爲例，M&m 數列如表 3 所示，在此例題中存在兩自然數$q=12$及$r=23$，其中$p>q>3$使得$a_r=a_{23}=a_q=a_{12}=10$，當然此數列是穩定的，且穩定值爲 10。

表 3

Columns 1 through 9

M&m	-2.0000	1.0000	2.0000	3.0000	3.5000	4.5000	5.5000	6.5000	5.2500
中位數	-2.0000	-0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.2500	3.5000

Columns 10 through 18

5.7500	9.0000	**10.0000**	9.3750	10.1250	7.1250	7.3750	7.6250	7.8750
4.0000	4.5000	4.8750	5.2500	5.3750	5.5000	5.6250	5.7500	6.1250

Columns 19 through 27

12.8750	13.6250	13.0625	13.6875	**10.0000**	10.2500	10.5000	10.7500	11.0000
6.5000	6.8125	7.1250	7.2500	7.3750	7.5000	7.6250	7.7500	7.8750

Columns 28 through 35

11.2500	24.1875	25.3125	14.8125	15.1875	19.6875	20.3125	10.0000
8.4375	9.0000	9.1875	9.3750	9.6875	10.0000	10.0000	10.0000

定理 1：如果 x 是任何大於或等於 41.625 的實數，則由(–x , 1 , x)起始所形成的 M&m 數列，一定會收斂在 41.625 且收斂長度爲 73。

證明：首先列出 x=41.625 所對應的 M&m 數列及中位數數列如表 4。

1. 由表 4 知若 x 等於 41.625 的實數，則由(–x , 1 , x)起始所形成的 M&m 數列會收斂在 41.625 且收斂長度爲 73。

2.令$\{a_k\}_{k\in N}$是一 M&m 數列，而且$a_1=-x$、a_2=1、$a_3=x$。定義$S_n\equiv\sum_{i=1}^{n}a_i$，且$m_k$表示前$k(\geq 3)$個數$\{a_k\}_{k\in N}$所成數列之中位數，則 $m_k=\frac{S_{k+1}}{k+1}$，$k\geq 3$，即 $S_{k+1}=m_k(k+1)$，$k\geq 3$。因此，我們有

$$a_{k+1}=S_{k+1}-S_k=(k+1)m_k-km_{k-1}=m_k+k(m_k-m_{k-1})\quad k\geq 4 \qquad (1)$$

3. 設$y=41.625+x_0$，其中$x_0>0$，可知$b_1=-y$、$m_1=-y$、b_2=1、$m_2=\frac{1}{2}(1-y)$、$b_3=y$、$m_3=1$、$b_4=3b_2-(b_1+b_3)=3$、$m_4=(1+3)/2=2$。由(1)知$b_5=5m_4-4m_3=6$，依此類推，$b_i=a_i\quad i=4,5,...,26$。

由表 4 知$a_{27}=a_{52}=41.625$，因此$b_{27}=a_{27}=41.625$。由引理 2 知$\{m_n\}$爲單調遞增且未達穩定值之前$a_n>m_{n-1}$。因爲$b_3=y$ >41.625 不會破壞$m_{27}=26$，所以由(1)知$b_i=a_i\quad i=28,29,...,52$，因爲$b_3=y$ >41.625 不會破壞$m_{52}=35.375$，由引理 4 知穩定值爲 41.625，又已知$b_3=y>41.625$，因此未達穩定之前$\{x_n\}$與$\{y_n\}$的$m_i\quad i=53,54,...,72$會一樣，所以由(1)知$b_i=a_i\quad i=53,54,...,73$，依此類推$b_i=a_i\quad i=4,5,...,73,.....$。

如此證得當 x 是任何大於或等於 41.625 的實數，則由(–x，1，x)起始所形成的 M&m 數列，一定會收斂在 41.625 且穩定長度爲 73。

表 4

Columns 1 through 7

M&m	-41.6250	1.0000	41.6250	3.0000	6.0000	8.0000	13.5000
中位數	-41.6250	-20.3125	1.0000	2.0000	3.0000	4.5000	6.0000

Columns 8 through 14

16.5000	15.0000	17.0000	38.2500	43.7500	23.2500	24.7500
7.0000	8.0000	10.7500	13.5000	14.2500	15.0000	15.7500

Columns 15 through 21

26.2500	27.7500	20.7500	21.2500	52.6250	56.3750	26.0000
16.5000	16.7500	17.0000	18.8750	20.7500	21.0000	21.2500

Columns 22 through 28

26.5000	44.2500	46.2500	42.0000	43.5000	**41.6250**	42.8750
22.2500	23.2500	24.0000	24.7500	25.3750	26.0000	26.1250

Columns 29 through 35

29.6250	29.8750	30.1250	30.3750	47.1250	48.3750	60.5625
26.2500	26.3750	26.5000	27.1250	27.7500	28.6875	29.6250

Columns 36 through 42

62.4375	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000
29.7500	29.8750	30.0000	30.1250	30.2500	30.3750	32.3125

Columns 43 through 49

113.6875	117.5625	39.8750	40.1250	40.3750	40.6250	40.8750
34.2500	34.3750	34.5000	34.6250	34.7500	34.8750	35.0000

Columns 50 through 56

41.1250	41.3750	**41.6250**	41.8750	42.1250	111.1250	113.8750
35.1250	35.2500	35.3750	35.5000	36.8750	38.2500	39.0625

Columns 57 through 63

84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500
39.8750	40.0000	40.1250	40.2500	40.3750	40.5000	40.6250

Columns 64 through 70

48.5000	48.7500	49.0000	49.2500	49.5000	49.7500	50.0000
40.7500	40.8750	41.0000	41.1250	41.2500	41.3750	41.5000

Columns 71 through 77

50.2500	50.5000	**41.6250**	41.6250	41.6250	41.6250	41.6250
41.6250	**41.6250**	41.6250	41.6250	41.6250	41.6250	41.6250

六、小於 41.625

透過多次的試驗及觀察，我們也得到底下的猜想。

猜想 1：如果 x 是任何小於 41.625 的實數，則由(–x , 1 , x)起始所形成的 M&m 數列，其收斂的特性無簡單規則可尋。

為了說明猜想 1 可能是正確的，我們改變 x 的值從 1~50，以觀察數列的收斂情形。如圖 6 所示，當 x 小於 41.625 時，即使我們將 x 的解析度不斷的增加，在某些點附近觀察不到穩定長度有簡單規則可尋。再舉兩個點當例子：(1) x=24.97，(2) x=24.98，分別得到(1)穩定長度 $p = 75$ 及(2)穩定長度 $p = 10941$。從這個例子隱約可以看到蝴蝶效應的影子，初始值只相差 0.01，經由簡單的 M&m 規則，其收斂長度竟然相差 10866!

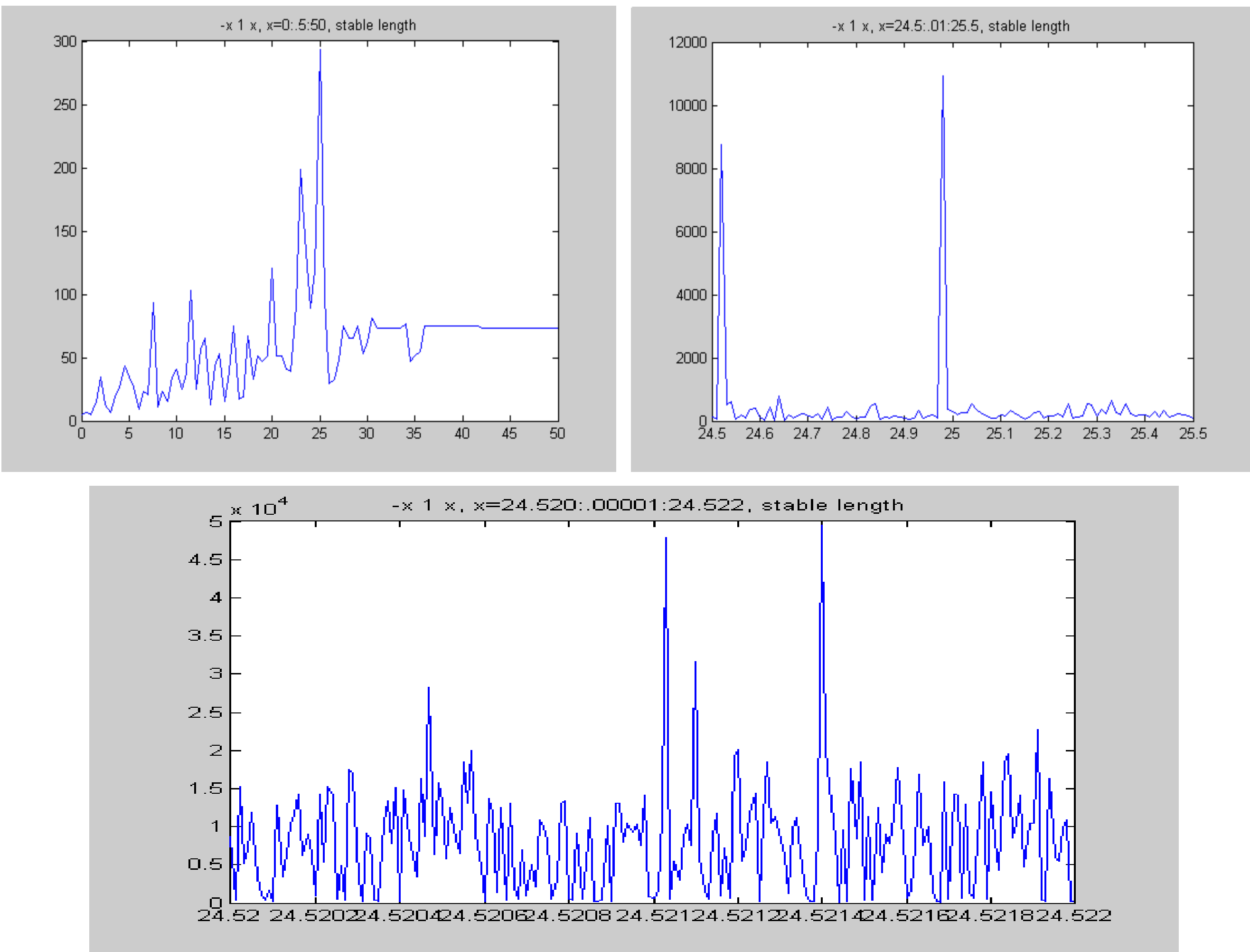


圖 6

七、節點與各分枝的穩定模式

如果 x 是任何小於 41.625 的實數，則由(–x , 1 , x)起始所形成的 M&m 數列，其收斂的特性無簡單規則可尋，但是我們還想透過實驗觀察來尋找一些蛛絲馬跡。由表 4 知當中位數數列下標爲奇數時，此數字會出現在 M&m 數列裡面，把這些數字稱爲節點(node)，由表 4 知節點如表 5 第二列所示，共有 35 個節點，由上述的引理 4 可知若 x 等於節點，其穩定值必爲節點 x，如表 5 第三列所示。例如：(1) $x = 41.3750$ 爲節點，其穩定值 $a_{71} = 41.3750$ ；(2) $x = 35$ 爲節點，其穩定值，$a_{51} = 35$ 。同時觀察到節點穩定長度成等差數列，且爲表 4 中中位數位置+2，如表 5 第四列所示，最後一個敘述其實是可以簡單證明的。

兩節點之間的實數線段稱爲分枝(branch)，共有 34 個分枝，針對不同的分枝(B1-B6, 及 B24-B34)做穩定長度的測試，結果得到圖 7。觀察圖 7 可以得知(1)越靠近 41.625 的分枝，除了邊緣地帶以外幾乎是常數值；(2) 越靠近 1 的分枝，其穩定長度分布越混亂。

表 5

Columns 1 through 8

表 4 中位數節點位置(K)	3.0000	5.0000	7.0000	9.0000	11.0000	13.0000	15.0000	17.0000
節點 x 值	1.0000	3.0000	6.0000	8.0000	13.5000	15.0000	16.5000	17.0000
穩定值	1.0000	3.0000	6.0000	8.0000	13.5000	15.0000	16.5000	17.0000
穩定長度	5.0000	7.0000	9.0000	11.0000	13.0000	15.0000	17.0000	19.0000

Columns 9 through 18

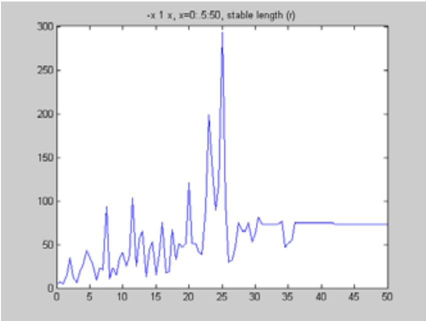
19.0000	21.0000	23.0000	25.0000	27.0000	29.0000	31.0000	33.0000	35.0000	37.0000
20.7500	21.2500	23.2500	24.7500	26.0000	26.2500	26.5000	27.7500	29.6250	29.8750
20.7500	21.2500	23.2500	24.7500	26.0000	26.2500	26.5000	27.7500	29.6250	29.8750
21.0000	23.0000	25.0000	27.0000	29.0000	31.0000	33.0000	35.0000	37.0000	39.0000

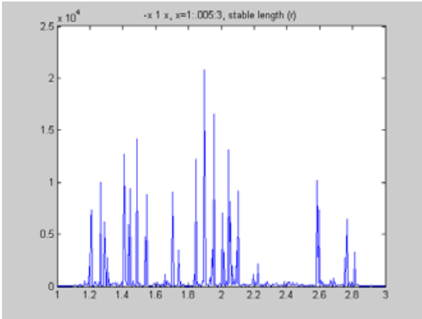
Columns 19 through 28

39.0000	41.0000	43.0000	45.0000	47.0000	49.0000	51.0000	53.0000	55.0000	57.0000
30.1250	30.3750	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	38.2500	39.8750
30.1250	30.3750	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	38.2500	39.8750
41.0000	43.0000	45.0000	47.0000	49.0000	51.0000	53.0000	55.0000	57.0000	59.0000

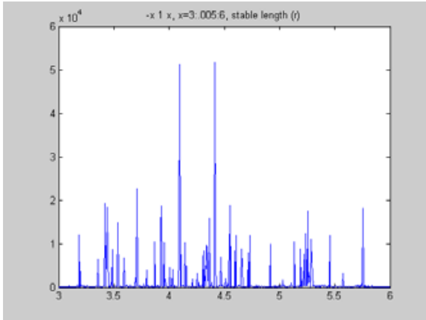
Columns 29 through 35

59.0000	61.0000	63.0000	65.0000	67.0000	69.0000	71.0000
40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250
40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250
61.0000	63.0000	65.0000	67.0000	69.0000	71.0000	73.0000





B1



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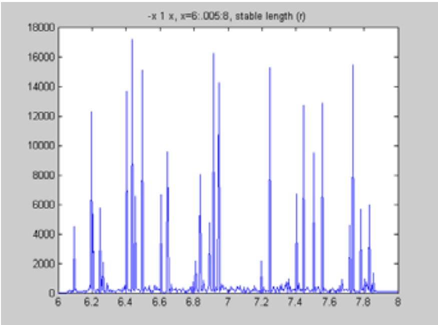

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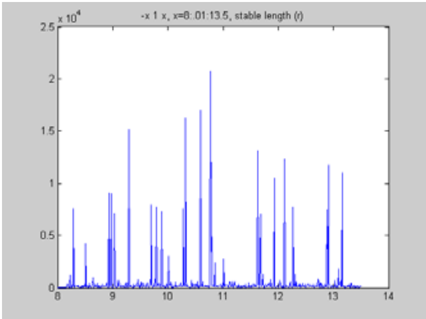

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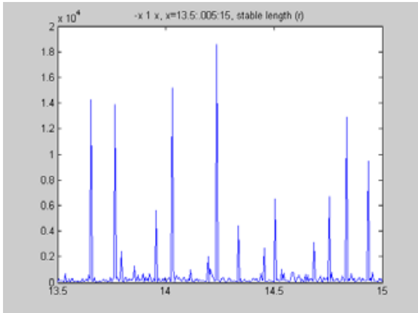

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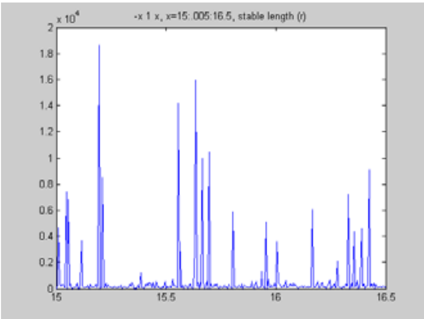
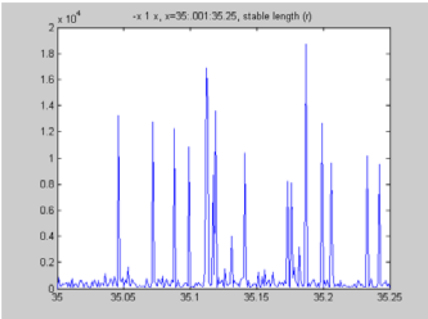


B6



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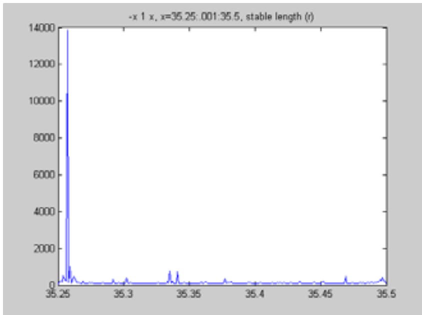

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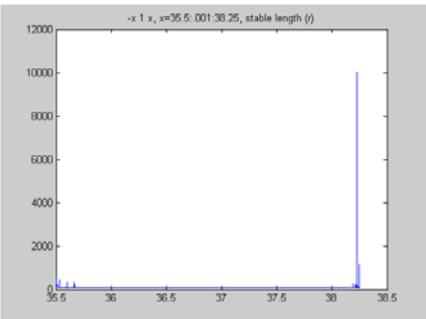

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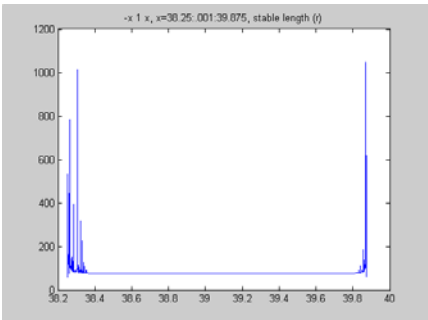

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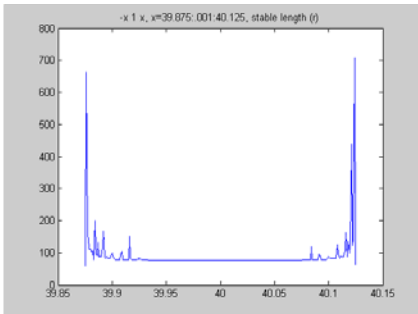

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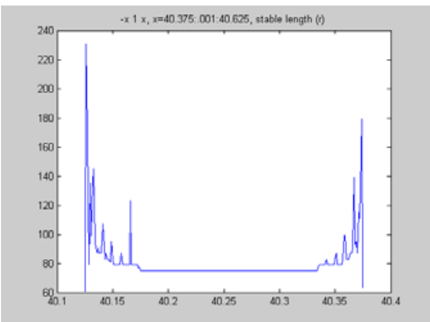

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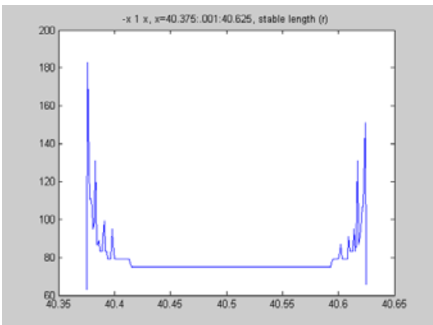
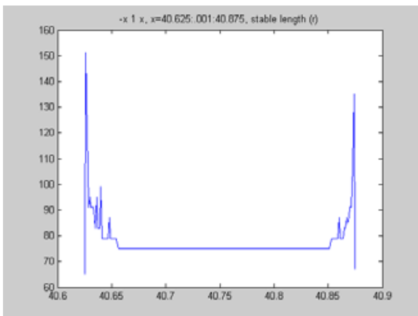
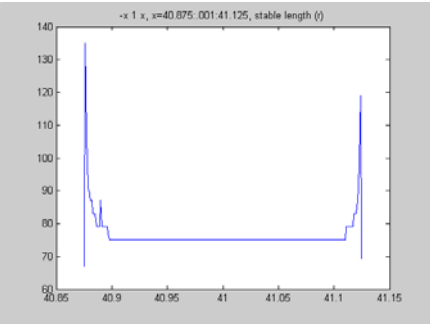


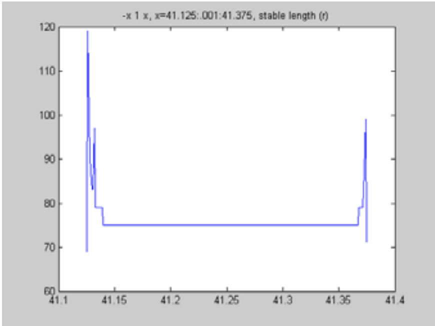
B30



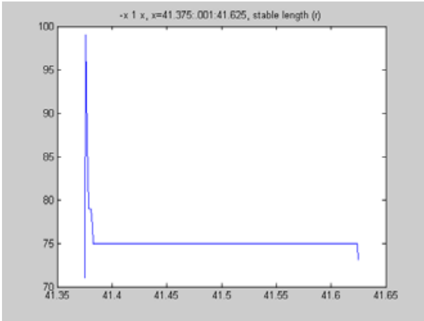
B32



B32



B33



B34

圖 7

八、節點鄰域的行為

接著我們想了解如果 M&m 數列的起始值 x 在節點附近時，這個 M&m 數列是否穩定及其穩定長度為何？

先以節點8為例，在圖 8 中顯示 $x=8$ 與 $x=8.00001$ 的 M&m 數列與中位數數列，前者穩定長度為 11，後者穩定長度為 61，在 11 以前兩張圖近似，在 11 以後 $x=8.00001$ 的 M&m 數列與中位數數列只有很小的差異，此微小的差異在圖中看不出來，經放大成圖 9，可以看出穩定前 M&m 數列與中位數數列的差與 0.00001 的級數(order)有關。

繼續透過實驗觀察發現只要初始值是 $x=8+2\varepsilon$ ，其中 $0<\varepsilon<<1$ ，則穩定長度皆為 61，於是我們又想找出**節點鄰域**的穩定長度模式。

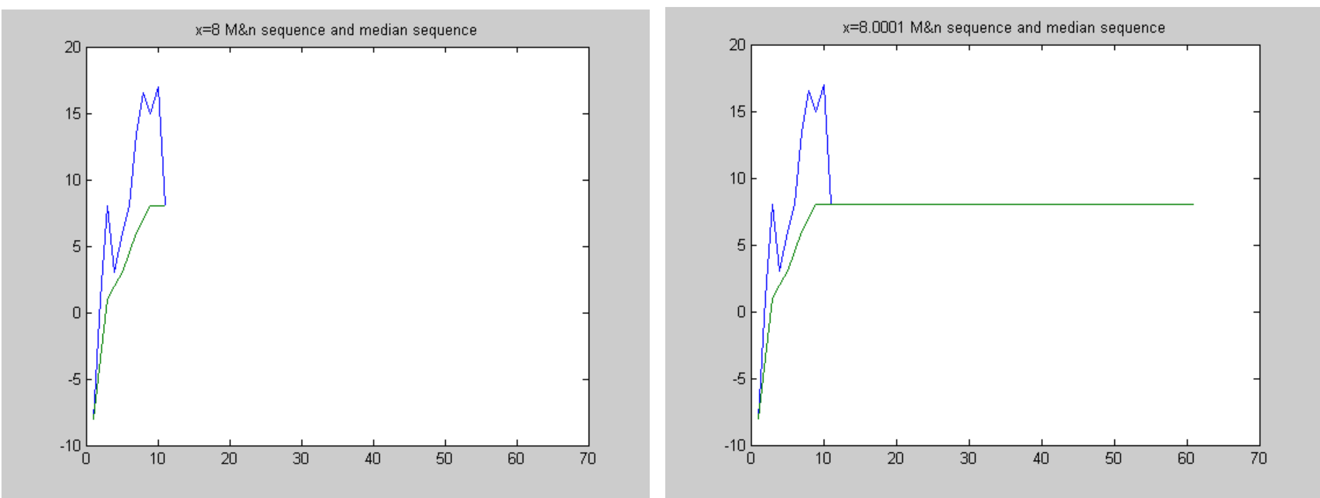


圖 8

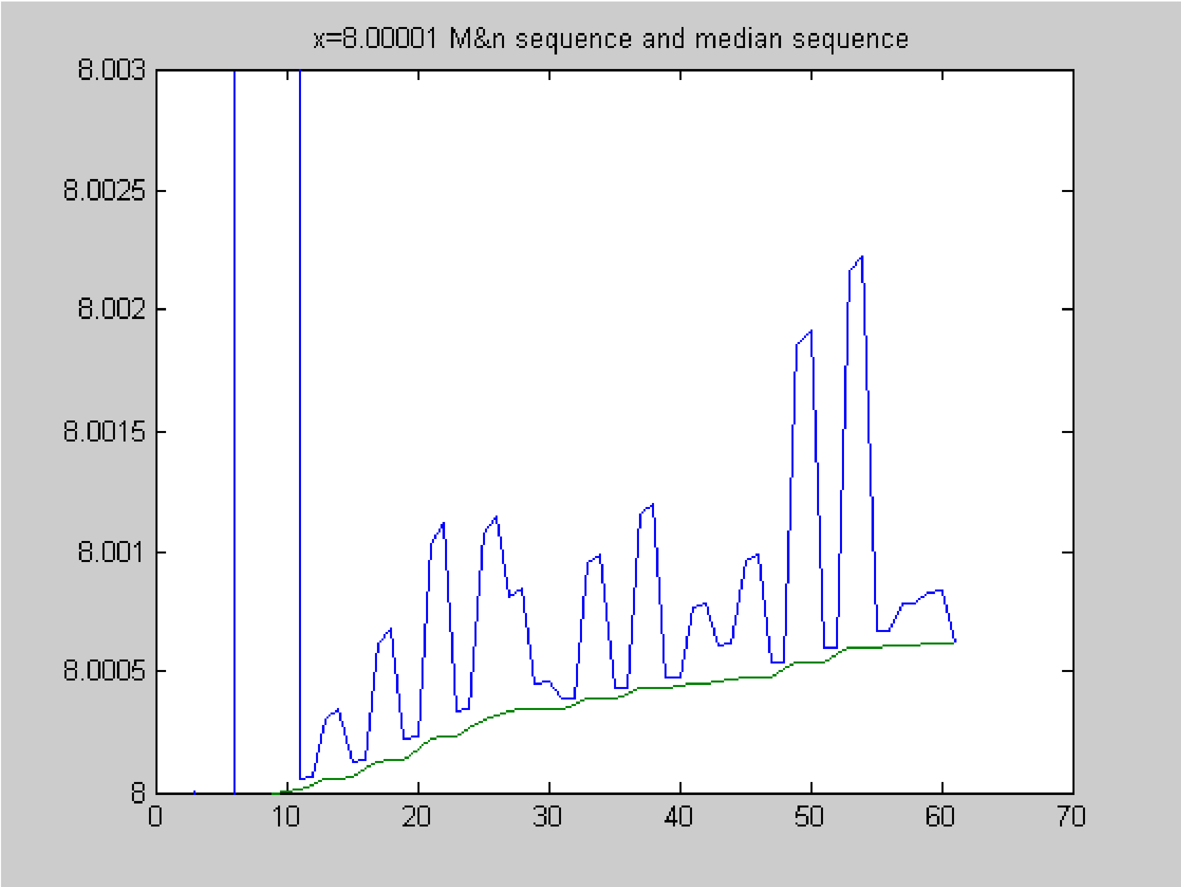


圖 9

以比節點8多一點的 $x=8+2\varepsilon$ 爲例，由表 5 知中位數節點位置 K=9，由表 4 知 $m_9=8$ ，利用(1)式 $a_{k+1}=m_k+k(m_k-m_{k-1})$ 知

$$a_{11}=m_{10}+10(m_{10}-m_9)=m_{10}+10\varepsilon=m_9+11\varepsilon=8+11\varepsilon$$

$$a_{12}=m_{11}+11(m_{11}-m_{10})=8+13\varepsilon \quad \cdots$$

觀察 a_{11} ， $a_{12}\cdots$後令 $b_1=0$ ， $m'_1=0$ ， $b_2=2$ ， $m'_2=1$

$$b_3=(a_{11}-m_9)/\varepsilon=11=m'_2+(9+2-1)(m'_2-m'_1) \text{，}$$

$$b_4=(a_{12}-m_9)/\varepsilon=13=m'_3+(9+3-1)(m'_3-m'_2) \text{，}$$

$$\cdots$$

$$b_{k+1}=m'_k+(9+k-1)(m'_k-m'_{k-1}))$$

也就是說比節點8多一點的 $x=8+2\varepsilon$ ，其收斂的長度爲節點8的穩定長度 11(K+2)加上新的 b 數列的穩定長度減 3。在這個例子中 b 數列的穩定長度爲 53，所以當 $x=8+0.01$，$x=8+0.001\cdots$ 或 $x=8+10^{-100}$ ，其穩定長度皆爲 53+11-3=61 或者說是 b 數列的穩定長度加 K(K=9)減 1。

將上述的推導擴充到任意 K 值 (K=3, 5, 7, ..., 67, 69) 知

$$a_{K+2}=m_{K+1}+(K+1)(m_{K+1}-m_K)=m_{K+1}+(K+1)\varepsilon=m_K+(K+2)\varepsilon$$

$$a_{K+3}=m_{K+2}+(K+2)(m_{K+2}-m_{K+1})=m_K+(K+4)\varepsilon \quad \cdots$$

令 $b_1=0$ ，$m'_1=0$ ，$b_2=2$ ，$m'_2=1$

$$b_3=(a_{K+2}-m_K)/\varepsilon=K+2=m'_2+(K+2-1)(m'_2-m'_1) ,$$

$$b_4=(a_{K+3}-m_K)/\varepsilon=K+4=m'_3+(K+3-1)(m'_3-m'_2) ,$$

$$\cdots$$

$$b_{k+1}=m'_k+(K+k-1)(m'_k-m'_{k-1}) \qquad (k>2) \qquad (2)$$

也就是說比節點多一點的 x ，其收斂的長度爲節點的穩定長度(K+2)加上新的 b 數列的穩定長度減 3 或者說是 b 數列的穩定長度加 K 減 1。

值得一提的是，在上述討論中的 $\varepsilon>0$ 只要滿足(充分條件)

$$8+\varepsilon\cdot\max\{b_k \mid 1\le k\le 53\}<13.5 \quad \text{(節點 8 的下一個節點值)},$$

那麼對所有這樣的 ε，當 $x\in(8,\,8+2\varepsilon)$ 時以(-x ,1, x)爲起始的 M&m 數列皆符合上述討論的結果。在此例中 $\max\{b_k \mid 1\le k\le 53=l(9)\}=444.875$ ，可以得到 $\varepsilon=(13.5-8)/444.875=0.012363$ ，其中 $l(9)$ 爲 K=9 時 b 數列的穩定長度，因此我們可以說當 $8\le x<8+2\varepsilon=8.024726$ ，其穩定長度皆爲 61 或者說是 b 數列的穩定長度加 K 減 1。在實際測試中即使 x 到達 8.0890，其穩定長度仍爲 61，如圖 10 所示，因此上式對 ε 的限制並非必要條件。

綜上所述我們可以寫成下列的性質：

性質 1： *對於所有的節點* $m_K=a\in\{1,3,6,8,13.5,...,41.375\}$, $K\in\{3,5,7,....,69\}$ (如表 5 所示) *都存在相對應的正數* ε *使得當* $x\in(a,\,a+2\varepsilon)$ *時，存在正整數* $l(K)$ *，使得，以(-x ,1, x)爲起始的 M&m 數列其收斂的穩定長度爲* $l(K)+K-1$

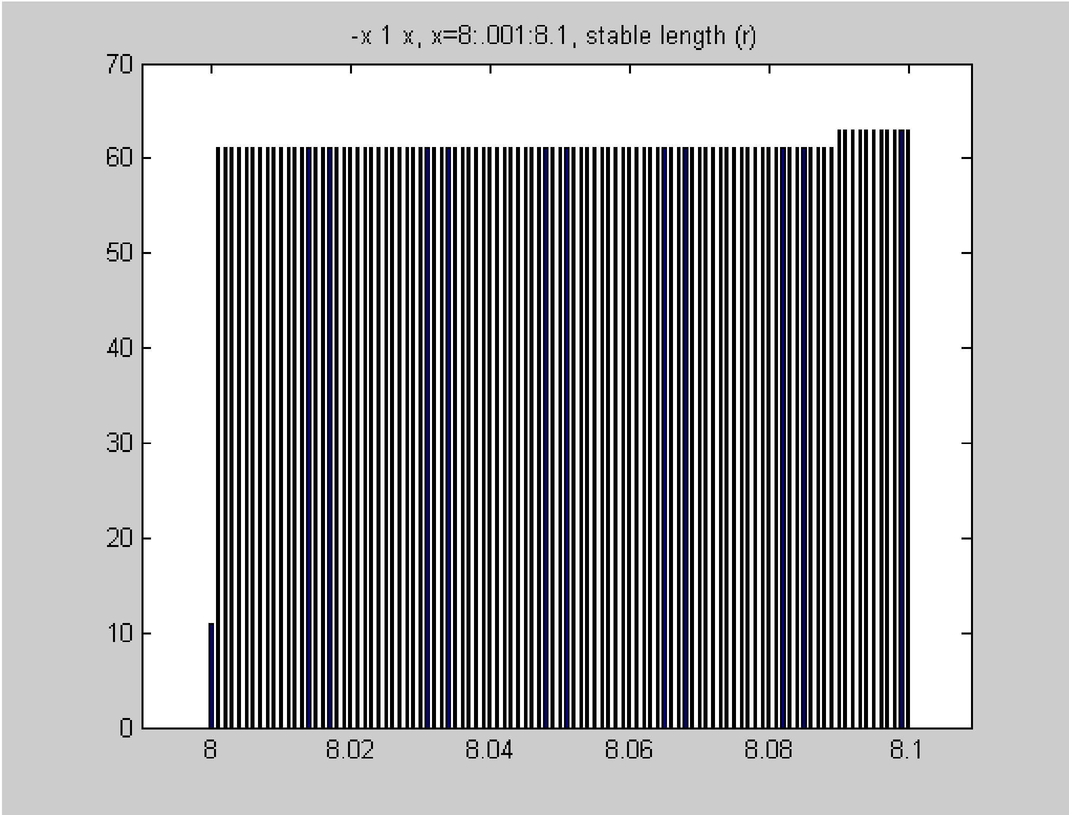


圖 10

接著我們把 K=3, 5, 7, ⋯, 67, 69 的 b 數列的穩定長度及比節點多一點的穩定長度列在表 6。例如：(1) $x = 41.3750$ (K=69)爲節點，b 數列的穩定長度爲 367，所以當 $x = 41.375001$，其穩定長度爲 203+69-1=271 ；(2) $x = 35$ (K=49)爲節點，b 數列的穩定長度爲 203，所以當 $x = 35.00001$，其穩定長度爲 367+49-1=415。

最後我們研究不同的節點其有效的鄰域範圍，有效的 ε 值滿足：

$$a_K + \varepsilon_K \cdot \max\{b_k \mid 1 \le k \le l(K)\} < a_{K+2} \quad \text{(節點} a_K \text{的下一個節點值)}$$

或者可以寫成

$$\varepsilon_K \le \frac{(a_{K+2} - a_K)}{\max\{b_k \mid 1 \le k \le l(K)\}}$$

由數學運算程式經過有限個步驟找出有效的 ε 值，結果如表 7 所示。使用表 5、表 6 與表 7 可以得知節點鄰域的收斂情形，例如由表 5 知 K=49 對應到節點 x=35，由表 7 知 $\varepsilon_{49} = 0.000026$，所以我們可以說當 $35 < x \le 35 + 2\varepsilon_{49} = 35.000052$ 時，由表 6 知道穩定長度爲 415。

表 6

Columns 1 through 10

K	3	5	7	9	11	13	15	17	19	21
b 之穩定長度($l(K)$)	71	59	45	53	249	249	237	141	171	215
節點 $x+2\varepsilon$ 之穩定長度	73	63	51	61	259	261	251	157	189	235

Columns 11 through 22

23	25	27	29	31	33	35	37	39	41	43	45
99	203	405	567	117	1147	1219	137	345	553	277	109
121	227	431	595	147	1179	1253	173	383	593	319	153

Columns 23 through 34

47	49	51	53	55	57	59	61	63	65	67	69
227	367	301	179	259	393	529	249	627	223	629	203
273	415	351	231	313	449	587	309	689	287	695	271

表 7

Columns 1 through 8

K	3	5	7	9	11	13	15	17
$(a_{K+2}-a_K)$	2.0000	3.0000	2.0000	5.5000	1.5000	1.5000	0.5000	3.7500
$\max\{b_k\}$	116.5625	318.50	325.1250	444.875	997.6250	1122.125	2138.625	2329.0625
ε_K	0.017158	0.009419	0.006151	0.012363	0.001504	0.001337	0.000234	0.001610

Columns 9 through 17

19	21	23	25	27	29	31	33	35
0.5000	2.0000	1.5000	1.2500	0.2500	0.2500	1.2500	1.8750	0.2500
2892.875	3007.5	1714.5	2471.9375	7491.5	3788.59375	7055.50	6167.1875	6520.750
0.000173	0.000665	0.000875	0.000506	0.000033	0.000066	0.000177	0.000304	0.000038

Columns 18 through 26

37	39	41	43	45	47	49	51	53
6842.5	6139.5	7762.375	15188.625	7870.25	8223.5	9549.8750	12527.09375	12233.5
0.2500	0.2500	3.8750	0.2500	0.2500	0.2500	0.2500	0.2500	2.7500
0.000037	0.000041	0.000499	0.000016	0.000032	0.000030	0.000026	0.000020	0.000225

Columns 27 through 34

55	57	59	61	63	65	67	69
17400.3125	27141.375	29075.0625	32780.5	4.206.5	28900.75	29266.75	21168.375
1.6250	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
0.000093	0.000009	0.000009	0.000008	0.000006	0.000009	0.000009	0.000012

伍、結論

本專題研究任取 3 個實數a_1、a_2、a_3爲起始的 M&m Sequences 之穩定性質。得到結果如下：

一、 當$a_1 = a_2 = a_3$或$a_1 < a_2 = a_3$或$a_1 = a_2 < a_3$，此 M&m Sequences 穩定。

二、當$a_1 < a_2 < a_3$時，若三數成等差則穩定。若三數不成等差，則透過線性變換$f(x) = \alpha x + \beta$轉換成以 –x , 1 , x (x>1)爲起始的數列。

我們證明經過線性變換$f(x) = \alpha x + \beta$轉換後的數列依然滿足 M&m Sequences 的定義。透過 MATLAB 程式試驗及觀察，可驗證我們得到的觀察結果：*當 $x \geq 41.625$，數列一定會收斂在 41.625 且收斂長度爲 73*，最後並提出定理及證明。接著我們猜測：如果 x 是任何小於 41.625 的實數，則由(–x , 1 , x) 爲起始所形成的 M&m 序列，其收斂的特性無簡單規則可尋，但是我們還是努力的找出一部份的結果，包括：

(1)若 x 等於節點，其穩定值必爲節點 x，同時節點穩定長度爲中位數節點位置+2；

(2)越靠近 41.625 的分枝，除了邊緣地帶以外幾乎是常數值，越靠近 1 的分枝，其穩定長度分布越混亂；

(3)在節點附近多一點(K= 3, 5, 7, …, 67, 69)的 x，即節點的鄰近區域，其收斂的穩定長度爲節點的穩定長度(K+2)加上新的 b 數列的穩定長度減 3，或者說是 b 數列的穩定長度加 K 減 1，最後並列出各節點的鄰域範圍。

陸、研究的困難度與未來展望

在這次研究中，我們已證明當$x > 41.625$及$x < 41.625$的某些部份必會穩定，同時找出其穩定性質。未來將繼續討論各分支穩定長度的模式，希望能證明所有以任意實數a_1、a_2、a_3爲起始的 M&m Sequences 都是穩定的，以解決 Shultz 及 Shiflett 在[1]中所提出的猜想。

研究的困難度與未來展望分述如下：

一、M&m Sequences 並不是一種完美的數列，例如$a_{k+1} = m_k + k(m_k - m_{k-1})$ 的運算中運用到中位數m_k及m_{k-1}，未穩定之前有一半的中位數m_q (q爲偶數)是來自於兩數相加除以 2，除以 2 的運算可能會增加一個有效位數，因此在某些情況下我們並不知道穩定值之有效位數是否會

趨近無窮大，我們猜想是不會趨近無窮大，但是目前卻無法證明。

二、本研究使用 MATLAB 當工具，他的有效位數是 15，因此在實驗過程我們要時時注意穩定是否是因爲有效位數有限所造成的假穩定現象，另一個誤差的來源爲計算機的二進位與人爲計算的十進位誤差，例如 MATLAB 計算(8.00001-8)/2 = 4.999999999810711e-006，但是我們都知道答案是 5 e-006。

三、我們還將繼續討論非節點加上一個很小的數。舉一個例子，假設 $x = 4 + \delta$ ，其中 $\delta = 10^{-20}$ ，目前透過理論及實驗我們都無法知道穩定值及穩定長度是多少，這個問題也許需要更創新的方法才能解決。

柒、參考文獻

[1] H. S. Shultz and R. C. Shiflett, " M&m Sequences", *The College Mathematics Journal*, May 2005 (NO.3 Vol.36), pp. 191-198.

[2] M. Chamberland, M. Martelli, "THE MEAN-MEDIAN MAP", *Journal of Difference Equations and Applications* ***13***, Jan 2007, pp. 577-583.

捌、附件

附件一(圖 1 的 MATLAB 程式)

```
% figure_1
clear, clf,
I=47;
A=[ 0      67      78 ];

a1=A(1); a2=A(2); a3=A(3);
for i=1:3
        B(i)=(a1+a3-2*A(i))/(a1+a3-2*a2);
end

Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
```

```
    Med(j)=median(A);
end

subplot(211), plot(A),
title('( 0      67      78 )')
ylabel('sequence value')

A=sort(B);
Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
    j=i+3;
    A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
    Med(j)=median(A);
end

subplot(212), plot(A),
title('(-1.3929      1.0000      1.3929)')
ylabel('sequence value')
```

附件二(圖 2 的 MATLAB 程式)

```
% figure_2
clear, clf

I=10000;
format long
X=1:.5:50; lx=length(X);
for k=1:lx
    A=[-X(k) 1 X(k)]; C(k)=-I/10;
    Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
    A(4)=Med(3)*4-sum(A); Med(4)=median(A);
    for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
        Med(j)=median(A);
        if A(j)==Med(j-1),   B(k)=A(j); C(k)=j; break, end
    end
end

subplot(211), plot(X,B), title('-x 1 x, x=1:.5:50, stable value')
subplot(212), plot(X,C), title('-x 1 x, x=1:.5:50, stable length')
```

附件三(圖 5 的 MATLAB 程式)

```
% figure_4
clear, clf

I=347;

x=25;
A=[-x 1 x];
Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
	j=i+3;
	A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
	Med(j)=median(A);
end

A(2,:)=Med;
plot(A'), title('A=[-25 1 25] M&n sequence and median sequence')
```

附件四 Excel 程式表格 1

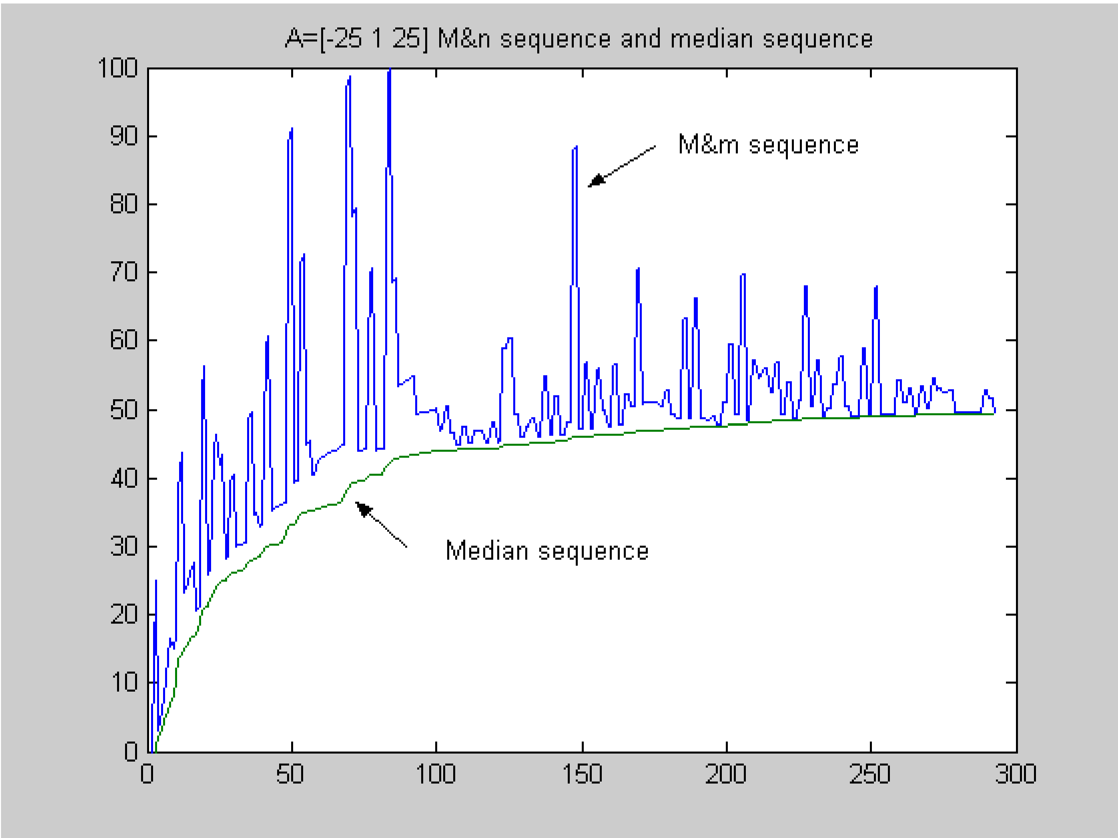
11		length =	13		
25		stable value =	40		
33					Median
31					25
40					28
46					31
38					32
40					33
55.5					35.5
60.5					38
49					39
51					40
40					40
40					40
40					40
40					40
40					40
40					40
40					40
40					40
40					40

評語

本作品討論 M&m 數列之收斂性與收斂長度，作者對文獻中所討論之(0,x,x+1)方法，提出改以(-x,1,x)來產生數列，這樣的改變使得此數列收斂相關性質的討論，遠較文獻的結果更爲完整。本作品結構相當清楚，原創性亦高，已具此類領域研究之一定水準，這對一高中生而言，更屬難得。

2008 ISEF / Mathematics
Research Paper

An Approach to Solving the M&m Conjecture



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Contents

I. M&m Sequence and M&m conjecture

Given any three real numbers a_1, a_2, a_3, let the 4[th] number a_4 satisfies "the Mean of the first 4 numbers equals the median of the first three numbers", i.e.,

$$(a_1 + a_2 + a_3 + a_4)/4 = median(a_1, a_2, a_3)\,, \tag{1}$$

thus

$$a_4 = 4 \cdot median(a_1, a_2, a_3) - (a_1 + a_2 + a_3)\,. \tag{2}$$

Similarly, for $n \geq 4$, let

$$a_{n+1} = (n+1) \cdot median\ (a_1, a_2, ..., a_n) - (a_1 + a_2 + ... + a_n)\,. \tag{3}$$

Sequence $< a_n >$ is called the M&m sequence with initial terms (a_1, a_2, a_3).

Note that in order to find the median, we need to place the numbers in increasing order, then the median is the number in middle place or the average of the middle two numbers.

For example, the M&m sequence $< a_n >$ with initial terms (11, 20, 27) is shown below:

11, 20, 27, 22, 25, 27, 32.5, 35.5, 34, 36, **27, 27, 27,** …

where $a_n = 27$ for $n \geq 11$.

The M&m sequence $< a_n >$ is stable if there is a constant *s* and an integer *k* such that $a_n = s$ for $n \geq k$ and $a_{k-1} \neq s$. The constant value *s* is called the stable value, and *k* is called the oscillating length. In the above example: stable value is 19 and the oscillating length is 11.

Based on Schultz and Shiflett's investigations [1], they provided the M&m conjecture: ***Every M&m sequence*** $< a_n >$ is **stable**.

II. Computer Experiment

For given initial terms (a_1, a_2, a_3), the remaining terms of the associated M&m sequence can be generated using a spreadsheet program such as Excel. Here is the procedure:

1. List the index 1..800 along Column A.
2. Enter the initial terms on cells B1, B2 and B3.
3. On Cell B4, enter the formula =$A4*MEDIAN(B$1:B3)-SUM(B$1:B3)
4. Copy the formula from B4 to B4..B800.
5. Column B then displays the numerical values of the sequence $a_1, a_2, ..., a_{800}$.

A new M&m sequence is regenerated whenever the initial terms are changed. This completes the computation setup of the computer experiment.

From computer experiment, we find the following three important properties:

1) Invariance under permutation: No matter how the initial terms are permuted, the remaining terms are unchanged.
2) Effect of scalar multiplication: The nth term of the M&m sequence with initial terms $(\alpha a_1, \alpha a_2, \alpha a_3)$ is αa_n for all $n \geq 1$.
3) Effect of termwise addition: The nth term of the M&m sequence with initial terms $(a_1+\beta, a_2+\beta, a_3+\beta)$ is $a_n+\beta$ for all $n \geq 1$.

III. Normalization of the M&m Sequence

Given any $a_1 \leq a_2 \leq a_3$, with $2a_2 \neq a_1 + a_3$. Let $\beta = -(a_1+a_3)/2$, $\alpha = 1/(a_2+\beta)$, and

$$x = \alpha(a_3+\beta) = \frac{a_3 - a_1}{2a_2 - a_3 - a_1}, \tag{4}$$

then the original M&m sequence with (a_1, a_2, a_3) can be normalized to a new M&m sequence with initial terms $-x = \alpha(a_1+\beta)$, $1 = \alpha(a_2+\beta)$, $x = \alpha(a_3+\beta)$, where $|x| \geq 1$.

For example, given $a_1 = 0$, $a_2 = 67$, $a_3 = 78$, then $x = 1.3929$, that is, the initial terms are normalized to (-1.3929, 1.0000, 1.3929). These two M&m sequences have the same pattern of behavior (as shown in Fig.1). Thus we focus on the M&m sequence with initial terms $(-x, 1, x)$.

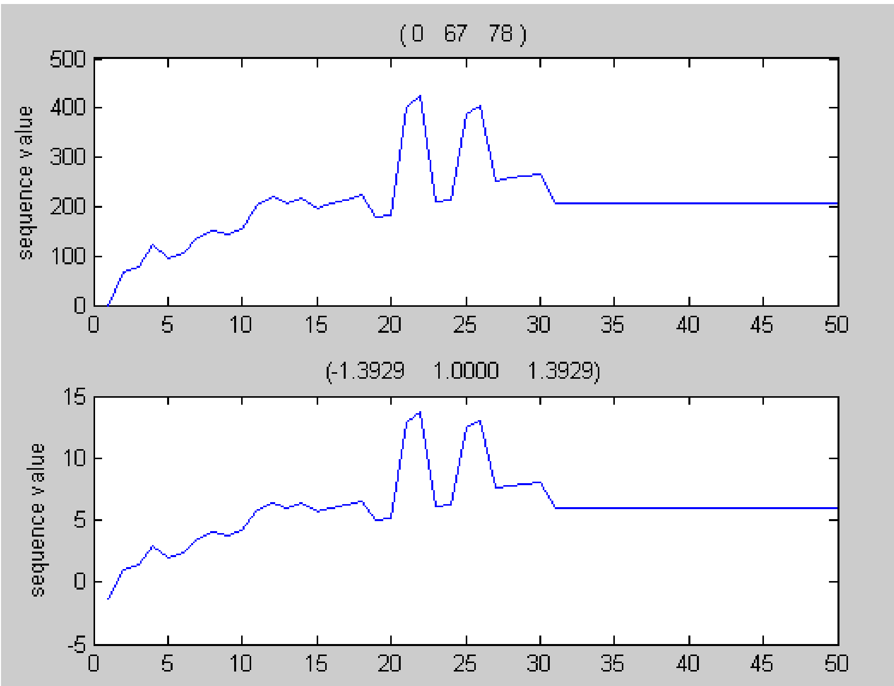


Fig.1 Behavior of M&m Sequences before and after norm

IV. Main Results

A. Three lemmas

Let $\{a_k\}_{k\in N}$ is an M&m sequence with initial terms $a_1 = -x$, a_2 =1, and $a_3 = x$, $a_4 = 1\times 4 - (-x+1+x) = 3$. Let $S_n \equiv \sum_{i=1}^{n} a_i$, and m_k denote the median of the first *k* values, in Eq. (A2), $m_k = \frac{S_k + a_{k+1}}{k+1}$, that is, $S_{k+1} = m_k(k+1)$, hence we have

$$a_{k+1} = S_{k+1} - S_k = (k+1)m_k - km_{k-1} = m_k + k(m_k - m_{k-1}) \quad k \geq 4 \tag{5}$$

Lemma 1： *When exist* $k > 4$, $k \in N$ ，*such that* $m_{k-1} = m_{k-2}$, *then the sequence is stable, and the stable length is* ∶ $p = \min\{k \mid k > 4 \text{ and } m_{k-1} = m_{k-2}\}$, *where p must be an odd number.*

Proof. Let $A = \{k \mid k > 4 \text{ and } m_{k-1} = m_{k-2}\}$ and $p = \min\{k \mid k > 4 \text{ and } m_{k-1} = m_{k-2}\}$, from Eq. (5) and the definition of *p*, we have

$$a_p = m_{p-1} + (p-1)(m_{p-1} - m_{p-2}) = m_{p-1} \tag{6}$$

Go ahead one step, we obtain

$$\begin{cases} m_p = \dfrac{a_p + m_{p-1}}{2} = m_{p-1} & p \text{ is even} \\ m_p = m_{p-1} & p \text{ is odd} \end{cases} \tag{7}$$

Similarly, $m_p = m_{p+1} = m_{p+2} = \cdots$ and from Eqs.(6) and (7), $a_p = a_{p+1} = a_{p+2} = \cdots$, so $m_{k-1} = m_{k-2}$ results in the sequence stable and the length *p*.

Next, we prove that *p* must be an odd number by the method of reduction ad absurdum. Assume *p* is even, and known $m_{p-1} = m_{p-2} \neq m_{p-3}$, since $p-2$ is also even, we have

$$m_{p-2} = \frac{1}{2}(m_{p-1} + m_{p-3}) \tag{8}$$

Following Eq. (8), we have $m_{p-2} = m_{p-3}$. This violate the true $m_{p-1} = m_{p-2} \neq m_{p-3}$, so *p* must be an odd number.

For example, let $x = 8$ ($a_1 = -8, a_2 = 1, a_3 = 8$), the M&m sequence and the median sequence are list in Table 1, respectively. In this example, $a_{11} = m_{10} = m_9 = 8$ and the stable length $p = 11$, it is an odd number definitely. Given two more examples: (1) $x = 41.3750$, $a_{71} = m_{70} = m_{69} = 41.3750$

and the stable length $p=71$ is odd; (2) $x=35$, $a_{51}=m_{50}=m_{49}=35$ and the stable length $p=51$ is also odd.

Table 1 M&m sequence and the corresponding median sequence for $x=8$

	1	2	3	4	5	6	7	8	9	10	**11**	12
M&m sequence	-8.0	1.0	8.0	3.0	6.0	8.0	13.5	16.5	15.0	17.0	**8.0**	8.0
Median sequence			1.0	2.0	3.0	4.5	6.0	7.0	**8.0**	**8.0**	8.0	8.0

Lemma 2： *The median sequence {* m_n *} monotonically increasing and* $a_n > m_{n-1}$ *until the M&m sequence reach stable state.*

Proof. Using mathematical induction,

1. We first prove $m_4 > m_3$.

Because the initial three numbers are $-x, 1, x\ (x>1)$, we have $m_3 = 1$ and then $a_4 = 3$.

(i) When $x \geq 3$, $m_4 = 2 > m_3$; (ii) When $x < 3$, $m_4 = \frac{1+x}{2} > 1 = m_3$.

Combining (i) and (ii) we get $m_4 > m_3$.

2. Suppose. $m_{n-1} > m_{n-2}$ (n>5), from Eq. (5) we obtain

$$a_n = S_n - S_{n-1} = m_{n-1} + (n-1)(m_{n-1} - m_{n-2}) > m_{n-1} \tag{9}$$

Since $a_n > m_{n-1}$, we get $m_n > m_{n-1}$.

3. Suppose. $m_{n-1} = m_{n-2}$ (n>5), from Lemma 1, we have

$$a_n = S_n - S_{n-1} = m_{n-1} + (n-1)(m_{n-1} - m_{n-2}) = m_{n-1} \tag{10}$$

By mathematical induction and Lemma 1, we conclude that the median sequence { m_n } monotonically increasing and $a_n > m_{n-1}$ until the M&m sequence reach stable state.

To describe the concept in Lemma 2, we take an example which starting three numbers (–25 , 1 , 25). The M&m sequence and the associated median sequence are both plotted in Fig. 2 (Matlab program is shown in Appendix D). From Fig. 2 we clearly see that the median sequence { m_n } monotonically increasing and $a_n > m_{n-1}$ until the M&m sequence reach stable state. The same result appeared in the previous example (Table 1).

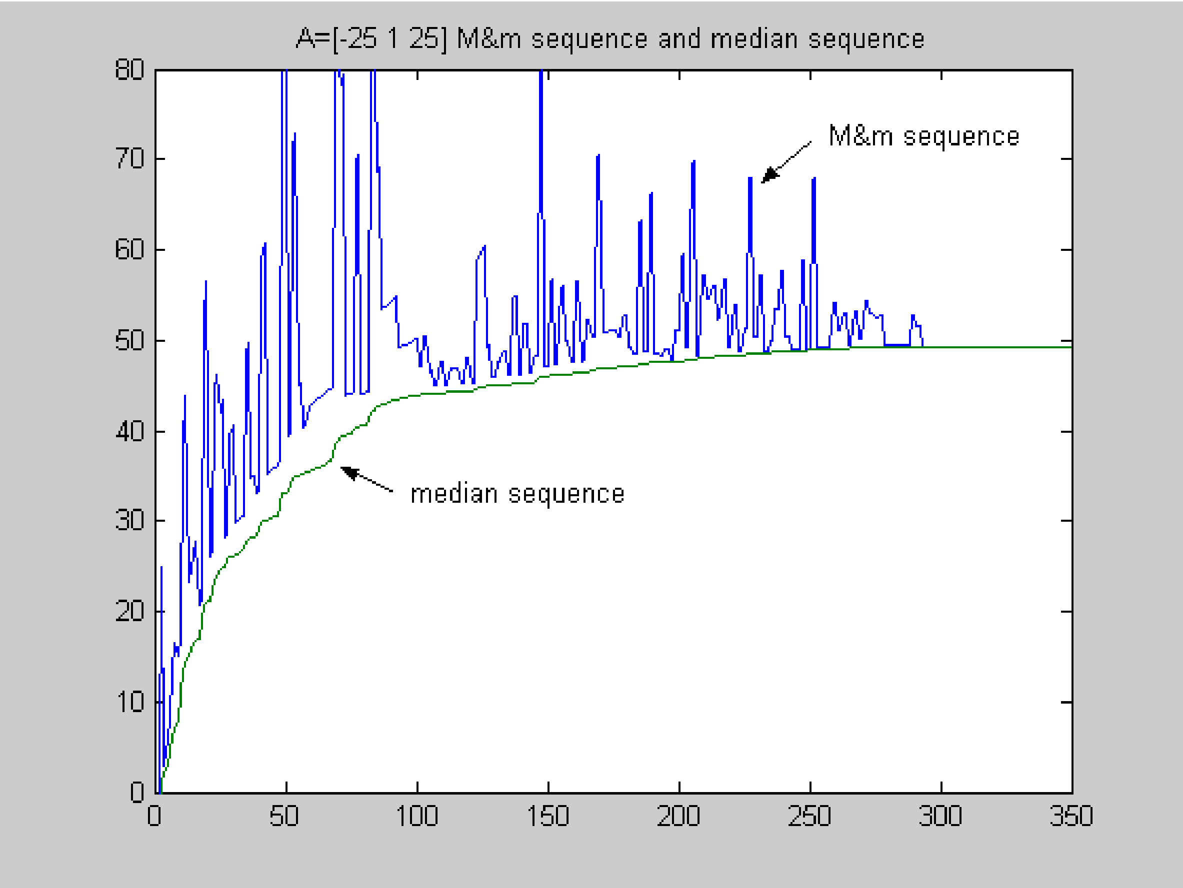


Fig. 2 M&m sequence and the corresponding median sequence for $x = 25$

Lemma 3： *Let* $\{a_k\}_{k \in N}$ *is an M&m sequence. Suppose there exist two nature number* r *and* q, *where* $r > q \geq 3$, *such that* $a_r = a_q$, *then the sequence is stable and if* $k' = \min A$ *where* $A = \{k \mid k \in N \text{ and } m_k = a_q\}$, *then the oscillating length is* $k'+2$ *and the stable value is* a_q.

Proof. According to the Theorem 2.4 in [2] we have the sequence is stable. From Eq. (5), we derive that if it doesn't reach stable state, $a_r > m_{r-1}$ and $a_q > m_{q-1}$. By Lemma 2, in the case when the median sequence reaches the value $m_{k'} = a_q$, then $m_{k_1+1} = \dfrac{m_{k'} + a_q}{2} = \dfrac{a_r + a_q}{2} = a_q$, two successive medians will be the same, that is, the oscillating length is $k'+2$ and from lemma 2 we obtain the stable value is a_q.

For example, the M&m sequence starting with $a_1 = -2, a_2 = 1, a_3 = 2, \ (x = 2)$ is listed in Table 2. In that there exist two nature numbers $q = 12$ and $r = 23$, where $r > q \geq 3$, such that $a_r = a_{23} = a_q = a_{12} = m_{k'} = m_{33} = 10$. It is obvious that this sequence is stable with oscillating length 33+2=35 and the stable value 10.

Table 2 M&m sequence and the corresponding median sequence for $x = 2$

Columns 1 through 9

M&m	-2.0000	1.0000	2.0000	3.0000	3.5000	4.5000	5.5000	6.5000	5.2500
median	-2.0000	-0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.2500	3.5000

Columns 10 through 18

5.7500	9.0000	**10.0000**	9.3750	10.1250	7.1250	7.3750	7.6250	7.8750
4.0000	4.5000	4.8750	5.2500	5.3750	5.5000	5.6250	5.7500	6.1250

Columns 19 through 27

12.8750	13.6250	13.0625	13.6875	**10.0000**	10.2500	10.5000	10.7500	11.0000
6.5000	6.8125	7.1250	7.2500	7.3750	7.5000	7.6250	7.7500	7.8750

Columns 28 through 35

11.2500	24.1875	25.3125	14.8125	15.1875	19.6875	20.3125	10.0000
8.4375	9.0000	9.1875	9.3750	9.6875	**10.0000**	10.0000	10.0000

B. Case 1: $x \geq 41.625$

Theorem 1: *If $x \geq 41.625$, M&m sequence with initial terms $-x, 1, x$ has stable value 41.625 and the oscillating length is 73.*

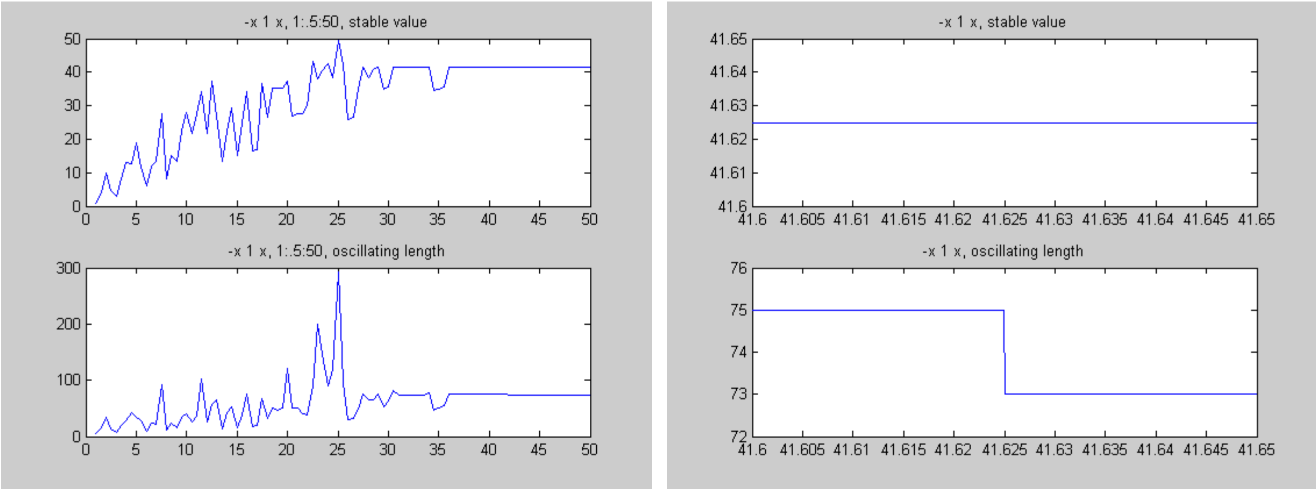
Proof. 1. Let $\{a_n\}$ and $\{m_n\}$ be the M&m and the median sequence with initial terms ($-41.625, 1, 41.625$). Table 3 shows that the stable value is 41.625 and the oscillating length is 73.

2. For $x > 41.625$, say $x = 41.625 + y$, where $y > 0$. Let $\{c_n\}$ be the M&m sequence with initial terms ($-x, 1, x$), and the median $m'_k = median(c_1, c_2, ..., c_k)$. Then we have $c_4 = 3c_2 - (c_1 + c_3) = 3 (= a_4)$, $m'_4 = (1+3)/2 = 2$.

3. Because of $m'_3 = m_3$, $m'_4 = m_4$ (since $c_3 > 41.625$ can't change the median sequence) and $a_{k+1} = m_k + k(m_k - m_{k-1})$, $c_{k+1} = m'_k + k(m'_k - m'_{k-1})$, $k \geq 3$, we have $c_5 = a_5$, Then $m'_5 = m_5$, similarly, $c_6 = a_6$, $m'_6 = m_6$, and so on.

That is, $c_n = a_n$ for $n \geq 4$. So $\{a_n\}$ and $\{c_n\}$ have the same stable value 41.625 and oscillating length 73.

Fig. 3(a) shows the stable value and the oscillating length of the M&m sequence with initial value ($-x, 1, x$) for $1 \leq x \leq 50$. Fig. 3(b) is the same as Fig. 3(b) except the range in x-axis. We observe that the stable value is always 41.625 and the oscillating length is always 73 for $x \geq 41.625$.



(a) (b)

Fig. 3 (a) Stable value and length for different x and (b) for different x close to 41.625.

Table 3 M&m sequence and the corresponding median sequence for $x = 41.625$

Columns 1 through 9

M&m	-41.6250	1.0000	41.6250	3.0000	6.0000	8.0000	13.5000	16.5000	15.0000
Median	-41.6250	-20.3125	1.0000	2.0000	3.0000	4.5000	6.0000	7.0000	**8.0000**

Columns 10 through 18

17.0000	38.2500	43.7500	23.2500	24.7500	26.2500	27.7500	20.7500	21.2500
10.7500	13.5000	14.2500	15.0000	15.7500	16.5000	16.7500	17.0000	18.8750

Columns 19 through 27

52.6250	56.3750	26.0000	26.5000	44.2500	46.2500	42.0000	43.5000	**41.6250**
20.7500	21.0000	21.2500	22.2500	23.2500	24.0000	24.7500	25.3750	26.0000

Columns 28 through 36

42.8750	29.6250	29.8750	30.1250	30.3750	47.1250	48.3750	60.5625	62.4375
26.1250	26.2500	26.3750	26.5000	27.1250	27.7500	28.6875	29.6250	29.7500

Columns 37 through 45

34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	113.6875	117.5625	39.8750
29.8750	30.0000	30.1250	30.2500	30.3750	32.3125	34.2500	34.3750	34.5000

Columns 46 through 54

40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	**41.6250**	41.8750	42.1250
34.6250	34.7500	34.8750	35.0000	35.1250	35.2500	35.3750	35.5000	36.8750

Columns 55 through 63

111.1250	113.8750	84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500
38.2500	39.0625	39.8750	40.0000	40.1250	40.2500	40.3750	40.5000	40.6250

Columns 64 through 72

48.5000	48.7500	49.0000	49.2500	49.5000	49.7500	50.0000	50.2500	50.5000
40.7500	40.8750	41.0000	41.1250	41.2500	41.3750	41.5000	**41.6250**	**41.6250**

Columns 73 through 77

41.6250	41.6250	41.6250	41.6250	41.6250
41.6250	41.6250	41.6250	41.6250	41.6250

For example, we choose $x = 41.625$ and $x = 999$ to run the oscillating lengths, and obtain the results shown in Fig. 4 and Table 4. In Fig. 4, two subfigures are obviously the same except the first three terms. Also in Table 4, two sequences are the same except the first three numbers. Hence we see that when x is greater than or equal to 41.625, all the M&m sequences are the same after the forth term and all stable values are 41.625, as well as the stable lengths 73.

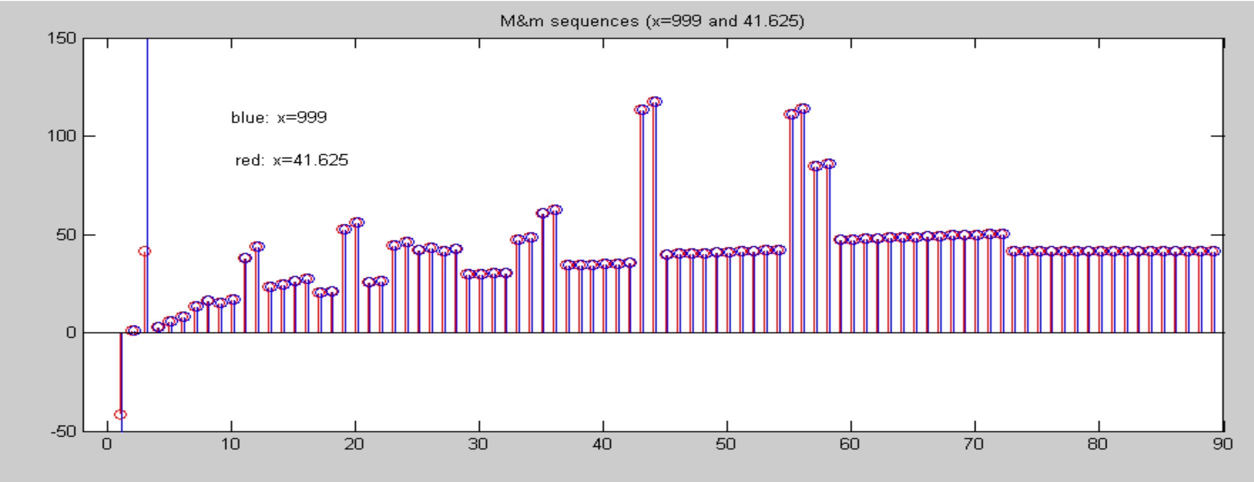


Fig. 4 Behavior of both both M&m Sequences for $x = 41.625$ and $x = 999$

Table 4 M&m Sequences for $x = 41.625$ and $x = 999$

Columns 1 through 11										
-41.6250	1.0000	41.6250	3.0000	6.0000	8.0000	13.5000	16.5000	15.0000	17.0000	38.2500
-999.0000	1.0000	999.0000	3.0000	6.0000	8.0000	13.5000	16.5000	15.0000	17.0000	38.2500
Columns 12 through 22										
43.7500	23.2500	24.7500	26.2500	27.7500	20.7500	21.2500	52.6250	56.3750	26.0000	26.5000
43.7500	23.2500	24.7500	26.2500	27.7500	20.7500	21.2500	52.6250	56.3750	26.0000	26.5000
Columns 23 through 33										
44.2500	46.2500	42.0000	43.5000	41.6250	42.8750	29.6250	29.8750	30.1250	30.3750	47.1250
44.2500	46.2500	42.0000	43.5000	41.6250	42.8750	29.6250	29.8750	30.1250	30.3750	47.1250
Columns 34 through 44										
48.3750	60.5625	62.4375	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	113.6875	117.5625
48.3750	60.5625	62.4375	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	113.6875	117.5625
Columns 45 through 55										
39.8750	40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250	41.8750	42.1250	111.1250
39.8750	40.1250	40.3750	40.6250	40.8750	41.1250	41.3750	41.6250	41.8750	42.1250	111.1250
Columns 56 through 66										
113.8750	84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500	48.5000	48.7500	49.0000
113.8750	84.5625	86.1875	47.2500	47.5000	47.7500	48.0000	48.2500	48.5000	48.7500	49.0000
Columns 67 through 77										
49.2500	49.5000	49.7500	50.0000	50.2500	50.5000	41.6250	41.6250	41.6250	41.6250	41.6250
49.2500	49.5000	49.7500	50.0000	50.2500	50.5000	41.6250	41.6250	41.6250	41.6250	41.6250

C. Case 2: $x < 41.625$

With thousands of computer experiments, we observe that if x is a positive real number less than 41.625, the M&m sequence is also stable but does not appear to follow any regular behavior. However, we obtain two more important theorems.

Select numbers which are less than 41.625 from the M&m sequence in Table 4, and form a **(node) set** $\Omega = \{a_k \mid a_k < 41.625, 2 \le k \le 72\}$. There are 34 elements in Ω (shown in Table 5). We call these elements **nodes,** presented as d_i, then arrange the elements, so that $d_1 < d_2 < ... < d_{34}$. It is obviously that $d_i = median(a_1, a_2, ..., a_{2i+1})$, for convenient we set $K = 2i + 1$. Then we have:

Theorem 2: *If* d_i *is a node,* $i \in \{1, 2, ..., 34\}$ *the M&m sequence with initial terms* $(-d_i, 1, d_i)$ *has the stable value* d_i *and the oscillating length* $K + 2$, *where* $K = 2i + 1$.

Proof. 1) Let $\{a_n\}$ and $\{m_n\}$ be the M&m and the median sequence with initial terms ($-41.625, 1, 41.625$), as shown in Table 4.

2) For $x = d_i$, let $\{c_n\}$ be the M&m sequence with initial terms ($-d_i, 1, d_i$), and the median $m'_k = median(c_1, c_2, ..., c_k)$.

3) Because $m'_3 = m_3$, $m'_4 = m_4$, … $m_n = m'_n$ for $3 \le n \le K$ and $a_{n+1} = m_n + n(m_n - m_{n-1})$, $c_{n+1} = m'_n + n(m'_n - m'_{n-1})$, then we have $c_n = a_n$ for $4 \le n \le K$ and $m_K = m'_K = d_i$. Since $m'_K = d_i$, $m'_{K+1} = (d_i + m'_K)/2 = d_i$. Based on property 1, we obtain that the stable value is d_i and the oscillating length is $(K+1)+1 = K+2$.

For example, (1) $x = 41.3750$ is a node, its stable value is $d_{34} = 41.3750$ and the oscillating length $K + 2 = 69 + 2 = 71$; (2) $x = 35$ is another node, its stable value is $d_{24} = 35$, is also the node $x = 35$ and the oscillating length $K + 2 = 49 + 2 = 51$.

Table 5 List of nodes in ascending order.

i	1	2	3	4	5	6	7	8	9	10
Node	1.0000	3.0000	6.0000	8.0000	13.5000	15.0000	16.5000	17.0000	20.7500	21.2500
i	11	12	13	14	15	16	17	18	19	20
Node	23.2500	24.7500	26.0000	26.2500	26.5000	27.7500	29.6250	29.8750	30.1250	30.3750
i	21	22	23	24	25	26	27	28	29	30
Node	34.2500	34.5000	34.7500	35.0000	35.2500	35.5000	38.2500	39.8750	40.1250	40.3750
i	31	32	33	34						
Node	40.6250	40.8750	41.1250	41.3750						

Now, building on the ideal of node, we would like to study the behavior of stable behavior as initial value x near the node. That is, we want to understand whether the M&m sequence is stable when x is a little more than a node. And what are their oscillating lengths?

We first observe an example, say node $d_4 = 8$, the M&m sequence and the median sequence are plotted in Fig. 5(a). And then we observe another example, say $x = 8.00001$ which is near the node $d_4 = 8$, the M&m sequence and the median sequence are plotted in Fig. 5(b), Both M&m sequences are partially listed in Table 6. The oscillating length of the former is 11, and the latter is 61. The behaviors of both are the same before a_{11}. After a_{11}, in the case of $x = 8.00001$, the difference between the M&m sequence and the median sequence is very small. The small difference is not clearly seen, but by amplifying this part, we renew plotting it as Fig. 6(a). From Fig. 6(a), we see that the difference before stable is related to the order (about 0.00001) of the small distance between the two initial values.

Now, we first calculate the difference, then minus 8 and divided by $0.000005\,(= 0.00001/2)$, then plot the result as Fig. 6(b), we see that it is similar to Fig. 6(a).

Based on computer experiment step by step, we find that if the initial value is $x = 8 + 2\varepsilon$, where $0 < \varepsilon << 1$, then the stable value is always 61, this result motivates us to go on searching the behavior of stability when initial values near the nodes.

For example, let $x = d_4 + 2\varepsilon = 8 + 2\varepsilon = 8.00001$ ($\varepsilon = 0.000005$), we know the index is $K = 2i + 1 = 9$, from Table 6, we have $m_9 = 8$. Using Eq. (5) $a_{k+1} = m_k + k(m_k - m_{k-1})$ we obtain

$$a_{11} = m_{10} + 10(m_{10} - m_9) = m_{10} + 10\varepsilon = m_9 + 11\varepsilon = 8 + 11\varepsilon = 8.000055 \quad (11)$$

$$a_{12} = m_{11} + 11(m_{11} - m_{10}) = 8 + 13\varepsilon = 8.000065 \quad (12)$$

$\dots$

Observe a_{11}, $a_{12} \dots$ and let $b_1 = 0$, $m'_1 = 0$, $b_2 = 2$, $m'_2 = 1$ to generate a modified sequence,

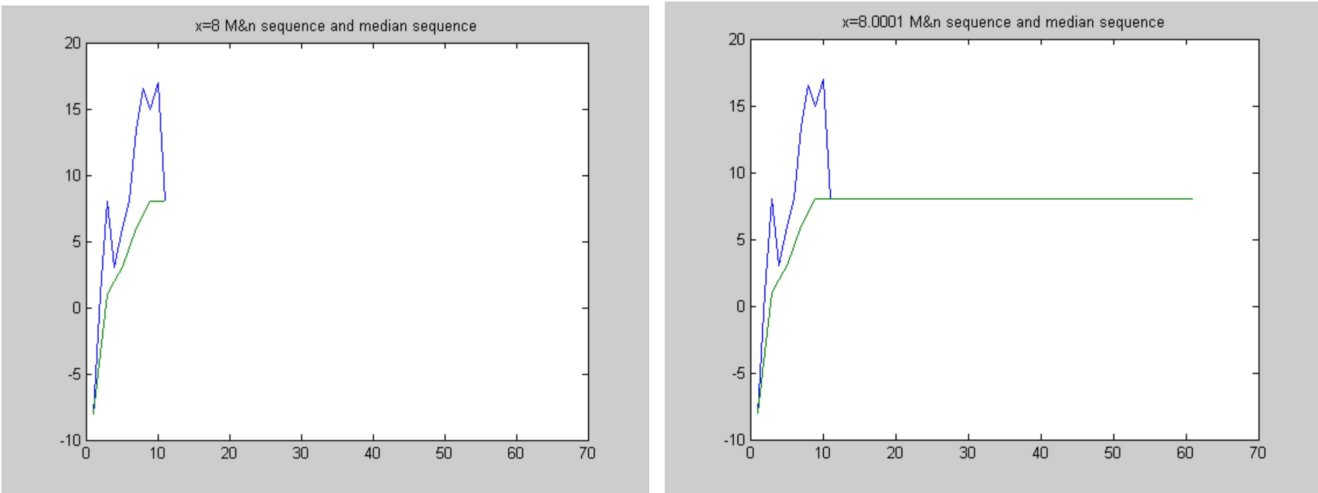
$$b_3 = (a_{11} - m_9)/\varepsilon = 11 = m'_2 + (9 + 2 - 1)(m'_2 - m'_1) \quad (13)$$

$$b_4 = (a_{12} - m_9)/\varepsilon = 13 = m'_3 + (9 + 3 - 1)(m'_3 - m'_2) \quad (14)$$

$\dots$

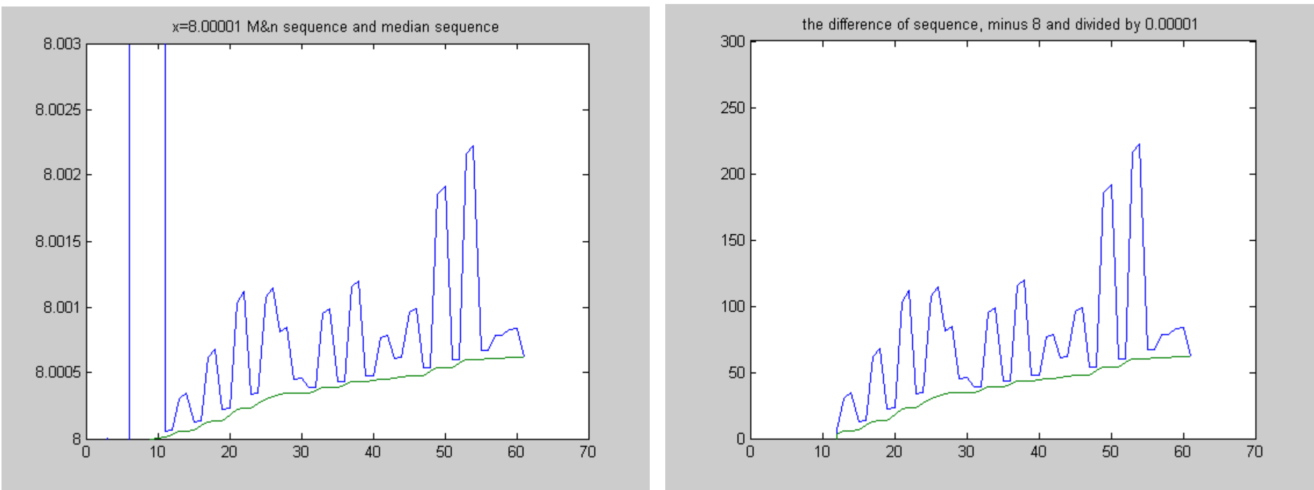
$$b_{k+1} = m'_k + (9 + k - 1)(m'_k - m'_{k-1})) \quad (15)$$

This processing gives us an insight into the questions. That is, if $x = 8 + 2\varepsilon$, its oscillating length is equal to the length of node $d_4 = 8$ ($K + 2 = 11$) plus the extra length (denoted as $l(K = 9)$) of the new sequence $\{b_n\}$ and then minus 3, since there are three extra numbers b_1, b_2, a_{K+2}. In the example, the length of $\{b_n\}$ is 53 (also shown in Table 7), so we can say whatever $x = 8 + 0.01$, $x = 8 + 0.001$, ..., or $x = 8 + 10^{-100}$, the stable lengths are all 53+11-3=61. In other words, we can say that its oscillating length is equal to the index ($K = 9$) plus $l(K)$ and then minus 1.



(a) (b)

Fig. 5 M&m sequences and the median sequences for $x = 8$ and $x = 8.00001$



(a) (b)

Fig. 6 (a) M&m sequence and the median sequence for $x = 8.00001$. (b) The new sequence.

Table 6 The M&m sequences for $x = 8$ and $x = 8.00001$.

	Columns 1 through 11										
M&m ($x = 8$)	-8.000	1.000	8.000	3.000	6.000	8.000	13.500	16.500	15.000	17.000	8.000
Median			1.000	2.000	3.000	4.500	6.000	7.000	8.0000	8.0000	8.0000

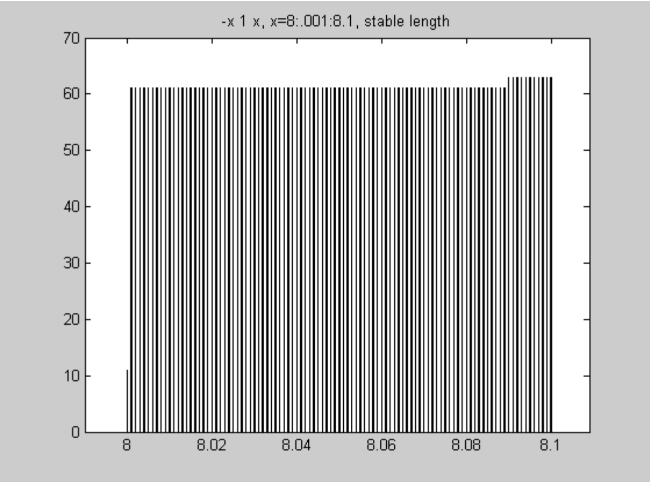
	Columns 1 through 11											
M&m ($x = 8.00001$)	-8.00001	1.00	8.00001	3.0	6.0	8.0	13.5	16.5	15.0	17.0	**8.000055**	**8.000065**
Median			1.0	2.0	3.0	4.5	6.0	7.0	8.0	8.000005	8.00001	8.0000325

Let $V_9 = b_{l(9)} = b_{53}$ denote the stable value of $\{b_n\}$, by computing we have $V_9 = 123.5000$, if

$$8 + 123.5 \times \varepsilon = d_4 + V_9 \cdot \varepsilon < d_5 = 13.5 \tag{16}$$

then we obtain that 2ε is bounded in $\tau_9 = 2(d_5 - d_4)/V_9 = 0.089069$.

For example, when x reaches to the value $x = 8.089$, their lengths are all the same as 61, the computer results are shown in Fig. 7(a). The stable values are shown in Fig. 7(b).



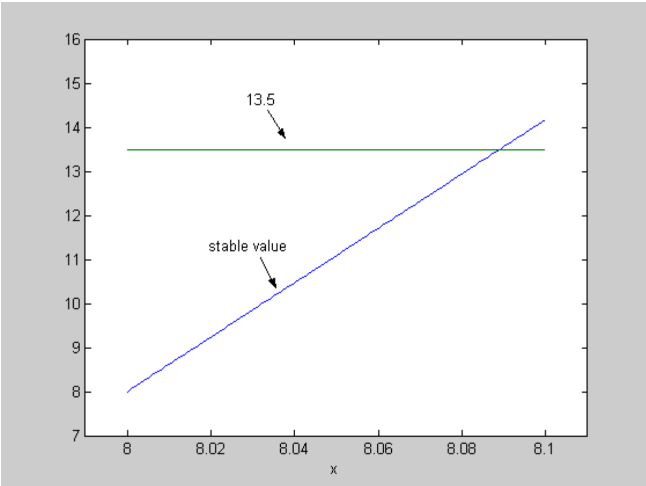


Fig. 7 (a) Oscillating lengths for $8 < x < 8.1$. (b) Stable values for $8 < x < 8.1$

In general, let $\{a_n\}$ denote the M&m sequence begin with the initial terms $(-d_i, 1, d_i)$, and consider the M&m sequence $\{a'_n\}$ begin with initial terms $(-x, 1, x)$, $x = d_i + 2\varepsilon$, for convenient we set $K = 2i + 1$, then we have that $a'_k = a_k$, for $k = 4, 5, ..., K+1$ and

$$a'_{K+2} = m'_{K+1} + (K+1)(m'_{K+1} - m'_K) = m'_{K+1} + (K+1)\varepsilon = m'_K + (K+2)\varepsilon \tag{17}$$

$$a'_{K+3} = m'_{K+2} + (K+2)(m'_{K+2} - m'_{K+1}) = m'_K + (K+4)\varepsilon \tag{18}$$

$\ldots$

To study $\{a'_n\}$, we introduce a new sequence $\{B_K(k)\}_{k \in N}$, which satisfies $B_K(1) = 0$, $B_K(2) = 2$, and

$$B_K(k+1) = \mu_k + (K + k - 1)(\mu_k - \mu_{k-1}),\ k > 2 \tag{19}$$

where $\mu_k = median(B_K(1), ..., B_K(k))$, $k \geq 2$. Let $l(K)$ denote the oscillating length of $\{B_K(k)\}_{k \in N}$ and the stable value is $V_K = B_K(l(K))$, if

$$m'_K + V_K \cdot \varepsilon = d_i + V_K \cdot \varepsilon < d_{i+1} \tag{20}$$

then it follows that

$$a'_{K+2} = m'_K + \varepsilon \cdot B_K(3), \tag{21}$$

$$a'_{K+3} = m'_K + \varepsilon \cdot B_K(4), \tag{22}$$

$$\cdots$$

$$a'_{K+l(K)-1} = m'_K + \varepsilon \cdot B_K(l(K)) \tag{23}$$

Note that if Eq. (20) holds, then we obtain that 2ε is bounded in $\tau_K = 2(d_{i+1} - d_i)/V_K$. From the above discussion we conclude that:

Theorem 3: *For* $i \in \{1,2,3,...,34\}$ *and* $d_i \in \Omega$, $\tau_K = 2(d_{i+1} - d_i)/V_K$, $K = 2i+1$, *if* $x \in (d_i, d_i + \tau_K)$, *then there exist a positive integer* $l(K)$ *defined as above, such that the M&m sequence* $\{a'_n\}$ *with initial terms* $(-x, 1, x)$ *is stable with oscillating length* $l(K) + K - 1$ *and stable value* $\varepsilon V_K + d_i$.

Proof. By calculate directly, for all $K = 2i+1$, $i \in \{1,2,3,...,34\}$, the corresponding sequence $\{B_K(k)\}_{k \in N}$ is stable with oscillating length $l(K)$. If $x = d_i + 2\varepsilon$, $0 < 2\varepsilon < \tau_K$, then we have $a_{K+l(K)-1} < d_{i+1}$. Hence, the Eq. (20) holds, then recall Eq. (21-23) holds. This implies the sequence $\{a'_n\}_{n=K+2}^{K+l(K)-1}$ and the sequence $\{B_K(k)\}_{k=3}^{l(K)}$ has the same structure. Then from Eq.(23) we have the oscillating length $l(K) + K - 1$ and note that $m'_K = d_i$ we obtain the stable value $\varepsilon \cdot V_K + d_i$.

By Matlab programs, we calculate the all results and shown in Table 7. For example, $d_4 = 8$, if $x = 8 + 2\varepsilon$ with $2\varepsilon < \tau_9 \cong 0.089069$, then its oscillating length is equal to $l(K) + K - 1 = 53 + 9 - 1 = 61$. So we can say whatever $x = 8 + 0.01$, $x = 8 + 0.001$, ..., or $x = 8 + 10^{-100}$, the oscillating lengths are all 61 and the stable values are $\varepsilon V_9 + d_i = 0.005 \times 123.5 + 8 = 8.6175$, 8.0615, ..., or $8 + 61.75 \times 10^{-100}$, respectively.

Table 7 Index K, $l(K)$, length for $x = d_i + 2\varepsilon$, τ_K and V_K

Index K	3	5	7	**9**	11	13	15	17	19
$l(K)$	71	59	45	**53**	249	249	237	141	171
Length	73	63	51	**61**	259	261	251	157	189
τ_K	**0.098462**	0.077921	0.054795	**0.089069**	0.007942	0.009084	0.001826	0.013394	0.000346
V_K	40.6250	77.0000	73.0000	**123.5000**	377.7500	330.2500	547.6250	559.9375	655.2500

Index K	21	23	25	27	29	31	33	35	37
$l(K)$	215	99	203	405	567	117	1147	3707	137
Length	235	121	227	431	595	147	1179	3741	173
τ_K	0.004155	0.006349	0.003781	0.000280	0.000345	0.001976	0.002457	0.000262	0.000350
V_K	962.7500	472.500	661.2500	1788.2813	1449.125	1265.50	1525.9531	1905.8223	1429.500

Index K	39	41	43	45	47	49	51	53	55
$l(K)$	345	553	277	109	227	367	301	179	259
Length	383	593	319	153	273	415	351	231	313
τ_K	0.000314	0.003276	0.000163	0.000287	0.000206	0.000181	0.000149	0.002027	0.000789
V_K	1590.50	2365.9688	3075.00	1742.00	2425.625	2756.625	3346.250	2714.00	4118.125

Index K	57	59	61	63	65	67	69
$l(K)$	393	529	249	627	223	629	203
Length	449	587	309	689	287	695	271
τ_K	0.000089	0.000100	0.000106	0.000069	0.000106	0.000079	0.000116
V_K	5588.00	5005.0625	4724.875	7283.375	4697.00	6306.25	4293.50

By similar derivation we also have:

Theorem 4: *For* $i \in \{2,3,...,34\}$ *and* $d_i \in \Omega$, $\lambda_K = \frac{2(d_{i+1} - d_i)}{V_K - 2}$, $K = 2i+1$, *if* $x \in (d_i - \lambda_K, d_i)$, *then there exist a positive integer* $l(K)$ *defined as above, such that the M&m sequence* $\{a'_n\}$ *with initial terms* $(-x, 1, x)$ *is stable with oscillating length* $l(K) + K - 1$ *and stable value* $\varepsilon(V_K - 2) + d_i$.

Proof. Let $\{a'_n\}$ denote the M&m sequence begin with the initial terms $(-d_i - 2\varepsilon, 1, d_i + 2\varepsilon)$, and consider the M&m sequence $\{a''_n\}$ begin with initial terms $(-x, 1, x)$, $x = d_i - 2\varepsilon$. By directive comparing we have

$$m''_K = m'_K - 2\varepsilon \tag{24}$$

$$m''_{K+1} = m'_{K+1} - 2\varepsilon \tag{25}$$

Using Eq. (5) $a_{k+1} = m_k + k(m_k - m_{k-1})$ we obtain

$$a''_{K+2} = m''_{K+1} + (K+1)(m''_{K+1} - m''_K) = m'_{K+1} - 2\varepsilon + (K+1)(m'_{K+1} - m'_K) = a'_{K+2} - 2\varepsilon \qquad (26)$$

And so on, we have $a''_k = a'_k - 2\varepsilon$ for $k \geq K+2$, this implies the sequence, $\{a'_n\}_{n=K+2}^{K+l(K)-1}$ is equivalent to $\{a''_n\}_{n=K+2}^{K+l(K)-1}$ except the small difference 2ε. Then we obtain that if $x \in (d_i - \lambda_K, d_i)$, with $\lambda_K = \dfrac{2(d_{i+1} - d_i)}{V_K - 2}$, from Eq.(23) we have the oscillating length $l(K) + K - 1$ and note that $m'_K = d_i$, we have the stable value $\varepsilon \cdot (V_K - 2) + d_i$.

For example, let $x = d_4 - 2\varepsilon = 8 - 2\varepsilon = 7.99999$ ($\varepsilon = 0.000005$), the M&m sequence and median sequence are plotted in Fig. 8, we see that it is similar to Fig. 6(a) for $x = d_4 + 2\varepsilon = 8 + 2\varepsilon = 8.00001$. The oscillating lengths for $x = d_4 - 2\varepsilon$ are shown in Fig. 9(a), and the stable values for $x = d_4 \pm 2\varepsilon$ are shown in Fig. 9(b). From Fig. 9(b) we see that the stable values are a little un-symmetry, since there is a small difference 2ε.

Another example is setting $x = d_3 \pm 2\varepsilon = 6 \pm 2\varepsilon$, Table 8 lists the M&m sequence and median sequence for $x = d_3 \pm 2\varepsilon = 6 \pm 0.001$ and Fig. 10 shows the oscillating length for $x = d_3 \pm 2\varepsilon = 6 \pm 2\varepsilon$, these let us easily understand the above concepts.

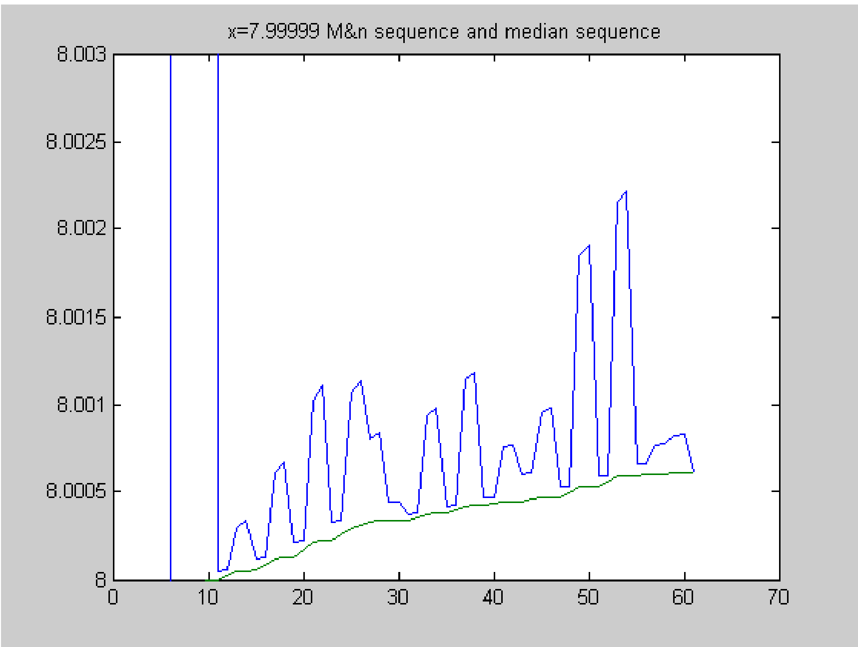
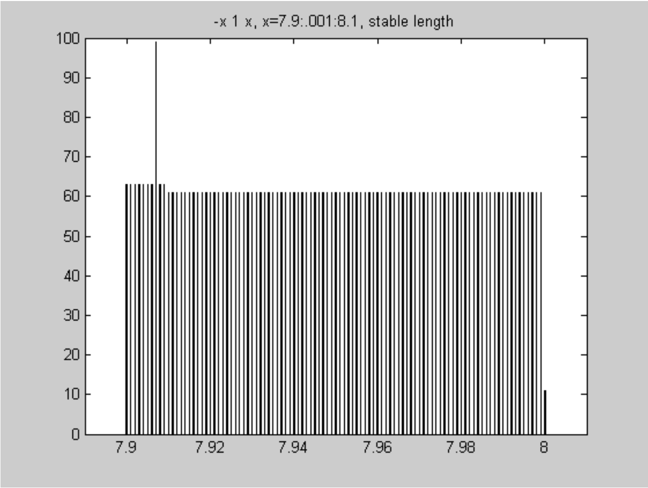


Fig. 8 M&m sequence and the median sequence for $x = 7.99999$.



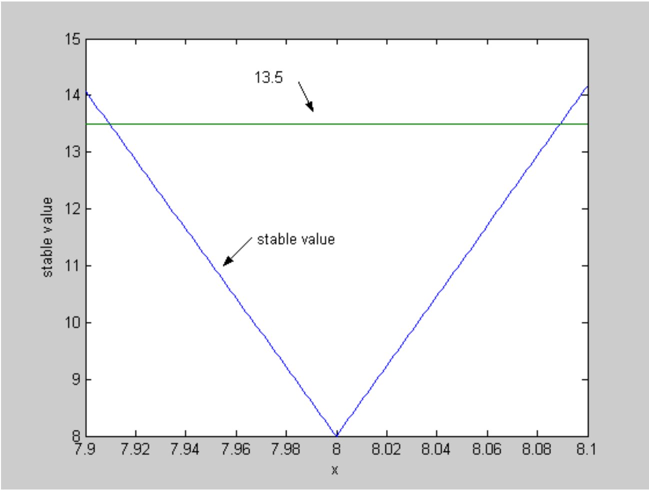


Fig. 9 (a) Oscillating lengths for $7.9 < x < 8$. (b) Stable values for $7.9 < x < 8.1$

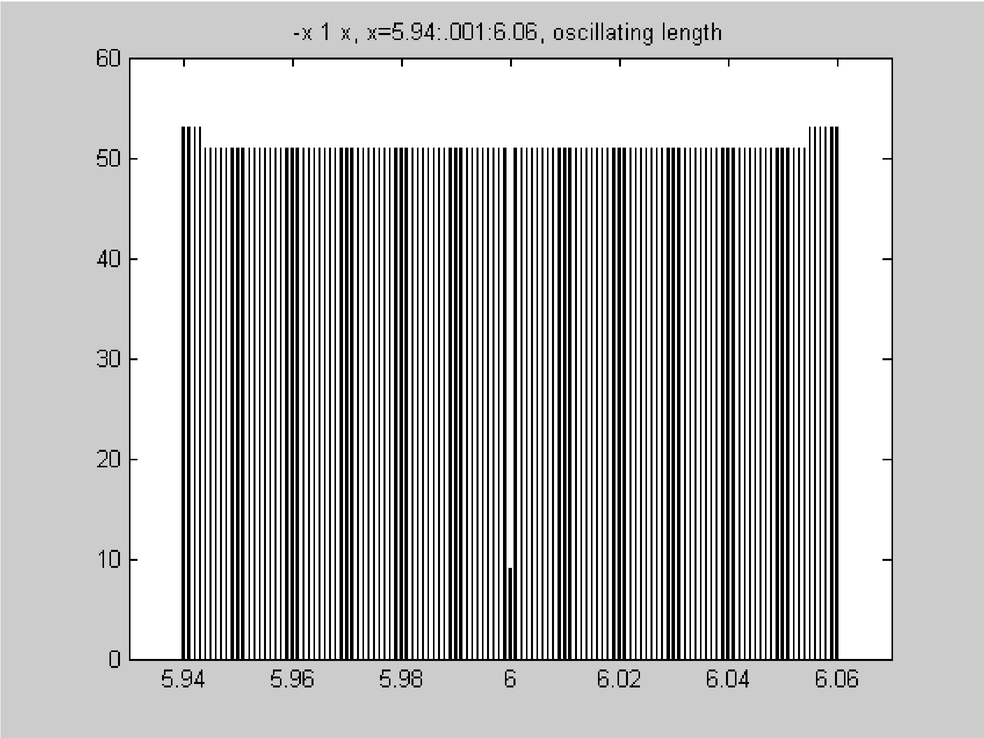


Fig. 10 Oscillating lengths for $5.94 < x < 6.06$.

Table 8(a) The M&m sequence and median sequence for $x = d_3 + 2\varepsilon = 6.001$

Columns 1 through 8

M&m	-6.001000	1.000000	6.001000	3.000000	6.000000	8.000000	13.500000	16.500000
median	-6.001000	-2.500500	1.000000	2.000000	3.000000	4.500000	**6.000000**	**6.000500**

Columns 9 through 17

6.004500	6.005500	6.020250	6.023750	6.011000	6.012000	6.046750	6.052250	6.019500
6.001000	6.002750	6.004500	6.005000	6.005500	6.008250	6.011000	6.011500	6.012000

Columns 18 through 26

6.020500	6.083250	6.090750	6.027375	6.028125	6.023125	6.023375	6.0533125	6.0559375
6.015750	6.019500	6.019875	6.020250	6.020375	6.020500	6.0218125	6.023125	6.023250

Columns 27 through 35

6.026500	6.026750	6.0288125	6.0291875	6.066375	6.069125	6.030625	6.030875	6.0376875
6.023375	6.0235625	6.023750	6.025125	6.026500	6.026625	6.026750	6.0270625	6.027375

Columns 36 through 43

6.0383125	6.041250	6.042000	6.04153125	6.04221875	6.036500	6.036875	6.06009375
6.027750	6.028125	6.02846875	6.0288125	6.029000	6.0291875	6.02990625	6.0306250

Columns 44 through 51

6.06153125	6.036250	6.036500	6.1571875	6.1625625	6.042375	6.042625	**6.036500**
6.030750	6.030875	6.0335625	6.036250	6.036375	**6.036500**	**6.036500**	6.036500

Table 8(b) The M&m sequence and median sequence for $x = d_3 - 2\varepsilon = 5.999$

Columns 1 through 8

M&m	-5.999000	1.000000	5.999000	3.000000	6.000000	8.000000	13.496500	16.495500
median	-5.999000	-2.499500	1.000000	2.000000	3.000000	4.499500	**5.999000**	**5.999500**

Columns 9 through 17

6.003500	6.004500	6.019250	6.022750	6.010000	6.011000	6.045750	6.051250	6.018500
6.000000	6.001750	6.003500	6.004000	6.004500	6.007250	6.010000	6.010500	6.011000

Columns 18 through 26

6.019500	6.082250	6.089750	6.026375	6.027125	6.022125	6.022375	6.0523125	6.0549375
6.014750	6.018500	6.018875	6.019250	6.019375	6.019500	6.0208125	6.022125	6.022250

Columns 27 through 35

6.025500	6.025750	6.0278125	6.0281875	6.065375	6.068125	6.029625	6.029875	6.0366875
6.022375	6.0225625	6.022750	6.024125	6.025500	6.025625	6.025750	6.0260625	6.026375

Columns 36 through 43

6.0373125	6.040250	6.041000	6.04053125	6.04121875	6.035500	6.035875	6.05909375
6.026750	6.027125	6.02746875	6.0278125	6.028000	6.0281875	6.02890625	6.029625

Columns 44 through 51

6.06053125	6.035250	6.035500	6.1561875	6.1615625	6.041375	6.041625	**6.035500**
6.029750	6.029875	6.0325625	6.035250	6.035375	**6.035500**	**6.035500**	6.035500

V. Comparison with the Known Result

In [1] Shultz and Shiflett proved:

Theorem: If x' is any real number great than or equal to 21.3125, then the sequence $\mu(0, x', x'+1)$ has length 73 and stable value $x'+20.3125$.

In terms of our terminology, this result states: for *x* lying in the interval $1.098\overline{461538} > x > 1$ (see Fig. 8), that is, the associate M&m sequence has oscillating length 73. The range of their result equals $1 = d_1 < x < d_i + \tau_3 = 1.098462$ in our study (see Fig. 9(a)). The stable values are shown in Fig. 9(b). In our current work, we have established stability for 34 more intervals than [1].

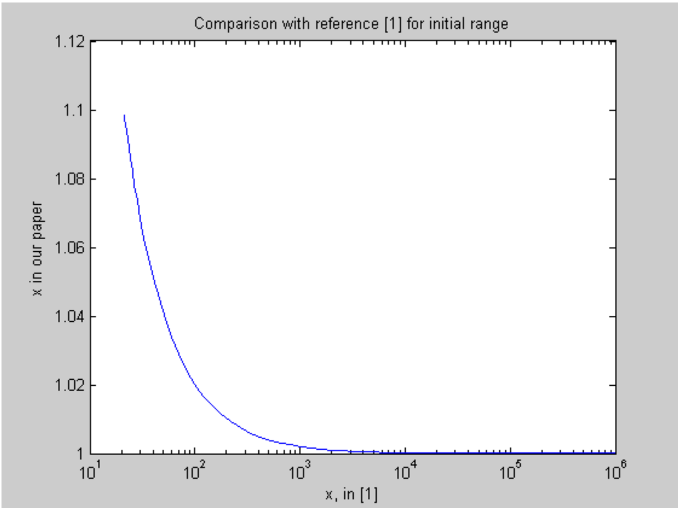


Fig. 8 Range $x' \geq 21.3125$ in [1] is mapped to x in our paper

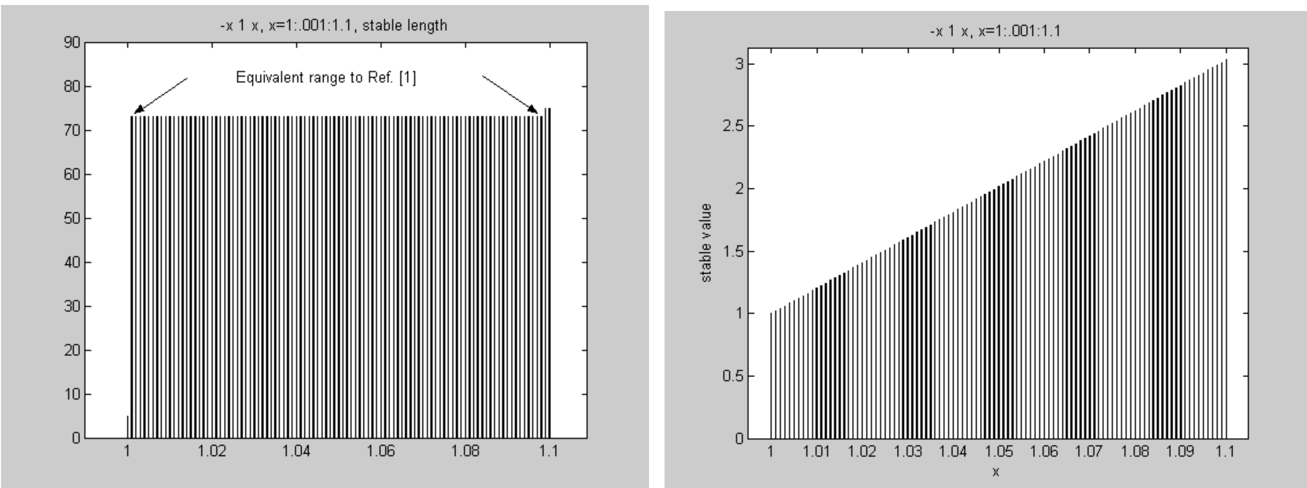


Fig. 9 (a)The behavior of lengths for $1.1 > x > 1$ and related to [1]. (b) The behavior of stable values.

VI. Discussion and Open Problems

1. The M&m Conjecture remains open.
2. The behavior of the M&m sequence except for x lying in the 34 semi-closed intervals and $[41.625, \infty)$ studied above remains unclear.
3. It is unknown at this stage if our method will be successful leading to the complete solution of the M&m conjecture. The result of this project encourages us to investigate if more intervals of convergence may be discovered.

VII. References

[1] H. S. Shultz and R. C. Shiflett, " M&m Sequences", *The College Mathematics Journal*, May 2005 (NO.3 Vol.36), pp. 191-198.
[2] M. Chamberland, M. Martelli, "The Mean-Median Map", *Journal of Difference Equations and Applications* ***13***, Jan 2007, pp. 577-583.

VIII. Appendix

A. Basic considerations and linear transformation

This article is in order to investigate of the M&m conjecture. We start with three numbers, a_1, a_2, a_3, and assume that $a_1 \le a_2 \le a_3$ in general since the order of the three given numbers does not affect subsequent terms. And then we have

$$a_4 = 4a_2 - (a_1 + a_2 + a_3) = 3a_2 - (a_1 + a_3) \qquad \text{(A1)}$$

Let $S_n \equiv \sum_{i=1}^{n} a_i$, M_k denote the mean and m_k denote the median of the first *k* values, then

$$m_k = M_{k+1} \equiv \frac{S_k + a_{k+1}}{k+1} \qquad \text{(A2)}$$

A.1 Consider the case: $a_1 = a_2 = a_3 = a$

If $a_1 = a_2 = a_3 = a$, we have

$$a_4 = 4a_2 - (a_1 + a_2 + a_3) = 4a - 3a = a \qquad \text{(A3)}$$

$$a_5 = 5(\frac{a_2 + a_3}{2}) - (a_1 + a_2 + a_3 + a_4) = 5a - 4a = a \qquad \text{(A4)}$$

And so on, we obtain that $a_1 = a_2 = a_3 = a_4 = a_5 = \cdots = a$, that is, the sequence is stable.

A.2 Consider the case: $a_1 < a_2 = a_3$ or $a_1 = a_2 < a_3$

1. If $a_1 < a_2 = a_3$, let $a_2 = a_3 = a$

$$a_4 = 3a_2 - (a_1 + a_3) = 3a - a - a_1 = 2a - a_1 > a \qquad \text{(A5)}$$

$$a_5 = 5a - (2a - a_1 + a + a + a_1) = a \qquad \text{(A6)}$$

$$a_6 = 6a - (2a - a_1 + a + a + a + a_1) = a \qquad \text{(A7)}$$

$$\Rightarrow a_5 = a_6 = \ldots\ldots = a_k = a$$

So that this sequence is stable with stable length 5 and stable value a_2.

2. If $a_1 = a_2 < a_3$, let $a_1 = a_2 = a$

$$a_4 = 3a_2 - (a_1 + a_3) = 3a - a - a_3 = 2a - a_3 < a \qquad \text{(A8)}$$

$$a_5 = 5a - (2a - a_3 + a + a + a_3) = a \qquad \text{(A9)}$$

$$a_6 = 6a - (2a - a_3 + a + a + a + a_3) = a \qquad \text{(A10)}$$

$$\Rightarrow a_5 = a_6 = \ldots\ldots = a_k \quad = a$$

So that this sequence is also stable with stable length 5 and stable value a_2.

A.3 Consider the case: $a_1 < a_2 < a_3$

1. If $a_3 - a_2 = a_2 - a_1$, the mean is equal to the median, then the sequence is stable with stable length 4 and stable value a_2.
2. If $a_3 - a_2 \neq a_2 - a_1$, we find that there exists a smart transformation $f(t) = \alpha t + \beta$ such that the M&m Sequence beginning with a_1, a_2, a_3 can be transformed into another M&m Sequence beginning with $-x, 1, x\ (x > 1)$. This transformation is better than that in [1] and [2], since in the mean operation, the initial values $-x$ and x will be mutually canceled.

Consider the function $f(t) = \alpha t + \beta$ such that $f(a_1) = -x$，$f(a_2) = 1$，$f(a_3) = x$. Firstly, from $f(a_1) + f(a_3) = 0$, we obtain $\alpha \cdot a_1 + \beta + \alpha \cdot a_3 + \beta = 0$, that is, $\alpha(a_1 + a_3) + 2\beta = 0$, then

$$\beta = \frac{-\alpha(a_1 + a_2)}{2} \tag{A11}$$

Substitute Eq.(16) into $f(a_2) = 1$, that is, $\alpha \cdot a_2 + \beta = 1$, we have

$$\alpha = \frac{-2}{a_1 + a_3 - 2a_2} \tag{A12}$$

and

$$\beta = \frac{a_1 + a_3}{a_1 + a_3 - 2a_2} \tag{A13}$$

Therefore, the function $f(t)$ can be represented as

$$f(t) = \frac{2t - (a_1 + a_3)}{2a_2 - (a_1 + a_3)} \tag{A14}$$

Because $a_3 - a_2 \neq a_2 - a_1$, that is, $2a_2 - (a_3 + a_1) \neq 0$, we can find a transformation $f(t) = \alpha t + \beta$ such that

(1) If $a_3 - a_2 < a_2 - a_1$, we have

$$f(a_1) = -x\ ，\ f(a_2) = 1\ ，\ f(a_3) = x = \frac{a_3 - a_1}{2a_2 - (a_1 + a_3)} > 1 \tag{A15}$$

(2) If $a_3 - a_2 > a_2 - a_1$, we have

$$f(a_1) = -y\ ，\ f(a_2) = 1\ ，\ f(a_3) = y = \frac{a_3 - a_1}{2a_2 - (a_1 + a_3)} < -1 \tag{A16}$$

By reordering the three numbers and interchanging x and $-y$, we have

$$f(a_3) = y = -x \text{,} \quad f(a_2) = 1 \text{,} \quad f(a_1) = -y = x \tag{A17}$$

Note that in Eq.(A17), x is also greater than 1.

A.4 Transformation of M&m Sequences

Now, we will show that given any M&m Sequence, $\{a_k\}_{k\in N}$, and by transformating into another M&m Sequence, $\{a'_k\}_{k\in N}$, where $a'_k = f(a_k), k = 1,2,3,\ldots$, the new sequence $\{a'_k\}_{k\in N}$ is also an M&m Sequence starting with $a'_1,\ a'_2,\ a'_3$.

Because $\{a_k\}_{k\in N}$ is an M&m Sequences, so that $m_k = M_{k+1}(=\frac{S_k + a_{k+1}}{k+1})$ and $m'_k = median(a'_1, a'_2, a'_3 \ldots a'_k)$, hence

$$a'_k = f(a_k) = \alpha a_k + \beta \tag{A18}$$

Therefore, the order of $a'_1, a'_2, a'_3, \ldots a'_k$ is the same as of $a_1, a_2, a_3, \ldots a_k$, then

$$m'_k = f(m_k) = f(M_{k+1}) = \alpha M_{k+1} + \beta = \alpha(\frac{a_1 + \ldots + a_{k+1}}{k+1}) + \beta = \frac{1}{k+1}\sum_{i=1}^{k} f(a_i) = M'_{k+1} \tag{A19}$$

From Eq.(A19) we conclude that the new sequence $\{a'_k\}_{k\in N}$ is also an M&m Sequence. This means that we may restrict our attention to the special case which beginning with $-x, 1, x\ (x>1)$.

For example, given three numbers (0, 67, 78), by transformation we obtain another three numbers (-1.3929, 1.0000, 1.3929), both have the same pattern of behavior, as shown in Fig. A1. (The Matlab program is shown in appendix D)

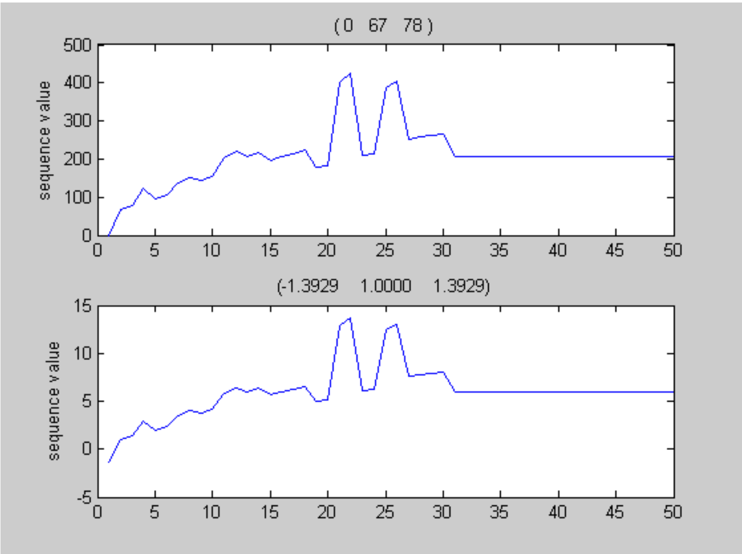


Fig. A1 Equivalent behavior of both M&m Sequences by transformation

B. Mapping uniformly distribution to our work

We define the interval between two neighbor nodes as **branch**. There are 34 branches, such as $B_i : d_i \sim d_{i+1}$, for example, $B_1 : 1 \sim 3$, $B_2 : 3 \sim 6$, $B_3 : 6 \sim 8$, ..., $B_{33} : 41.125 \sim 41.375$, $B_{34} : 41.375 \sim 41.625$. For different branches we test their oscillating lengths, we find: (1) when the branch nears $x = 41.625$, the oscillating length is almost constant except the edge area, and (2) when the branch near $x = 1$, the behavior of the oscillating length is chaotic.

A possible reason for the above phenomenon is that the transformation from any three numbers to $(-x, 1, x)$ is extremely non-uniform. Thus, we try to understand the relation between them. First, we let three numbers belong to $\{0, 0.01, 0.02, \cdots, 0.99, 1\}$, by ordering and using Eq. (A14), we obtain the distribution of branch is shown in Fig. B1, where left-side and right-side are with different ordinate only. Note that $B_{35} : 41.625 \sim \infty$ and B_{36} includes four cases: $a_1 = a_2 = a_3$, $a_1 < a_2 = a_3$, $a_1 = a_2 < a_3$, and $2a_2 = a_1 + a_3$. We see that branch 1 appears most times, that is, branch 1 is the most important among these branches.

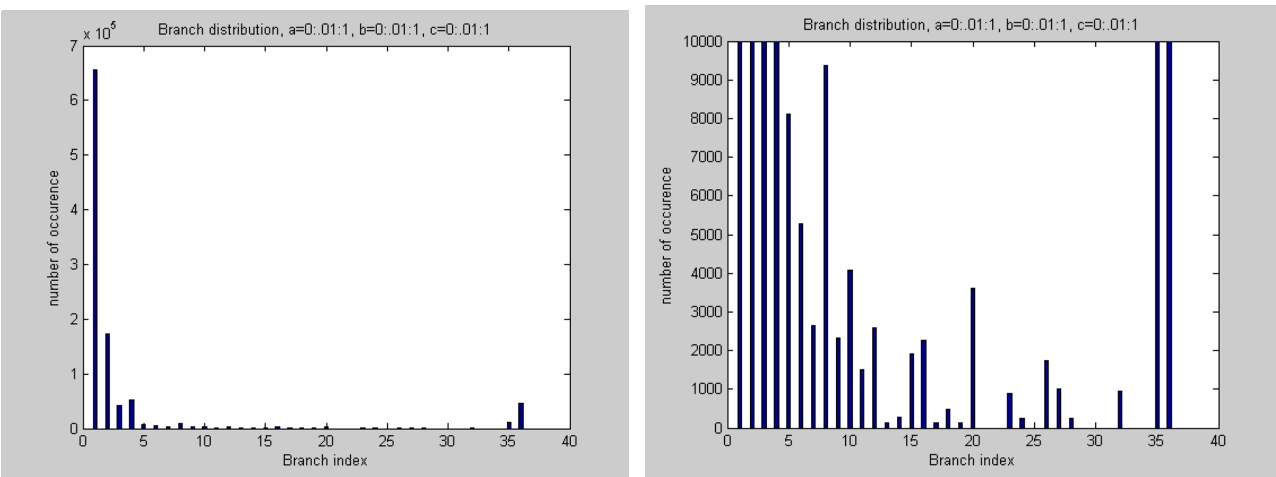


Fig. B1 Distribution of branch mapped by three numbers belong to $\{0, 0.01, 0.02, \cdots, 0.99, 1\}$

Next, we let the three numbers are all uniformly distributed in $(0, 1)$ randomly and try 10000 times, by ordering (a, b, c) and using Eq. (A14), we obtain the distribution of branch is shown in Fig. B2. It can be seen that the behavior is similar to Fig. B1 except the number of occurrence for B_{36} is zero. This reason is in random condition, the probability of four cases ($a_1 = a_2 = a_3$, $a_1 < a_2 = a_3$, $a_1 = a_2 < a_3$, or $2a_2 = a_1 + a_3$) occurring approaches to zero.

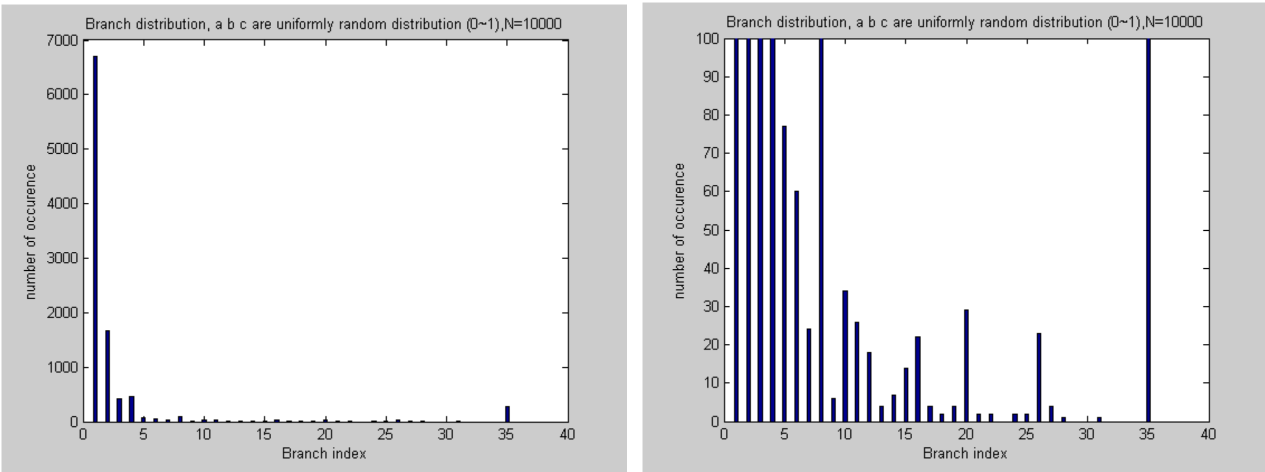


Fig. B2 Distribution of branch mapped by three random numbers uniformly distributed in $(0,1)$

Finally, we let the three numbers are all positive integral and uniformly distributed in $(1,10000)$ randomly and try 10000 times, by ordering (a, b, c) and using Eq. (A14), we obtain the distribution of branch is shown in Fig. B3. It can be seen that the behavior is also similar with Figs. B1 and B2.

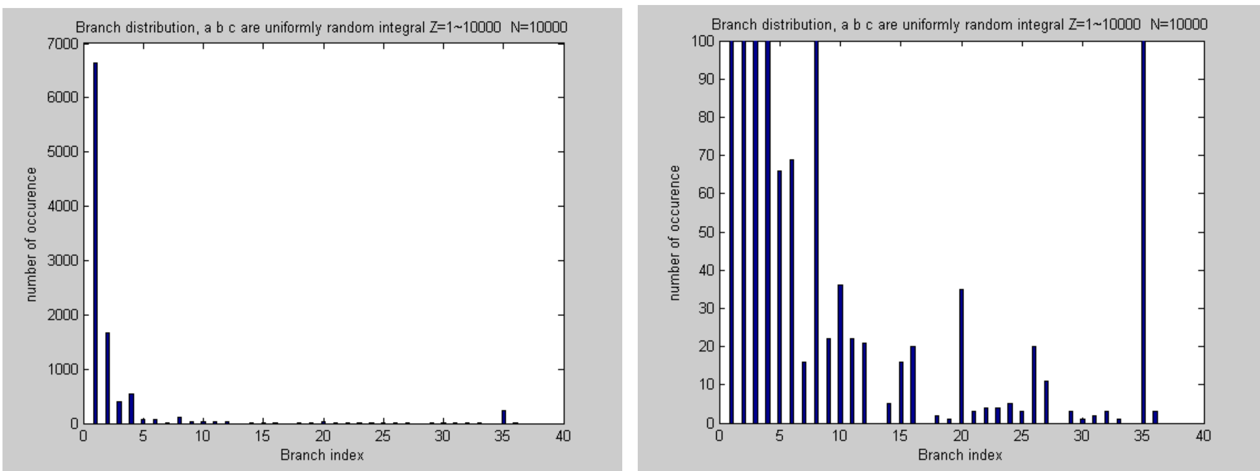


Fig. B3 Distribution of branch mapped by three random positive integral distributed in $(1,10000)$

Now, we focus on the case of distributed in $(1,10000)$ randomly and try to find the distribution in some branches, such as the most important $B_1 \sim B_4$ with resolution of 0.1. These results are plotted in Fig. B4, from Fig. B4 we see that the distribution is similarly decayed by exponential law. This maybe can explain the above phenomenon.

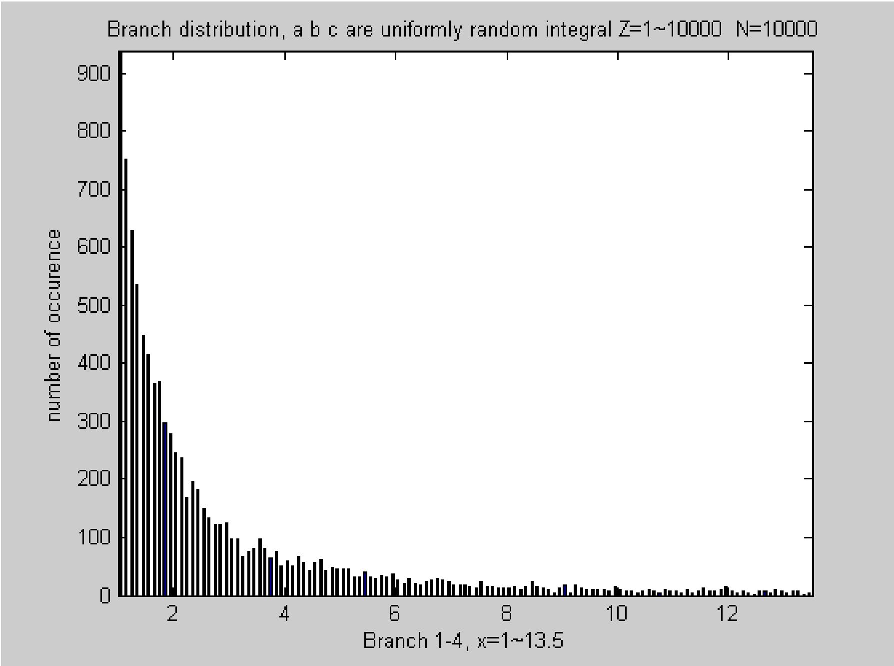


Fig. B4 Distribution of branch $B_1 \sim B_4$ (with resolution of 0.1) mapped by three random positive integral distributed in $(1, 10000)$

C. Discussion and Future Works

In this research we have proven that when $x \geq 41.625$ and some part of $x < 41.625$ will be stable, and found the some properties of stable values and oscillating lengths. In future, we will focus on the stable pattern for any branches. If we can do it, we will finish the great work. Hope we can make progress in solving the M&m conjecture and complete the hope of Shultz and Shiflett.

Some drawbacks in study are as following:

1. The M&m Sequence is not a perfect sequence. For example, the computation of sequence may use the recursion formula $a_{k+1} = m_k + k(m_k - m_{k-1})$. In that operation we use both medium numbers m_k and m_{k-1}. There are half medium numbers, m_q (q is even), are resulted from the addition of two numbers and divided by 2. Note that divided by 2 perhaps increases one decimal place. So, for some special cases, we don't know whether or not the effective digits of stable values will be infinite. We think it would not happen, but we can not prove it.

2. Any computer program has finite digits. For example, the number of effective digits in Matlab is 15, so in the stable testing, we must take attention to that is the sequence truly stable?

3. The middles of the branch close to 1, such as B_1, B_2, etc, are hard to process, for example, if initial value $x = 4 + \delta$, where $\delta = 10^{-100}$, we don't have any ideal to know its stable value and the oscillating length, no matter by theory or by computer since $x = 4$ is not a node. It maybe needs new or modified method to solve this question.

D. Some Matlab programs

(Matlab program for Figure 1 or A1)

```
% figure_1
clear, clf,
I=47;
A=[ 0      67      78 ];

a1=A(1); a2=A(2); a3=A(3);
for i=1:3
        B(i)=(a1+a3-2*A(i))/(a1+a3-2*a2);
end

Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
        Med(j)=median(A);
end

subplot(211), plot(A),
title('( 0      67      78 )')
ylabel('sequence value')

A=sort(B);
Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
        Med(j)=median(A);
end

subplot(212), plot(A),
title('(-1.3929        1.0000        1.3929)')
ylabel('sequence value')
```

(Matlab program for Figure 2)

```
% figure_2
clear, clf

I=347;

x=25;
A=[-x 1 x];
Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
A(4)=Med(3)*4-sum(A); Med(4)=median(A);
for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
        Med(j)=median(A);
end

A(2,:)=Med;
plot(A'), title('A=[-25 1 25] M&n sequence and median sequence')
```

(Matlab program for Figure 3(a))

```
% figure_2
clear, clf

I=10000;
format long
X=1:.5:50; lx=length(X);
for k=1:lx
    A=[-X(k) 1 X(k)]; C(k)=-I/10;
    Med(1)=A(1); Med(2)=(A(1)+(2))/2; Med(3)=median(A);
    A(4)=Med(3)*4-sum(A); Med(4)=median(A);
    for i=2:I
        j=i+3;
        A(j)=Med(j-1)+(j-1)*(Med(j-1)-Med(j-2));
        Med(j)=median(A);
        if A(j)==Med(j-1),    B(k)=A(j); C(k)=j; break, end
    end
end

subplot(211), plot(X,B), title('-x 1 x, x=1:.5:50, stable value')
subplot(212), plot(X,C), title('-x 1 x, x=1:.5:50, oscillating length')
```