臺灣二ＯＯ七年國際科學展覽會

科　　　別：數學

作 品 名 稱：「圖形板」的圖形軌跡之探討及其延伸

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Hello! My name is Fanny Wang. I'm in grade three in junior high now. I'm glad that I can introduce myself here.

I've been learning English for 7 years. My parents thought that cultivating one's language skill should be as early as possible . I'm really grateful that so many nice teachers helped me during these years. In fact, I've been fascinated with Engish and math. Although my math scores in the elementary were extremely bad, I still enjoyed the interaction with the teachers. In the first semester of junior high, however, I met a great math teachers. He's really humorous and smart. His enthusiasm toward math seemed to encourage me. Since then,I've put more and more emphasis, energy on this subject. Throughout the past two years, my attitude toward learning remained active. I wanted to seize any opportuninty to display myself. In the future, I hope I can learn some other foreign languages and some more researches about math.

「圖形板」的圖形軌跡之探討及其延伸

Probing And Extending The Locus of The Plate

Abstract：

Starting from the problem in AMC competition of Australia, we try to find out the locus and its length when a point in a regular polygon rolls in a circle. The result is that the locus has a wonderful and regular cycle.

Next, we discuss the regularity of the cycle when a regular polygon（n sides）rolls in another regular polygon. Furthermore,we discuss the the equation of the locus by changing the radius and the angle of rolling. we find out the argument function of the locus of a point inside when a a regular polygon（n sides）rolls in another regular polygon （m sides）： $\begin{cases} x_j = \overline{A_j B_j}\cos\theta + r\cos\dfrac{2(j-1)\pi}{m} \\ y_j = \overline{A_j B_j}\sin\theta + r\sin\dfrac{2(j-1)\pi}{m} \end{cases}$

（ $\theta_i - (\dfrac{360^0}{n} - \dfrac{360^0}{m}) \le \theta \le \theta_i$ ， A_j is the summits of the regular polygon（m sides），B_j corresponds A_j when a point inside the regular polygon （n sides）rolls, $r = \overline{OA_j}$ ）

And then, we do some moving simulation with some computer math software, such as Cabri Geometry、Mupad, etc. We discuss the regularity of the locus and its equation of a point inside when some special cycloids, like asteroids, cardioids, etc, roll in a certain condition. Moreover, with the result of research 2, we create the “plate” and apply for a patent on it. We hope to study math by playing games.

摘要：

從澳洲 AMC 競賽題出發，嘗試探討一正 n 邊形中的一點在單位圓內滾動軌跡及其軌跡長度，發現該軌跡均會產生奇妙的循環規律。

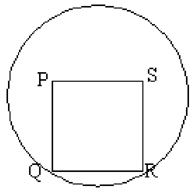
接下來，推廣探討正 n 邊形在其他正多邊形中滾動時循環的規律，並利用旋轉半徑及角度之間的變化深入探討其滾動軌跡方程式，發現正 n 邊形繞正 m 邊形滾動時其內部一點軌跡參數式爲

$$\begin{cases} x_j = \overline{A_jB_j}\cos\theta + r\cos\dfrac{2(j-1)\pi}{m} \\ y_j = \overline{A_jB_j}\sin\theta + r\sin\dfrac{2(j-1)\pi}{m} \end{cases}$$

，其中 $\theta_i - (\dfrac{360^0}{n} - \dfrac{360^0}{m}) \le \theta \le \theta_i$， A_j 爲正 m 邊形之各頂點、 B_j 爲正 n 邊形中內部一點旋轉時對應 A_j 之點， $r = \overline{OA_j}$ 。

進一步想嘗試使用數學電腦軟體如：Cabri Geometry、Mupad 等對以上研究去做一些動態模擬，並再探討一些特殊擺線如：星狀線、心臟線…等，在條件下相切滾動時，圖中某一點的軌跡規律性及其方程式。另外，應用研究二中的結果，創造出寓數學於遊戲的「圖形板」，並申請了新型專利。

壹、研究動機：



在澳洲 AMC 競賽題中出現一個題目：**『如圖，一邊長爲 1 米正方形置於一半徑爲一米的圓內，該正方形以如下的方式在圓內運動：繞點 P 順時針旋轉直到點 S 接觸到圓，繞點 S 順時針旋轉直到點 P 接觸到圓，以此類推，直到正方形有 2 個點到達最初 Q、R 所在的位置，求 P 點的軌跡長？』**結果發現 P 點滾動軌跡有一規律性，嘗試將圓內正方形推廣至正 n 邊形加以討論，這使我對多邊形的滾動產生好奇心，進而想深入研究探討其他滾動和圖形、角度、大小的關係。

貳、研究目的：

一、探討一正 n 邊形（邊長爲 1 單位）中的一點在單位圓內滾動的軌跡。

二、探討一正 n 邊形（邊長爲 1 單位）中心點繞一正 m 邊形（邊長爲 1 單位）的滾動軌跡。

三、探討各式擺線相切滾動時，滾動圖形中某一點的軌跡規律性及其方程式。

參、解釋名詞：

圖形板：由研究者申請專利之繪圖工具名稱。（詳見陸、結論與應用）

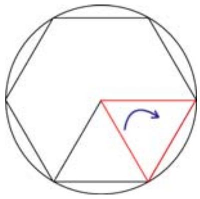
肆、研究器材及設備：

Cabri Geometry 軟體、Mupad 軟體、圓規、量角器。

伍、研究過程：

首先我們先探討一正 n 邊形中的任一點在單位圓內滾動的軌跡。

一、探討一正 n 邊形（邊長為 1 單位）中的一點在單位圓內滾動的軌跡

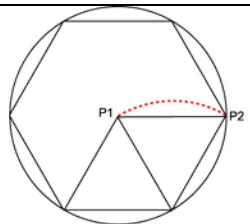
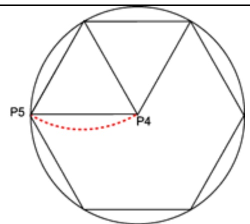
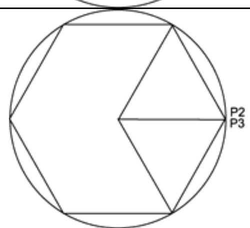
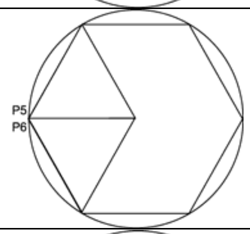
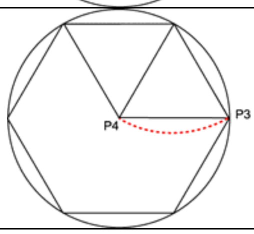
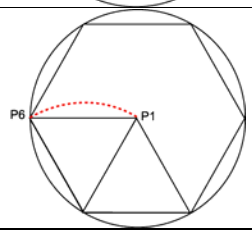


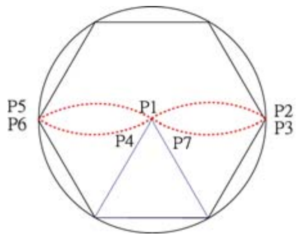
規則：將一邊長為 1 的正多邊形至入半徑為 1 的圓中，沿圓周滾動一週（直到又有兩頂點達到最初圖形頂點的位置）

我們依照規則滾動正多邊形，嘗試探討一正 n 邊形的『頂點』、『中心點』、『內部任一點』滾動後之軌跡圖形及其長度。

（一）探討一正 n 邊形（邊長為 1 單位）的『頂點』在單位圓內滾動的軌跡及其旋轉長度

1.探討一正三角形（邊長為 1 單位）的『頂點』在單位圓內滾動的軌跡及其旋轉長度

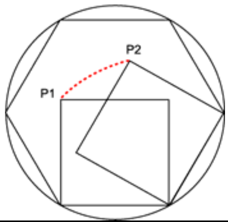
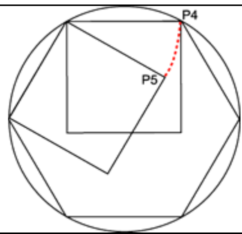
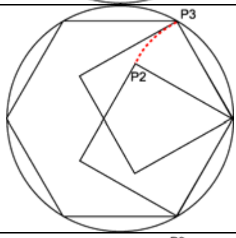
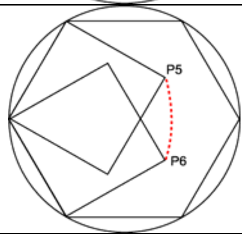
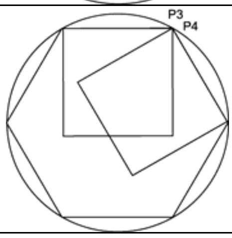
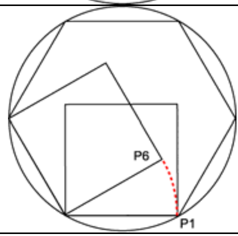
步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	1	$\frac{1}{3}\pi$	4		P4 ➔ p5	1	$\frac{1}{3}\pi$
2		P2 ➔ P3	0	0	5		P5 ➔ P6	0	0
3		P3 ➔ P4	1	$\frac{1}{3}\pi$	6		P6 ➔ P7	1	$\frac{1}{3}\pi$



發現：1.正三角形（邊長為 1 單位）的『頂點』在單位圓內滾動的軌跡如圖所示。

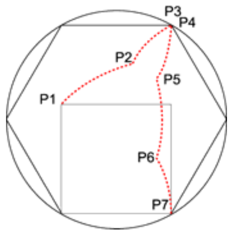
2.由於正三角形的旋轉角度為 120－60＝60 度，以 P 點計算其軌跡長度，軌跡長度總和＝$\frac{4}{3}\pi$

2.探討一正方形（邊長爲 1 單位）的頂點在單位圓內滾動的軌跡及其旋轉長度

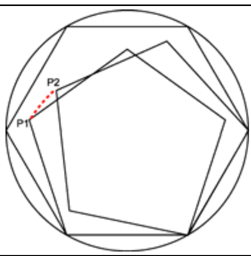
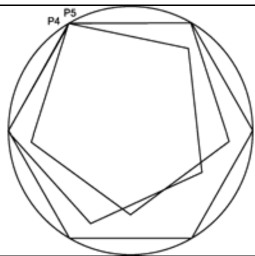
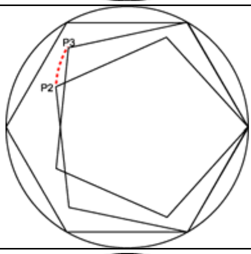
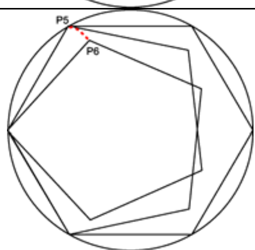
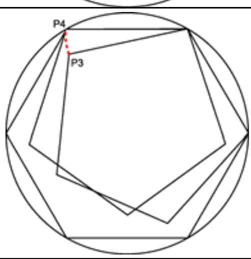
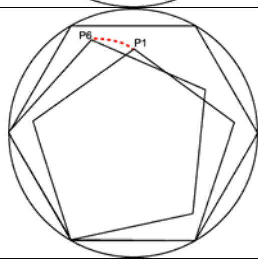
步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	$\sqrt{2}$	$\frac{\sqrt{2}}{6}\pi$	4		P4 ➔ p5	1	$\frac{1}{6}\pi$
2		P2 ➔ P3	1	$\frac{1}{6}\pi$	5		P5 ➔ P6	$\sqrt{2}$	$\frac{\sqrt{2}}{6}\pi$
3		P3 ➔ P4	0	0	6		P6 ➔ P1	1	$\frac{1}{6}\pi$

發現：1.正方形（邊長爲 1 單位）的『頂點』在單位圓內滾動的軌跡如圖。

2.由於正方形的旋轉角度爲 120－90＝30°度，以 P 點計算其軌跡長度，軌跡長度總和＝$(\frac{\sqrt{2}}{3}+\frac{1}{2})\pi$

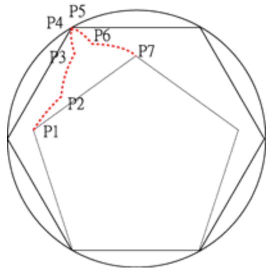


3.探討一正五邊形（邊長爲 1 單位）的頂點在單位圓內滾動的軌跡及其旋轉長度

步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$	4		P4 ➔ p5	0	0
2		P2 ➔ P3	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$	5		P5 ➔ P6	1	$\frac{1}{15}\pi$
3		P3 ➔ P4	1	$\frac{1}{15}\pi$	6		P6 ➔ P1	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$

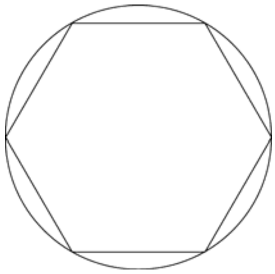
發現：1.正五邊形（邊長為 1 單位）的『頂點』在單位圓內滾動的軌跡如圖所示。

2.由於正五邊形的旋轉角度為 $120-108=12^\circ$，以 P 點計算其軌跡長度，軌跡長度總和＝$(\frac{7}{30}+\frac{\sqrt{5}}{10})\pi$



4.探討一正六邊形（邊長為 1 單位）的頂點在單位圓內滾動的軌跡及其旋轉長度

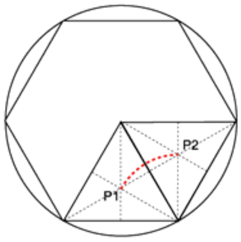
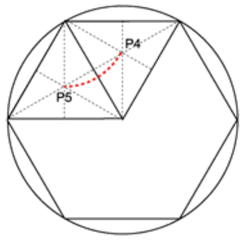
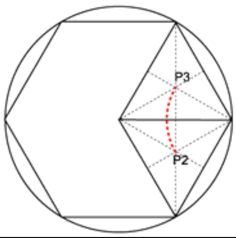
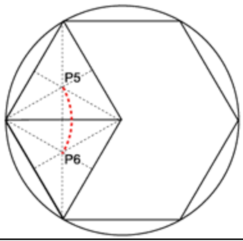
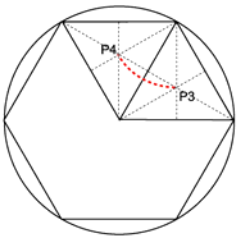
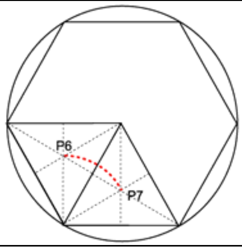
發現：由於正六邊形的旋轉角度為 $120-120=0^\circ$，故正六邊形（邊長為 1 單位）的『頂點』在單位圓內無法滾動，軌跡長度為 0，如圖所示。



※註：以下因為正六邊形（邊長為 1 單位）無法在單位圓內滾動，故不再加以討論。

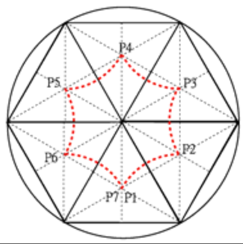
（二）探討一正 n 邊形（邊長為 1 單位）的中心點在單位圓內滾動的軌跡

1.探討一正三邊形（邊長為 1 單位）的中心點在單位圓內滾動的軌跡

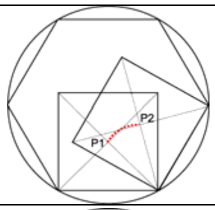
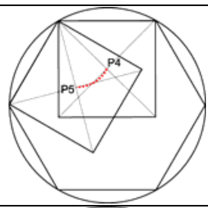
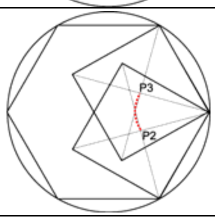
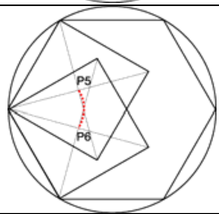
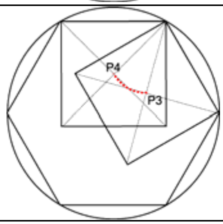
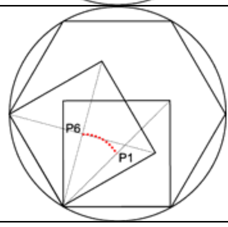
步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	4		P4 ➔ p5	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$
2		P2 ➔ P3	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	5		P5 ➔ P6	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$
3		P3 ➔ P4	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	6		P6 ➔ P1	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$

發現： 1.正△（邊長為 1 單位）的『頂點』在單位圓內滾動的軌跡如圖。

2.由於正△的旋轉角度為 60 度，以 P 點計算其軌跡長度，軌跡長度總和＝$\frac{2\sqrt{3}}{3}\pi$

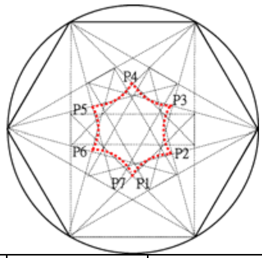


2.探討一正方形（邊長為 1 單位）的中心點在單位圓內滾動的軌跡

步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	4		P4 ➔ p5	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$
2		P2 ➔ P3	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	5		P5 ➔ P6	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$
3		P3 ➔ P4	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	6		P6 ➔ P1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$

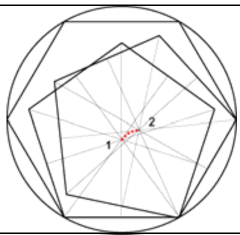
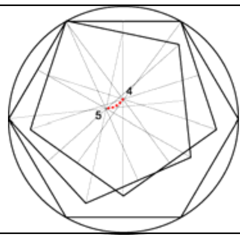
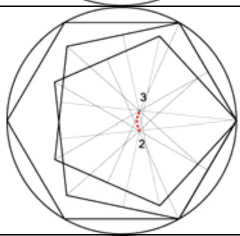
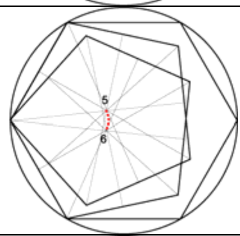
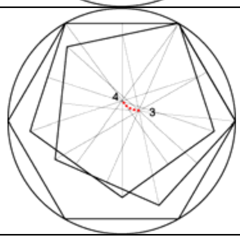
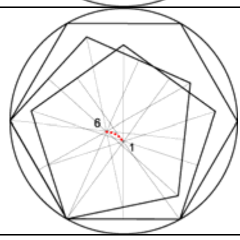
發現：1.正方形（邊長爲 1 單位）的『頂點』在單位圓內滾動的軌跡如圖所示。

2.由於正方形的旋轉角度爲 30 度，以 P 點計算其軌跡長度，



軌跡長度總和＝$\frac{\sqrt{2}}{2}\pi$

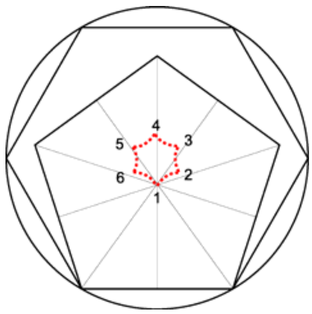
3.探討一正五邊形（邊長爲 1 單位）的中心點在單位圓內滾動的軌跡

步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 ➔ P2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	4		P4 ➔ p5	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$
2		P2 ➔ P3	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	5		P5 ➔ P6	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$
3		P3 ➔ P4	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	6		P6 ➔ P1	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$

發現：1.正五邊形（邊長爲 1 單位）的『頂點』在單位圓內滾動的軌跡如圖所示。

2.由於正五邊形的旋轉角度爲 12 度，以 P 點計算其軌跡長度，軌跡長度總和

$=\frac{8}{5\sqrt{10-2\sqrt{5}}}\pi$



（三）探討一正 n 邊形（邊長為 1 單位）內部任一點在單位圓內滾動的軌跡

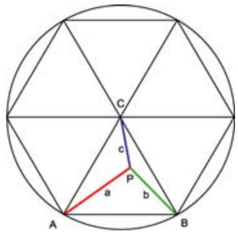
1.探討一正三邊形（邊長為 1 單位）的內部任一點在單位圓內滾動的軌跡

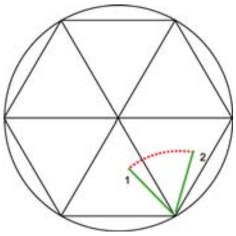
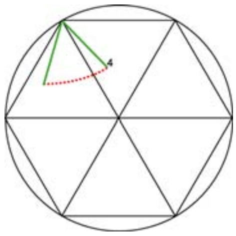
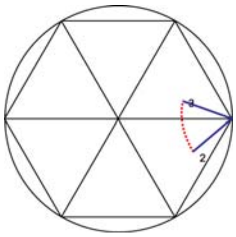
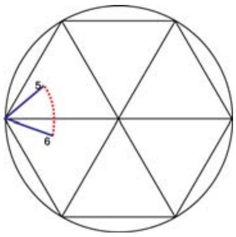
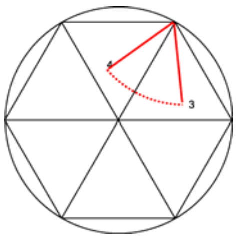
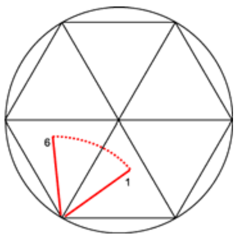
由於探討的是內部任一點，半徑未特定，

故不失一般性地假設點 P 至 A 點的半徑為 a，

點 P 至 B 點的半徑為 b，

點 P 至 C 點的半徑為 c（如圖所示）

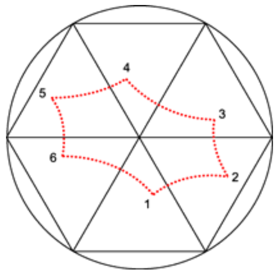


步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 → P2	b	$\frac{b\pi}{3}$	4		P4 → p5	b	$\frac{b\pi}{3}$
2		P2 → P3	c	$\frac{c\pi}{3}$	5		P5 → P6	c	$\frac{c\pi}{3}$
3		P3 → P4	a	$\frac{a\pi}{3}$	6		P6 → P1	a	$\frac{a\pi}{3}$

發現：1.正三角形（邊長為 1 單位）的『內部任一點』在單位圓內滾動的軌跡如圖所示。

2.由於正三角形的旋轉角度為 60 度，以 P 點計算其軌跡長度，

軌跡長度總和＝$\frac{2(a+b+c)\pi}{3}$



2.探討一正方形（邊長為 1 單位）的內部任一點在單位圓內滾動的軌跡

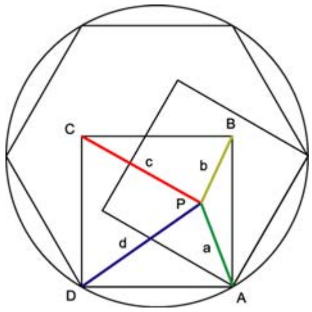
由於探討的是內部任一點，半徑未特定，

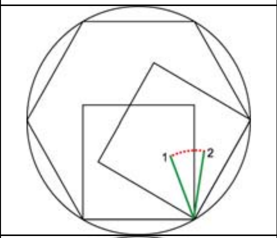
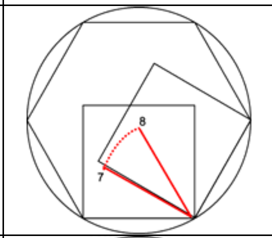
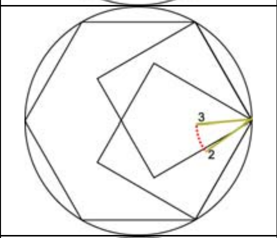
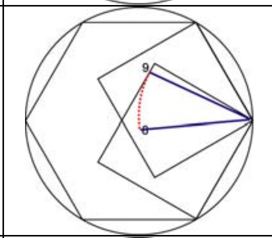
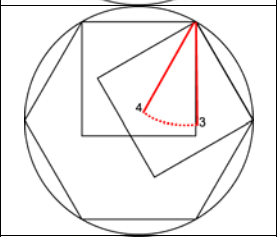
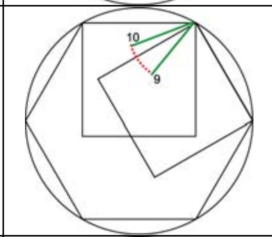
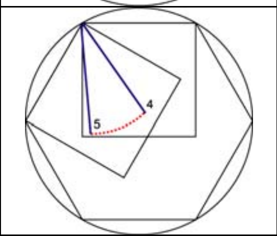
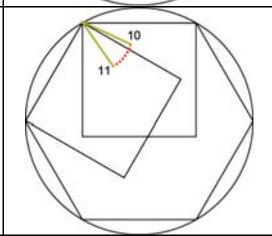
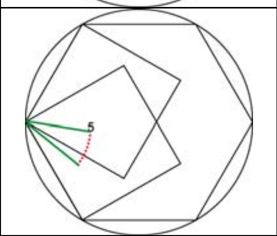
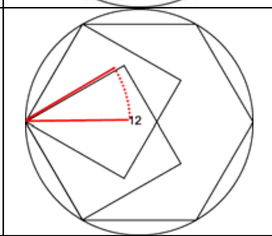
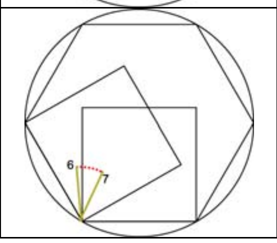
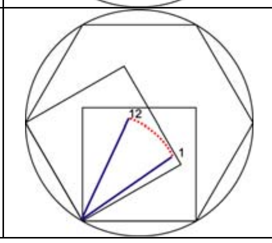
故不失一般性地假設點 P 至 A 點的半徑為 a，

點 P 至 B 點的半徑為 b，

點 P 至 C 點的半徑為 c

點 P 至 D 點的半徑為 d（如圖所示）

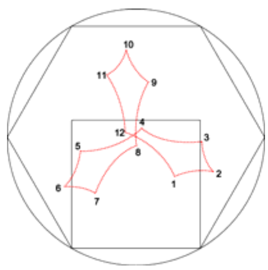


步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 → P2	a	$\frac{a\pi}{6}$	7		P7 → P8	c	$\frac{c\pi}{6}$
2		P2 → P3	b	$\frac{b\pi}{6}$	8		P8 → P9	d	$\frac{d\pi}{6}$
3		P3 → P4	c	$\frac{c\pi}{6}$	9		P9 → P10	a	$\frac{a\pi}{6}$
4		P4 → p5	d	$\frac{d\pi}{6}$	10		P10 → P11	b	$\frac{b\pi}{6}$
5		P5 → P6	a	$\frac{a\pi}{6}$	11		P11 → P12	c	$\frac{c\pi}{6}$
6		P6 → P7	b	$\frac{b\pi}{6}$	12		P12 → P1	d	$\frac{d\pi}{6}$

發現：1.正方形（邊長為 1 單位）的『內部任一點』在單位圓內滾動的軌跡如圖所示。

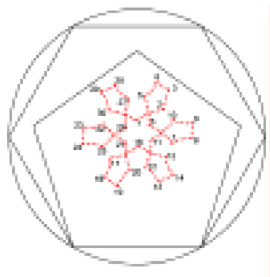
2.由於正方形的旋轉角度為 30 度，以 P 點計算其軌跡長度，

軌跡長度總和$=\frac{(a+b+c+d)\pi}{2}$

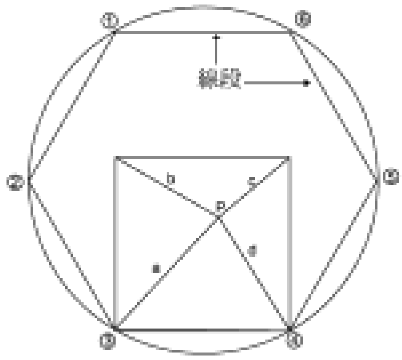


3.探討一正五邊形（邊長爲 1 單位）的『內部任一點』在單位圓內滾動的軌跡

以下分解步驟圖共有 20 步驟，故僅列出分解總圖表示



結論：一正 n 邊形（n=3,4,5）置於圓內滾動直到圖形中任一點回到原位置所滾動的軌跡皆形成一個循環，而圖形中任一點回到原位置所滾動的圈數 $=\frac{[n,6]}{6}$ 圈

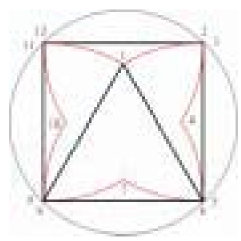


說明：在正 n 邊形內任取一點,此點至圓形頂點的連線共有 n 條，令其長度分別爲 a，b，c…，因重回原點代表那一點至同一頂點的長度相等，所以當線段 A 連接初始頂點，或線段 B 連接初始頂點時即重回原點，因此若將正 n 邊形的 n 條線段分別與正六邊形一一對應，則利用最小公倍數關係可求出其線段重新連接原起點所經過的頂點數，也就是將結果除以 6 可得圖形滾動圈數。

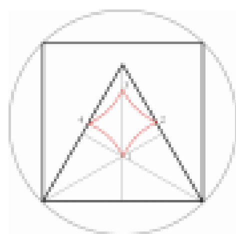
例：右圖，以 d 點爲準，當 d 點重新接到④時所走過的線段除以 6 即爲滾動圈數：$\frac{[4,6]}{6}=2$ 圈

二、探討一正 m 邊形（邊長爲 1 單位）中的一點在正 n 邊形（邊長爲 1 單位）滾動的軌跡及長度，其中 m＜n。

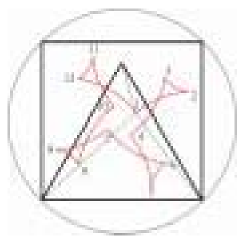
(一) 探討一正三角形（邊長爲 1 單位）的一點在正 n 邊形（邊長爲 1 單位）內滾動的軌跡



1.正三角形（邊長爲 1 單位）的『頂點』在正方形（邊長爲 1 單位）內滾動的軌跡長度爲 $\frac{4}{3}\pi$ 單位長，軌跡如右。



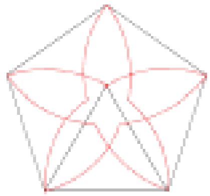
2.正三角形（邊長爲 1 單位）的『中心點』在正方形（邊長爲 1 單位）內滾動的軌跡長度: $\frac{2\sqrt{3}}{9}\pi$ 單位長，軌跡如右。



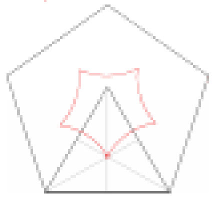
3.正三角形（邊長爲 1 單位）的『內部任一點』在正方形（邊長爲 1 單位）內滾動的軌跡如右

設任一點至三頂點分別爲 a,b,c，則軌跡長度: $\frac{2(a+b+c)\pi}{3}$ 單位長

4.正三角形（邊長爲 1 單位）的『頂點』在正五邊形（邊長爲 1 單位）內滾動的軌跡長度爲 $\frac{8}{3}\pi$ 單位長，軌跡如右。

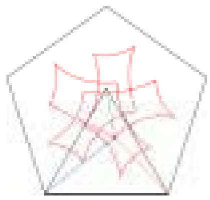


5.正三角形（邊長爲 1 單位）的『中心點』在正五邊形（邊長爲 1 單位）內滾動的軌跡長度: $\frac{4\sqrt{3}}{9}\pi$ 單位長，軌跡如右。



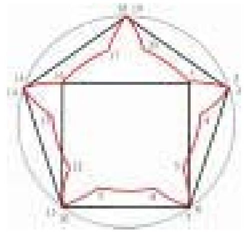
6.正三角形（邊長爲 1 單位）的『內部任一點』在正五邊形（邊長爲 1 單位）內滾動的軌跡如右

設任一點至三頂點分別爲 a,b,c，則軌跡長度: $\frac{2(a+b+c)\pi}{3}$ 單位長

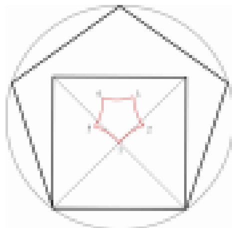


(二) 探討一正方形（邊長爲 1 單位）的一點在正 n 邊形（邊長爲 1 單位）內滾動的軌跡

1.正方形（邊長爲 1 單位）的『頂點』在正五邊形（邊長爲 1 單位）內滾動的軌跡長度: $\left(1+\frac{\sqrt{2}}{2}\right)\pi$ 單位長，其軌跡如右。

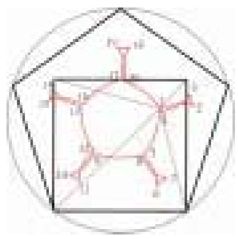


2.正方形（邊長爲 1 單位）的『中心點』在正五邊形（邊長爲 1 單位）內滾動的軌跡長度: $\frac{\sqrt{2}}{4}\pi$ 單位長，其軌跡如右。



3.正方形（邊長爲 1 單位）的『內部任一點』在正五邊形（邊長爲 1 單位）內滾動的軌跡

設任一點至三頂點分別爲 a,b,c,d，軌跡長度: $\frac{(a+b+c+d)\pi}{2}$ 單位長。

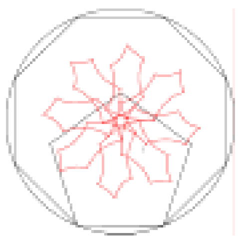


結論：根據上述圖形觀察，所形成的軌跡圖形邊數爲[m，n]（即 m,n 的最小公倍數），以圖形中的各點(中心點、頂點、其他位置的點)所畫出的軌跡圖形都不同。

備註：我們利用上述結論製作了「圖形板」，詳見陸、結論與應用。

如：1. 正方形的內部任一點在正七邊形內滾動軌跡，邊數爲[4，7]=28

2. 正五邊形的內部任一點在正八邊形內滾動軌跡，邊數爲[5，8]=40



(三) 探討一正 n 邊形（邊長爲 1 單位）的一點在正 m 邊形（邊長爲 1 單位）內滾動的軌跡參數式

以「正方形」（邊長爲 1 單位）的任一點在正六邊形（邊長爲 1 單位）內滾動的軌跡參數式爲例

不失一般性地假設 A_1 爲正六邊形的頂點，B_1 爲正方形內部任一點（如圖）

$$A_1=(a_1,b_1)=(r\cos\frac{2(1-1)\pi}{6},r\sin\frac{2(1-1)\pi}{6})$$

$$B_1=(c_1,d_1)$$

$$\alpha=\tan^{-1}\frac{b_1-d_1}{a_1-c_1}=\tan^{-1}\frac{0-d_1}{a_1-c_1}\text{(此時}90^0<\alpha<180^0\text{)}$$

$$\overline{A_1B_1}=\sqrt{(c_1-a_1)^2+(d_1-b_1)^2}$$

此時以 A_1 爲圓心，$\overline{A_1B_1}$ 爲旋轉半徑，B_1 順時針繞 30 度至 B_2

假設 B_2 坐標爲 (c_2,d_2)

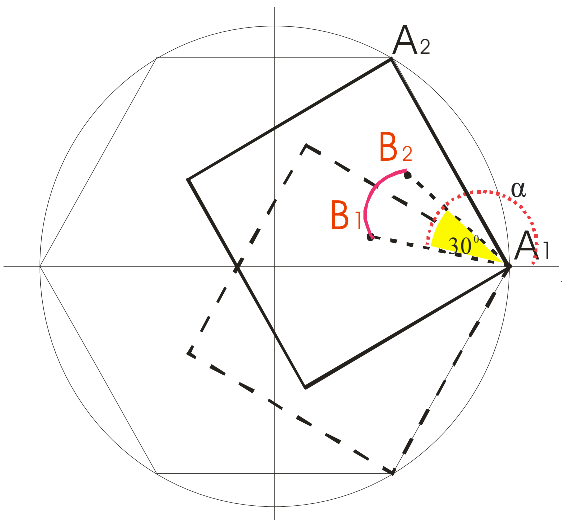
其中 $c_2=\overline{A_1B_1}\cos(\alpha-30^0)+r\cos\frac{2\times0\times\pi}{6}=\overline{A_1B_1}\cos(\alpha-30^0)+a_1$

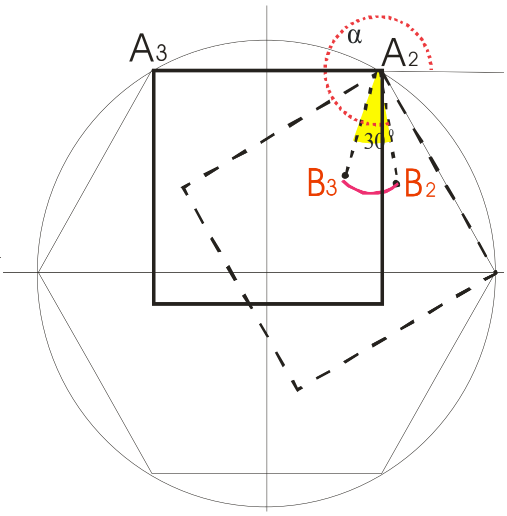
$$d_2=\overline{A_1B_1}\sin(\alpha-30^0)+r\sin\frac{2\times0\times\pi}{6}=\overline{A_1B_1}\sin(\alpha-30^0)+b_1$$

而 $\overset{\frown}{B_1B_2}$ 的軌跡參數式爲

$$x=\overline{A_1B_1}\cos\theta+r\cos\frac{2\times0\times\pi}{6}=\overline{A_1B_1}\cos\theta+a_1$$

$$y=\overline{A_1B_1}\sin\theta+r\sin\frac{2\times0\times\pi}{6}=\overline{A_1B_1}\sin\theta+b_1\text{，}\alpha-30^0\le\theta\le\alpha$$





承上，假設 A_2 爲正六邊形 A_1 的下一個頂點，$B_2=(c_2,d_2)$（如圖）

$$A_2=(a_2,b_2)=(r\cos\frac{2(2-1)\pi}{6},r\sin\frac{2(2-1)\pi}{6})$$

$$\alpha = \tan^{-1}\frac{b_2 - d_2}{a_2 - c_2}$$

$$\overline{A_2B_2} = \sqrt{(c_2 - a_2)^2 + (d_2 - b_2)^2}$$

此時以 A_2 爲圓心，$\overline{A_2B_2}$ 爲旋轉半徑，B_2 順時針繞 30 度至 B_3

假設 B_3 坐標爲 (c_3, d_3)

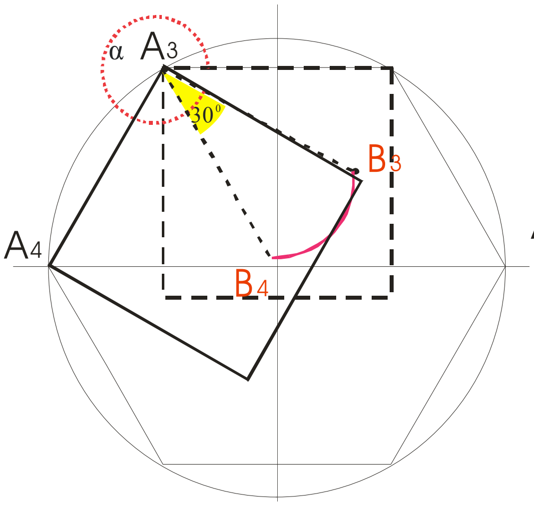
其中 $c_3 = \overline{A_2B_2}\cos(\alpha - 30^0) + r\cos\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\cos(\alpha - 30^0) + a_2$

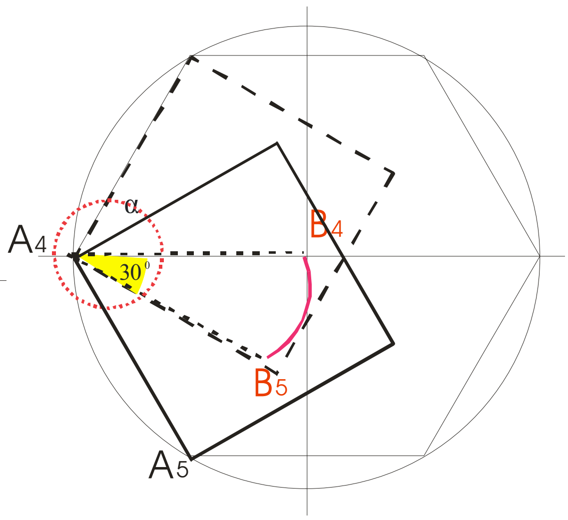
$$d_3 = \overline{A_2B_2}\sin(\alpha - 30^0) + r\sin\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\sin(\alpha - 30^0) + b_2$$

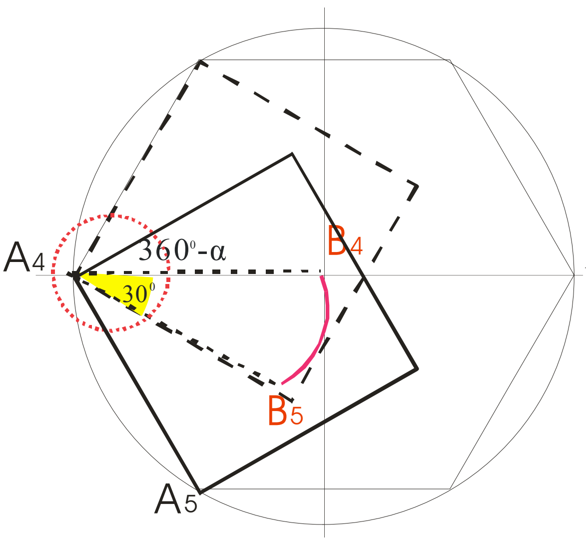
而 $\widehat{B_2B_3}$ 的軌跡參數式爲

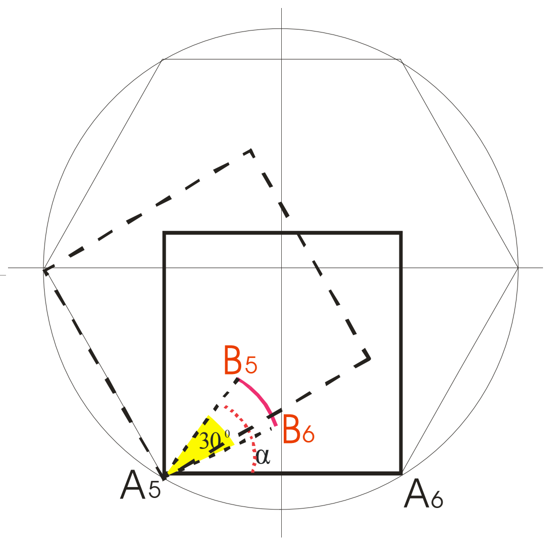
$$x = \overline{A_2B_2}\cos\theta + r\cos\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\cos\theta + a_2$$

$$y = \overline{A_2B_2}\sin\theta + r\sin\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\sin\theta + b_2 \text{，} \alpha - 30^0 \le \theta \le \alpha$$









不失一般性地假設 Aj 爲正六邊形的頂點，Bj 爲正方形內部任一點（如圖）

其中 $r=\overline{OA_j}$

$$A_j=(a_j,b_j)=(r\cos\frac{2(j-1)\pi}{6},r\sin\frac{2(j-1)\pi}{6})\ ，\ j=1,2,\ldots,12$$

$$B_j=(c_j,d_j)\quad ，c_j=\overline{A_{j-1}B_{j-1}}\cos(\theta_i-30^0)+r\cos\frac{2(j-1)\pi}{6}\ ，\ i=1,2,3,4$$

$$d_j=\overline{A_{j-1}B_{j-1}}\sin(\theta_i-30^0)+r\sin\frac{2(j-1)\pi}{6}\ ，\ j=2,3,\ldots,12$$

$$\overline{A_jB_j}=\sqrt{(c_j-a_j)^2+(d_j-b_j)^2}\ ，\ j=1,2,\ldots,12$$

此時以 A_j 爲圓心，$\overline{A_jB_j}$ 爲旋轉半徑，B_j 順時針繞 30 度至 B_{j+1}

此時 $\widehat{B_jB_{j+1}}$ 的軌跡參數式爲

$$\begin{cases}x_j=\overline{A_jB_j}\cos\theta+r\cos\dfrac{2(j-1)\pi}{6}\\ y_j=\overline{A_jB_j}\sin\theta+r\sin\dfrac{2(j-1)\pi}{6}\end{cases}\ ，\ \theta_i-30^0\le\theta\le\theta_i$$

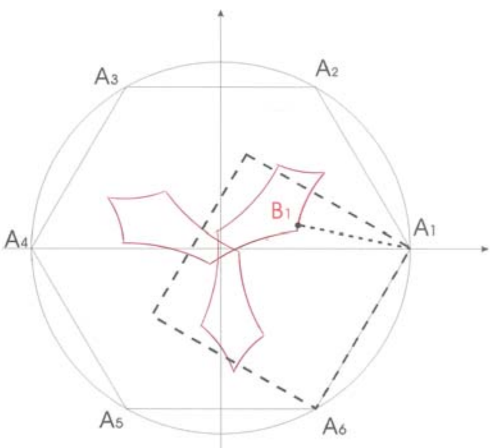
θ_i 的判定方式：是 Aj 、 Bj 兩點對應時產生的角度關係

情形一、若 $c_j-a_j<0,d_j-b_j<0\Rightarrow\theta_1=270^0-\beta$ ，$\beta=\tan^{-1}\left|\dfrac{c_j-a_j}{d_j-b_j}\right|$

情形二、若 $c_j-a_j>0,d_j-b_j<0\Rightarrow\theta_2=360^0-\beta$ ，$\beta=\tan^{-1}\left|\dfrac{d_j-b_j}{c_j-a_j}\right|$

情形三、若 $c_j-a_j>0,d_j-b_j>0\Rightarrow\theta_3=90^0-\beta$ ，$\beta=\tan^{-1}\left|\dfrac{c_j-a_j}{d_j-b_j}\right|$

情形四、若 $c_j-a_j<0,d_j-b_j>0\Rightarrow\theta_4=180^0-\beta$ ，$\beta=\tan^{-1}\left|\dfrac{d_j-b_j}{c_j-a_j}\right|$



推廣至正 n 邊形中的一點繞正 m 邊形滾動時，產生之軌跡爲一循環，其軌跡參數式爲：

若 Aj 爲正 m 邊形的頂點，Bj 爲正 n 邊形內部任一點（如圖），其中 $r=\overline{OA_j}$

$$A_j=(a_j,b_j)=(r\cos\frac{2(j-1)\pi}{m},r\sin\frac{2(j-1)\pi}{m})\text{ ， } j=1,2,\ldots,[m,n]$$

$$B_j=(c_j,d_j)\text{ ， } c_j=\overline{A_{j-1}B_{j-1}}\cos[\theta_i-(\frac{360^0}{n}-\frac{360^0}{m})]+r\cos\frac{2(j-1)\pi}{m}\text{ ， } i=1,2,3,4$$

$$d_j=\overline{A_{j-1}B_{j-1}}\sin[\theta_i-(\frac{360^0}{n}-\frac{360^0}{m})]+r\sin\frac{2(j-1)\pi}{m}\text{ ， } j=2,3,\ldots,[m,n]$$

$$\overline{A_jB_j}=\sqrt{(c_j-a_j)^2+(d_j-b_j)^2}\text{ ， } j=1,2,\ldots,[m,n]$$

此時以 A_j 爲圓心，$\overline{A_jB_j}$ 爲旋轉半徑，B_j 順時針繞 $(\frac{360^0}{n}-\frac{360^0}{m})$ 度至 B_{j+1}

此時 $\overset{\frown}{B_jB_{j+1}}$ 的軌跡參數式爲

$$\begin{cases} x_j=\overline{A_jB_j}\cos\theta+r\cos\frac{2(j-1)\pi}{m} \\ y_j=\overline{A_jB_j}\sin\theta+r\sin\frac{2(j-1)\pi}{m} \end{cases}\text{ ， } \theta_i-(\frac{360^0}{n}-\frac{360^0}{m})\le\theta\le\theta_i$$

θ_i 的判定方式：是 Aj 、 Bj 兩點對應時產生的角度關係

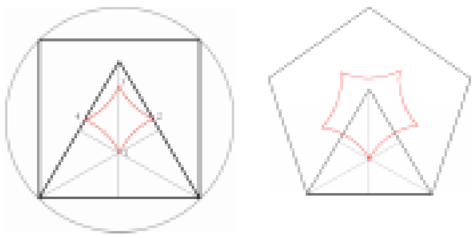
情形一、若 $c_j-a_j<0,d_j-b_j<0\Rightarrow\theta_1=270^0-\beta$ ，$\beta=\tan^{-1}\left|\frac{c_j-a_j}{d_j-b_j}\right|$

情形二、若 $c_j-a_j>0,d_j-b_j<0\Rightarrow\theta_2=360^0-\beta$ ，$\beta=\tan^{-1}\left|\frac{d_j-b_j}{c_j-a_j}\right|$

情形三、若 $c_j-a_j>0,d_j-b_j>0\Rightarrow\theta_3=90^0-\beta$ ，$\beta=\tan^{-1}\left|\frac{c_j-a_j}{d_j-b_j}\right|$

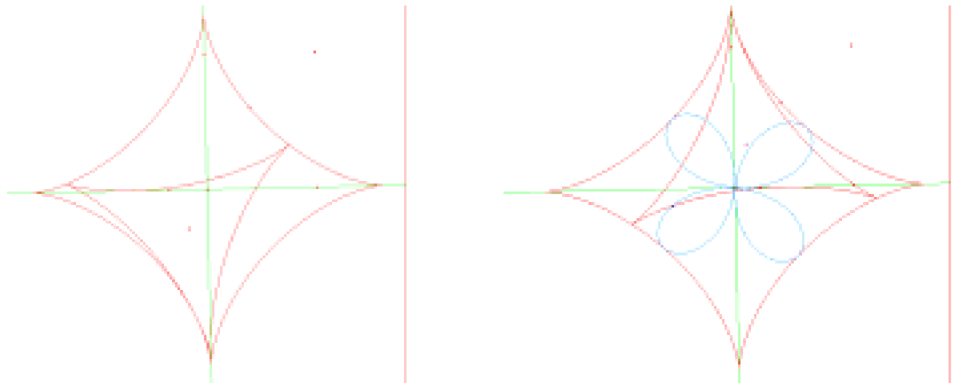
情形四、若 $c_j-a_j<0,d_j-b_j>0\Rightarrow\theta_4=180^0-\beta$ ，$\beta=\tan^{-1}\left|\frac{d_j-b_j}{c_j-a_j}\right|$

觀察了一下研究二中的結果，當正三角形中心點繞正方形滾動後、正三角形中心點繞正五邊形滾動後，發現會產生一似星狀之圖案，這些圖形是不是和擺線中的星狀線一樣呢？讓我不禁想探討如果像這樣的圖形，是否可以也像正多邊形滾動並找出滾動圖形中一點的軌跡並探討其軌跡方程式



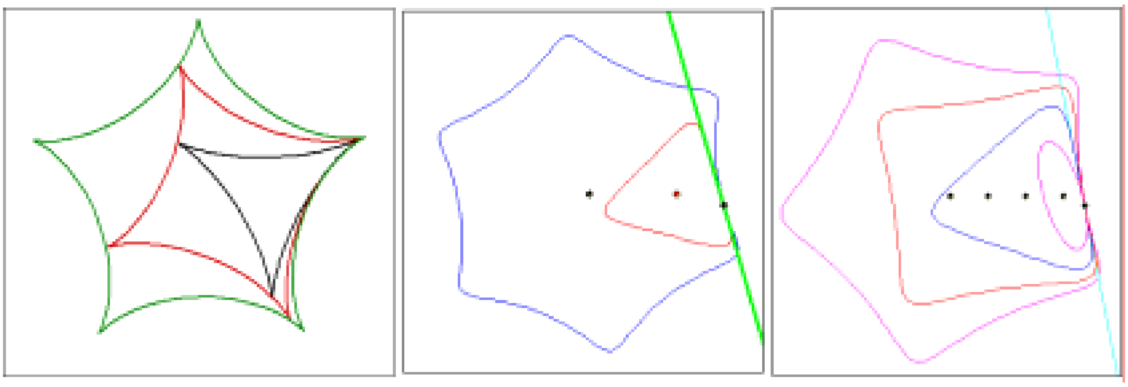
三、探討各式擺線相切滾動時，滾動圖形中某一點的軌跡規律性及其方程式。

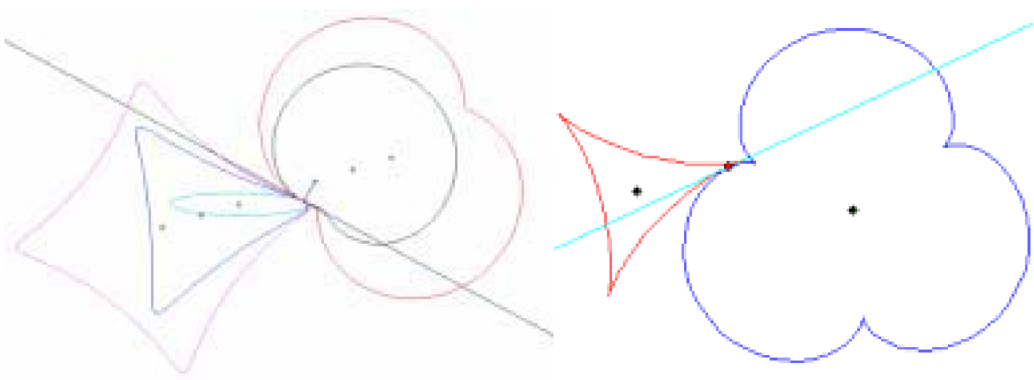
利用 Cabri Geometry 動態模擬三角形線在星狀線中滾動之情形。



探討三角形線上一點在星狀線中滾動所產生之軌跡（如上面右圖中藍色部份）。

未來嘗試利用 Cabri Geometry 動態模擬各式曲線（如下圖等等），滾動圖形中一點產生之軌跡及其方程式。





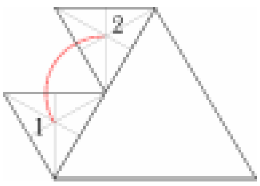
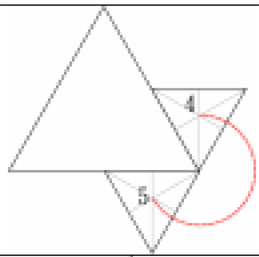
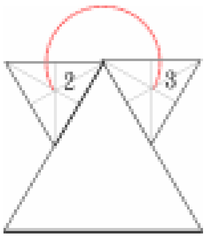
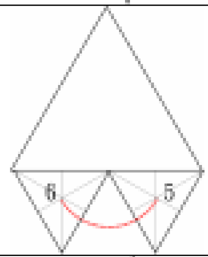
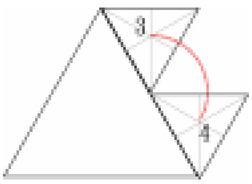
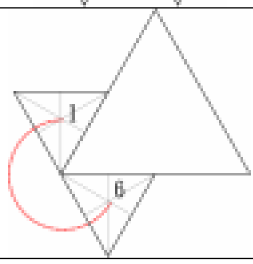
陸、討論：

一、探討一正 n 邊形（邊長爲 1 單位）中心點繞一正 n 邊形（邊長爲 m 單位）的滾動軌跡及其長度。

(一) 探討一正三角形（邊長爲 1 單位）中心點繞一正三邊形（邊長爲 m 單位）的滾動軌跡及其長度。

首先，探討一小三角形（邊長爲 1 單位）中心點在一大三角形（邊長爲 1 單位）外滾動的軌跡，發現軌跡長度: $\frac{4\sqrt{3}}{3}\pi$ 單位長

1.探討一小三角形（邊長爲 1 單位）中心點在一大三角形（邊長爲 2 單位）外滾動的軌跡

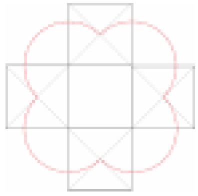
步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度	步驟	圖形軌跡變化分解圖	旋轉	旋轉半徑	旋轉長度
1		P1 → P2	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{9}\pi$	4		P4 → P5	$\frac{\sqrt{3}}{3}$	$\frac{4\sqrt{3}}{9}\pi$
2		P2 → P3	$\frac{\sqrt{3}}{3}$	$\frac{4\sqrt{3}}{9}\pi$	5		P5 → P6	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{9}\pi$
3		P3 → p4	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{9}\pi$	6		P6 → P1	$\frac{\sqrt{3}}{3}$	$\frac{4\sqrt{3}}{9}\pi$

發現：軌跡如右，長度: $2\sqrt{3}\pi$ 單位長

推廣發現：一小三角形（邊長爲 1 單位）中心點在一大三角形（邊長爲 m 單位）外滾動的軌跡長度: $\frac{2\sqrt{3}}{3}(m+1)\pi$ 單位長

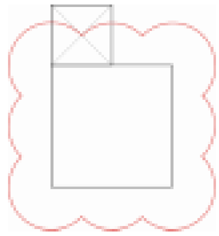
(二) 探討一正方形（邊長爲 1 單位）中心點繞一正方形（邊長爲 m 單位）的滾動軌跡及其長度。

1.探討一小正方形（邊長爲 1 單位）中心點在一大正方形（邊長爲 1 單位）外滾動的軌跡



發現：軌跡如右，長度: $2\sqrt{2}\pi$ 單位長

2.探討一小正方形（邊長爲 1 單位）中心點在一大正方形（邊長爲 2 單位）外滾動的軌跡

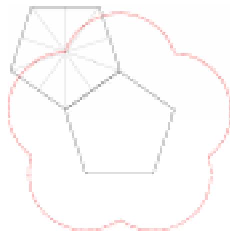


發現：軌跡如右，長度: $3\sqrt{2}\pi$ 單位長

推廣發現：探討一小正方形（邊長爲 1 單位）中心點在一大正方形（邊長爲 m 單位）外滾動的軌跡長度: $(m+1)\sqrt{2}\pi$ 單位長

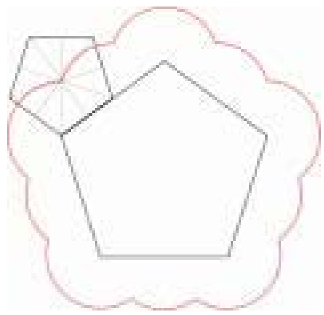
(三) 探討一正五邊形（邊長爲 1 單位）中心點繞一正五邊形（邊長爲 m 單位）的滾動軌跡及其長度。

1.探討一小正五邊形（邊長爲 1 單位）中心點在一大正五邊形（邊長爲 1 單位）外滾動的軌跡



發現：軌跡如右，長度: $\dfrac{16}{\sqrt{10-2\sqrt{5}}}\pi$ 單位長

2.探討一小正五邊形（邊長爲 1 單位）中心點在一大正五邊形（邊長爲 2 單位）外滾動的軌跡



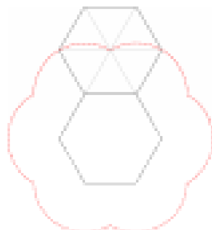
發現：軌跡如右，長度: $\dfrac{24}{\sqrt{10-2\sqrt{5}}}\pi$ 單位長

推廣發現：一小正五邊形（邊長爲 1 單位）中心點在一大正五邊形（邊長爲 m 單位）外滾動的軌跡長度:

$\dfrac{8(m+1)}{\sqrt{10-2\sqrt{5}}}\pi$ 單位長

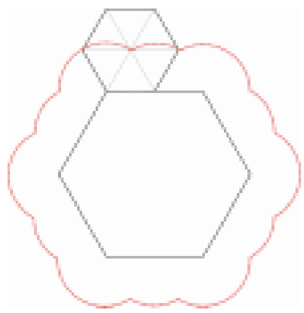
(四) 探討一正六邊形（邊長爲 1 單位）中心點繞一正六邊形（邊長爲 m 單位）的滾動軌跡及其長度。

1.探討一小正六邊形（邊長爲 1 單位）中心點在一大正六邊形（邊長爲 1 單位）外滾動的軌跡



發現：軌跡如右，長度: 4π 單位長

2.探討一小正六邊形（邊長爲 1 單位）中心點在一大正六邊形（邊長爲 2 單位）外滾動的軌跡



發現：軌跡如右，長度: 6π 單位長

推廣發現：一小正六邊形（邊長爲 1 單位）中心點在一大正六邊形（邊長爲 m 單位）外滾動的軌跡長度:

$2\pi(m+1)$ 單位長

(五) 探討一正 n 邊形（邊長爲 1 單位）中心點繞一正 n 邊形（邊長爲 m 單位）的滾動軌跡及其長度。

設正 n 邊形的重心至頂點的長度爲 k

正 n 邊形之內角爲 $\frac{180(n-2)}{n}$

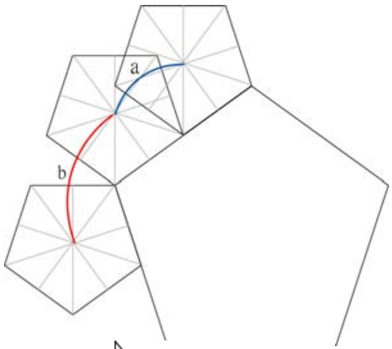
如圖，$a = 2k\pi \times \frac{180-\frac{180(n-2)}{n}}{360}$ 共 n(m-1)個

$\therefore$ 全部的 $a = 2k\pi \times n(m-1)\times \frac{180-\frac{180(n-2)}{n}}{360} = 2k\pi(m+1)$

$b = 2k\pi \times \frac{360-\frac{360(n-2)}{n}}{360}$ 共 n 個

$\therefore$ 全部的 $b = 2k\pi \times n \times \frac{360-\frac{360(n-2)}{n}}{360} = 4k\pi$

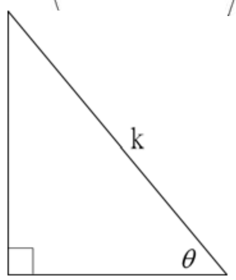
總長=$2k\pi(m-1)+4k\pi = 2k\pi(m+1)$



正 n 邊形的重心至頂點的長度 k 之算法

如圖，$\theta = \frac{180(n-2)}{n} \div 2 = 90 - \frac{180}{n}$

$\cos\theta = \frac{\frac{1}{2}}{k} = \frac{1}{2k}$，$2k\cos\theta = 1$，$k = \frac{1}{2\cos\theta}$



結論： 若$\theta = 90 - \frac{180}{n}$ (θ是正 n 邊形之內角的一半)

一正 n 邊形（邊長爲 1 單位）中心點繞一正 n 邊形（邊長爲 m 單位）的滾動軌跡總長：$2\pi(m+1)\times\frac{1}{2\cos\theta} = \frac{(m+1)\pi}{\cos\theta}$ 。

柒、結論及應用：

一、一正 n 邊形（n=3,4,5）置於圓內滾動直到圖形中任一點回到原位置所滾動的軌跡皆形成一個循環，而圖形中任一點回到原位置所滾動的圈數=$\frac{[n,6]}{6}$圈

二、一正 n 邊形（邊長爲 1 單位）中一點繞正 m 邊形（邊長爲 1 單位），所形成的軌跡圖形邊數爲[m，n]（即 m,n 的最小公倍數），其中 m＜n。而其軌跡參數式爲：

若 Aj 爲正 m 邊形的頂點，Bj 爲正 n 邊形內部任一點（如圖），其中 $r = \overline{OA_j}$

$A_j = (a_j, b_j) = (r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m})$，$j = 1,2,\ldots,[m,n]$

$y_j = \overline{A_jB_j}\sin\theta + r\sin\dfrac{2(j-1)\pi}{m}$

$B_j = (c_j, d_j)$ ，$c_j = \overline{A_{j-1}B_{j-1}}\cos[\theta_i - (\dfrac{360^0}{n} - \dfrac{360^0}{m})] + r\cos\dfrac{2(j-1)\pi}{m}$ ，$i = 1,2,3,4$

$d_j = \overline{A_{j-1}B_{j-1}}\sin[\theta_i - (\dfrac{360^0}{n} - \dfrac{360^0}{m})] + r\sin\dfrac{2(j-1)\pi}{m}$ ，$j = 2,3,\ldots,[m,n]$

$\overline{A_jB_j} = \sqrt{(c_j - a_j)^2 + (d_j - b_j)^2}$ ，$j = 1,2,\ldots,[m,n]$

此時以 A_j 為圓心，$\overline{A_jB_j}$ 為旋轉半徑，B_j 順時針繞 $(\dfrac{360^0}{n} - \dfrac{360^0}{m})$ 度至 B_{j+1}

此時 $\overparen{B_jB_{j+1}}$ 的軌跡參數式為

$$\begin{cases} x_j = \overline{A_jB_j}\cos\theta + r\cos\dfrac{2(j-1)\pi}{m} \\ y_j = \overline{A_jB_j}\sin\theta + r\sin\dfrac{2(j-1)\pi}{m} \end{cases} ，\theta_i - (\dfrac{360^0}{n} - \dfrac{360^0}{m}) \le \theta \le \theta_i$$

θ_i 的判定方式：$\beta = \tan^{-1}\left|\dfrac{c_j - a_j}{d_j - b_j}\right|$ 是 Aj 、Bj 兩點對應時產生的角度關係

情形一、若 $c_j - a_j < 0, d_j - b_j < 0 \Rightarrow \theta_1 = 270^0 - \beta$

情形二、若 $c_j - a_j > 0, d_j - b_j < 0 \Rightarrow \theta_2 = 360^0 - \beta$

情形三、若 $c_j - a_j > 0, d_j - b_j > 0 \Rightarrow \theta_3 = 90^0 - \beta$

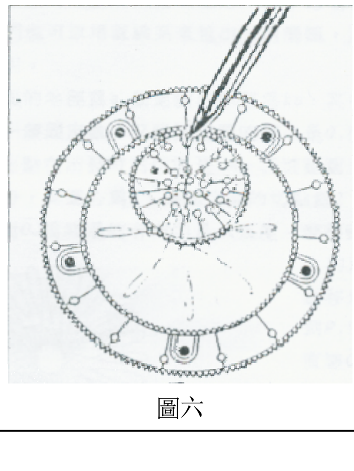
情形四、若 $c_j - a_j < 0, d_j - b_j > 0 \Rightarrow \theta_4 = 180^0 - \beta$

三、一正 n 邊形（邊長為 1 單位）中心點繞一正 n 邊形（邊長為 m 單位）的滾動軌跡總長為 $2\pi(m+1)\times\dfrac{1}{2\cos\dfrac{\theta}{2}} = \dfrac{(m+1)\pi}{\cos\dfrac{\theta}{2}}$（其中 θ 是正 n 邊形的內角度數）

四、應用：

※原有的擺線繪製圖形板：

市面上曾經流行過一種可繪製曲線的器具，它包含一個在圓周上刻滿鋸齒的小圓形板，以及一個在內外圓周上都刻有鋸齒的大圓環形板（見圖六）。把玩之時，將小圓板放在大圓環板內部，並讓鋸齒套合而使小圓板沿著大圓環板滾動。將筆插入小圓板上的一個小洞，隨著小圓板的滾動，鉛筆就會描繪出一條曲線。這曲線是什麼形狀呢？
(本文資料摘自「幾何學中的海倫」)

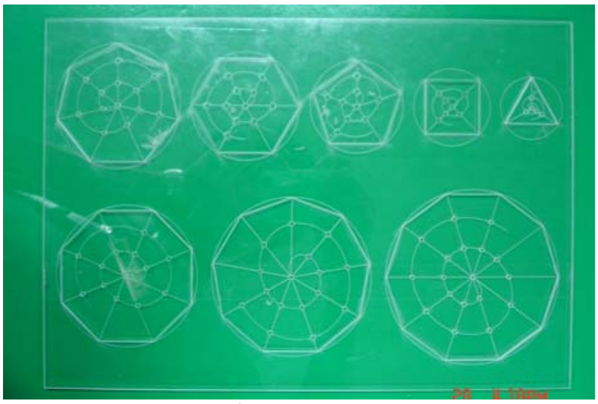


圖六

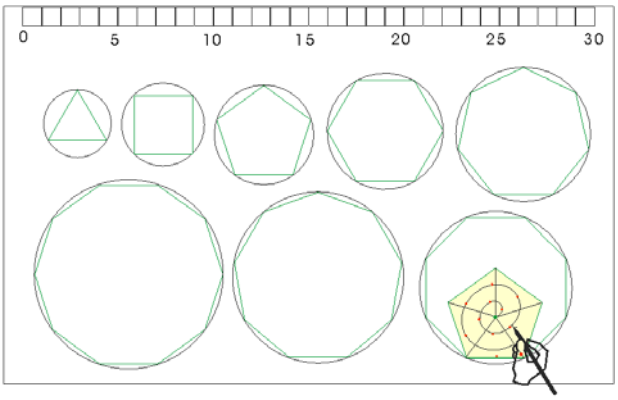
※新型設計曲線繪製圖形板：

設計出發點：每次要畫正多邊形滾動軌跡時不是要利用圓規、量角器作圖，就是要用電腦繪圖，相當麻煩。是否能利用研究二的結果，結合舊型擺線繪製圖形板之概念，製作出新型又方便的圖形板呢？

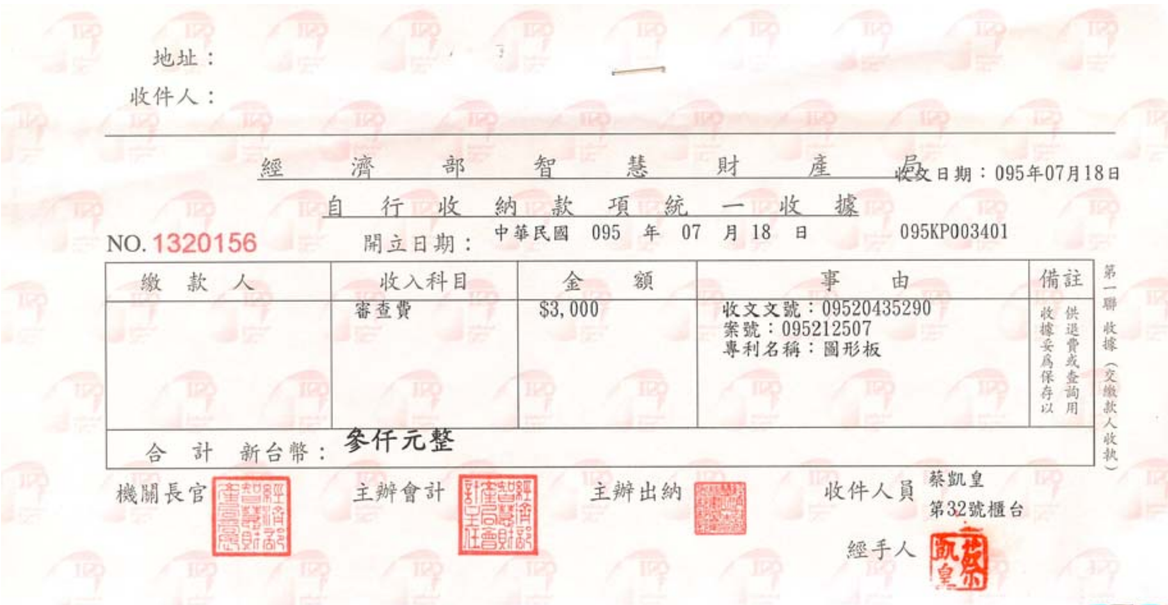
設計原理：利用研究二中的條件，此器具包含「邊長相等」的大小正多邊形洞，以及「和洞邊長相等」的大小正多邊形圖形板（如圖形板最初設計圖）。經試驗發現因正多邊形角度的關係，無法像舊型小圓版之圓周上刻鋸齒，故在繪製軌跡圖形時亦非常容易產生滑動（如圖）。嘗試各種不同防滑設計（如圖）失敗後，最後終於設計出一個滾動時不易滑動並且能迅速畫出完美的研究二結果之「圖形板」（見圖形說明），並向行政院經濟部智慧財產局申請了「新型專利」。



（原來會滑動的圖形板失敗之實品）



（最初圖形板使用－易滑動）



地址：

收件人：

經濟部智慧財產局　收文日期：095年07月18日

自行收納款項統一收據

NO. 1320156　開立日期：中華民國 095 年 07 月 18 日　095KP003401

繳款人	收入科目	金額	事由	備註
	審查費	$3,000	收文文號：09520435290 案號：095212507 專利名稱：圖形板	收據妥爲保存以供退費或查詢用
合計 新台幣：參仟元整				

第一聯 收據（交繳款人收執）

機關長官　主辦會計　主辦出納　收件人員 蔡凱皇 第32號櫃台

經手人

（行政院經濟部智慧財產局申請收據）

（圖形板之實品）

捌、參考資料：

一、 參考書目：

1.馮緒寧、袁向東譯。澳洲數學競賽試題解析 1985~1991。九章出版社。P.87，第 12 題。

2.趙文敏。幾何中的海倫。科學月刊。第 20 卷第 11 期、第 20 卷第 12 期。1989 年。

二、參考網站：

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http://www.qhsms.com.cn/baixian/index.htm

2.基本繪圖
http://elearning.emath.pu.edu.tw/mkuo2003/2003sdmgsp/cgsp/1_all.htm

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http://poncelet.math.nthu.edu.tw/chuan/cycloid/cycloid.html

4.外擺線實驗器完整版
http://web.chsh.chc.edu.tw/bee/oldmath/flash/910430.htm

5.內擺線實驗器完整版
http://web.chsh.chc.edu.tw/bee/oldmath/flash/910428.htm

6.The Cycloid Family of Curves
http://curvebank.calstatela.edu/cycloidmaple/cycloid.htm

7.MathWorld Cycloid
http://mathworld.wolfram.com/Cycloid.html

8.Cycloid
http://www-groups.dcs.st-and.ac.uk/~history/Curves/Cycloid.html

9.Cabri Java Applets
http://www.math.ntnu.edu.tw/~jcchuan/demo/cabrijava/index.htm

10.Cycloid
http://en.wikipedia.org/wiki/Cycloid

11.Special Plane Curves：Cycloid
http://xahlee.org/SpecialPlaneCurves_dir/Cycloid_dir/cycloid.html

12.Cycloid Curves
http://www.javaview.de/demo/PaCycloid.html

13.Property of Cycloid
http://www.ies.co.jp/math/java/calc/cycle_ang/cycle_ang.html

14.cycloid
http://www.2dcurves.com/roulette/roulettec.html

(1) Research Motive:

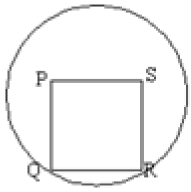


figure 1.

We started from a question in the AMC competition in Australia, "As in figure 1, a square sided exactly in total to 1 meter is placed in a circle which radius is also 1 meter. This square is moved according to the explanations below: Roll the point P clockwise till the point S contacts the circle. Roll the point S clockwise till the point P contacts with the circle. Roll this square this way until two of the square's vertices are back to the original positions which are point Q and point R. What is the length of point P's locus?"

The result of the rotation is that the locus formed by point P has a regularity, so I extended the square to a regular n sided polygon and examined it further. This made me gain a curiosity about rotations of regular polygons and I have researched deeply into other rotations as well as the relation between figures and angles.

(2) Research Goals:

1. To discover the loci and their lengths formed when a regular n sided polygon (side length=1 unit) rolls upon a unit circle.
2. To discover the loci and their lengths formed when a point in a regular n sided polygon (side length=1 unit) rolls upon another regular m sided polygon (side length=1 unit).

**We assumed that the lengths of all sides of the polygons equal 1 unit, so it's more convenient to formulate. Afterwards, (side=1) refers to (side length=1 unit).

(3) Research Equipment:

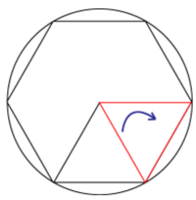
"Cabri Geometry" software, compasses and protractors.

(4) Research Process:

A. Probing into the locus when a point in a regular n sided polygon (side=1) rolls upon a unit circle.

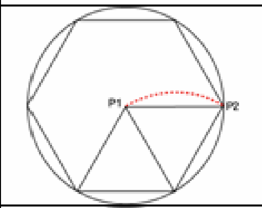
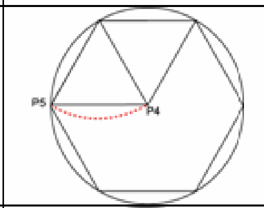
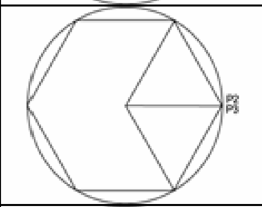
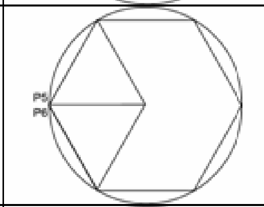
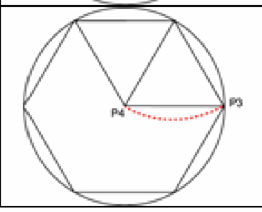
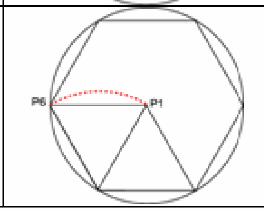
Trace rule: Place a regular polygon which side length equals 1 unit in a circle which radius is also 1 unit. Rotate the polygon along the circumference until two vertices go back to the original positions.

We roll the regular polygon clockwise according to the rule and try to find out the trace formed and the lengths when its "vertex", "center", and "any point inside" rotate.



a. To probe into the rotation locus and its length when **a “vertex” of a regular n sided polygon** rolls in a unit circle.

1. In the graphing table below, a vertex of a **regular triangle** (side=1) rolls upon a unit circle.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	1	$\frac{1}{3}\pi$	4		P4 → p5	1	$\frac{1}{3}\pi$
2		P2 → P3	0	0	5		P5 → P6	0	0
3		P3 → P4	1	$\frac{1}{3}\pi$	6		P6 → P7	1	$\frac{1}{3}\pi$

Results discovered:

(1) The rotation locus graph created when a vertex of a regular triangle rolls upon a unit circle is presented as in figure 2.

(2) The limits are set on the ground that the rotation angle of the triangle is 120－60＝60°, and the length of the locus $=\frac{4}{3}\pi$.

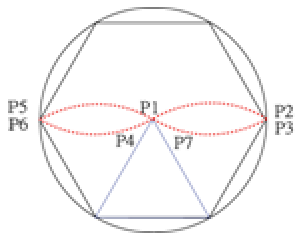
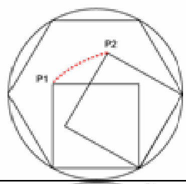
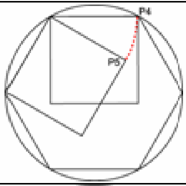
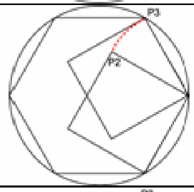
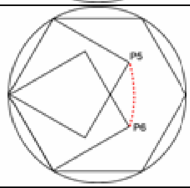
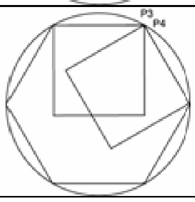
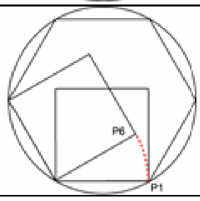


figure 2.

2. In the graphing table below, a vertex of a **square** (side=1) rolls upon a unit circle.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	$\sqrt{2}$	$\frac{\sqrt{2}}{6}\pi$	4		P4 → p5	1	$\frac{1}{6}\pi$
2		P2 → P3	1	$\frac{1}{6}\pi$	5		P5 → P6	$\sqrt{2}$	$\frac{\sqrt{2}}{6}\pi$
3		P3 → P4	0	0	6		P6 → P1	1	$\frac{1}{6}\pi$

Results discovered:

(1) The rotation locus graph created when a vertex of a square rolls upon a unit circle is presented as in figure 3.

(2) The limits are set on the ground that the rotation angle of the square is 120－90＝30°, and the length of the locus = $(\frac{\sqrt{2}}{3}+\frac{1}{2})\pi$.

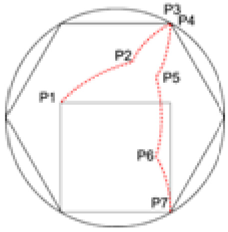
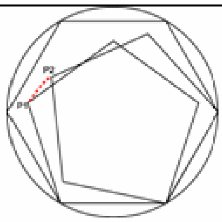
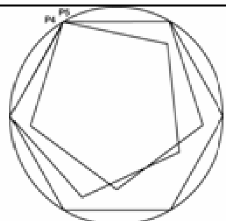
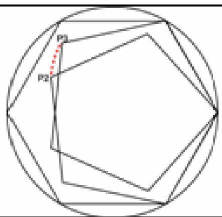
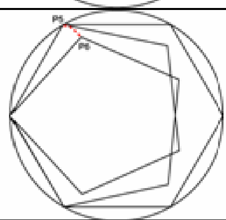
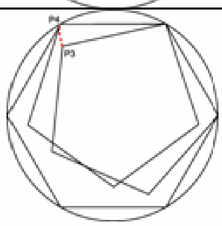
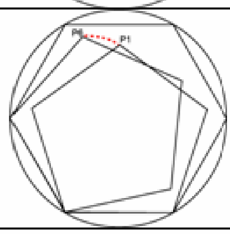


figure 3.

3. In the graphing table below, a vertex of a **regular pentagon** (side=1) rolls upon a unit circle.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$	4		P4 → p5	0	0
2		P2 → P3	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$	5		P5 → P6	1	$\frac{1}{15}\pi$
3		P3 → P4	1	$\frac{1}{15}\pi$	6		P6 → P1	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+1}{30}\pi$

Results discovered:

(1) The rotation locus graph created when a vertex of a pentagon rolls upon a unit circle is presented as in figure 4.

(2) The limits are set on the ground that the rotation angle of the pentagon is 120－108＝12°, and the length of the locus ＝ $(\frac{7}{30}+\frac{\sqrt{5}}{10})\pi$.

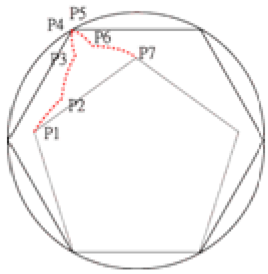
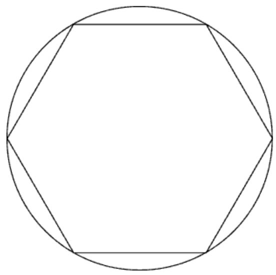


figure 4.

4. A vertex of a **regular hexagon** (side=1) rolls upon a unit circle.

Results discovered:

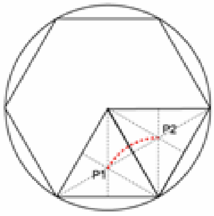
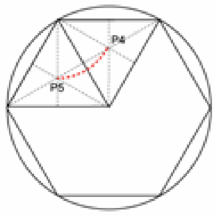
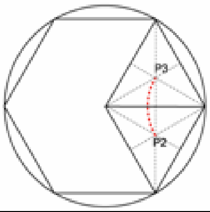
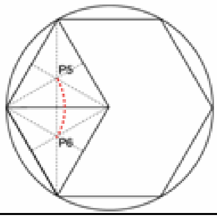
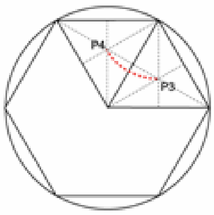
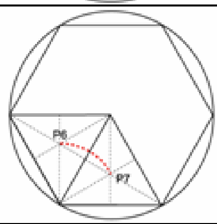
The limits are set on the ground that the rotation angle of the hexagon 120－120＝0°, and the vertex of the regular hexagon can not rotate upon the unit circle. The length of the locus = 0.



Note: Because a regular hexagon can not rotate upon a unit circle, no more explanations will follow.

b. To probe into the rotation locus and its length when **a “center” of a regular n sided polygon** rolls in a unit circle.

1. In the graphing table below, a center of a **regular triangle** (side=1) rolls upon a unit circle.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	4		P4 → p5	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$
2		P2 → P3	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	5		P5 → P6	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$
3		P3 → P4	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$	6		P6 → P1	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{9}\pi$

Results discovered:

(1) The rotation locus graph created when a center of a regular triangle rolls upon a unit circle is presented as in figure 5.

(2) The limits are set on the ground that the rotation angle of the triangle is 60°, and the length of the locus $= \frac{2\sqrt{3}}{3}\pi$.

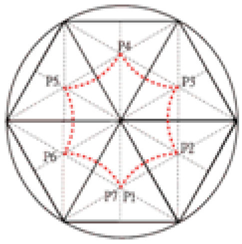
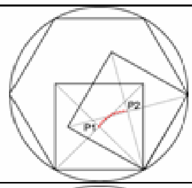
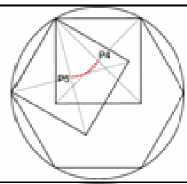
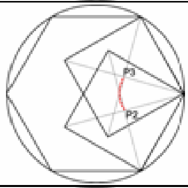
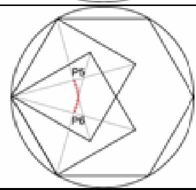
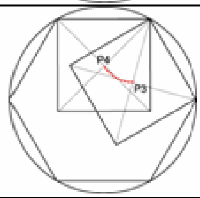
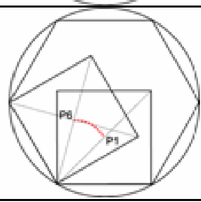


figure 5.

2. In the graphing table below, a center of a **square** (side=1) rolls upon a unit circle.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	4		P4 → p5	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$
2		P2 → P3	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	5		P5 → P6	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$
3		P3 → P4	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$	6		P6 → P1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{12}\pi$

Results discovered:

(1) The rotation locus graph created when a center of a square rolls upon a unit circle is presented as in figure 6.

(2) The limits are set on the ground that the rotation angle of the square is 30°, and the length of the locus $=\frac{\sqrt{2}}{2}\pi$.

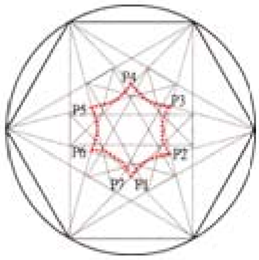
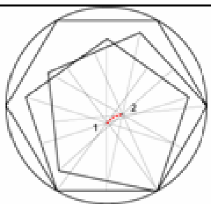
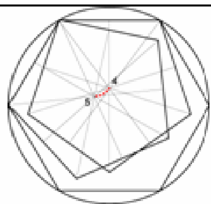
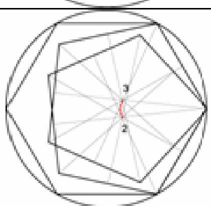
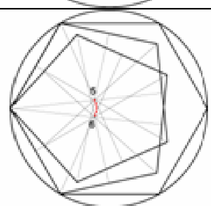
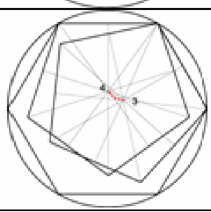
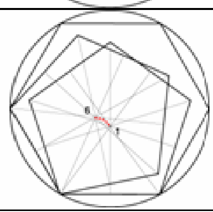


figure 6.

3. In the graphing table below, a center of a pentagon (side=1) rolls upon a unit

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	4		P4 → p5	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$
2		P2 → P3	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	5		P5 → P6	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$
3		P3 → P4	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$	6		P6 → P1	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\frac{4}{15\sqrt{10-2\sqrt{5}}}\pi$

Results discovered:

(1) The rotation locus graph created when a center of a pentagon rolls upon a unit circle is presented as in figure 7.

(2) The limits are set on the ground that the rotation angle of the pentagon is 12°, the length of the locus $= \dfrac{8}{5\sqrt{10-2\sqrt{5}}}\pi$.

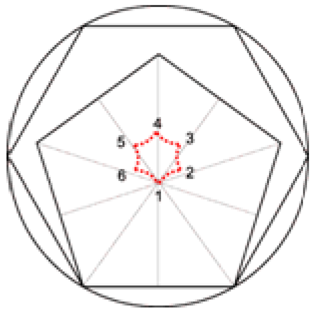


figure 7.

c. To probe into the rotation locus and its length when **"any point" inside a regular n sided polygon** rolls in a unit circle.

1. When any point inside a **regular triangle** (side=1) rolls upon a unit circle, the radius is not specified since the explanation includes any point inside. Therefore, we assume the radius of point P to A as "a",

P to B as "b",

P to C as "c" (as shown in figure 8).

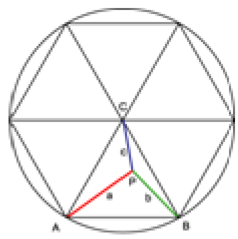
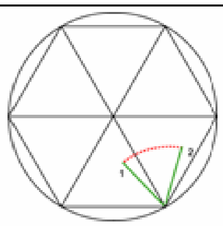
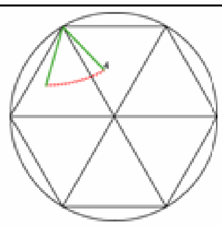
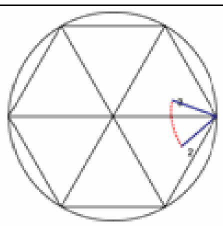
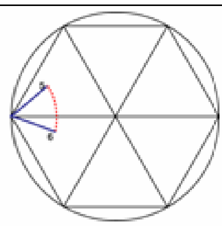
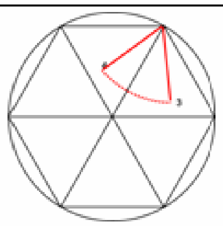
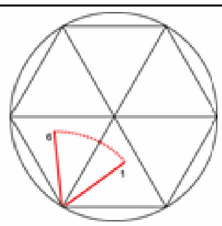


figure 8.

In the graphing table below, results are noted.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	b	$\frac{b\pi}{3}$	4		P4 → p5	b	$\frac{b\pi}{3}$
2		P2 → P3	c	$\frac{c\pi}{3}$	5		P5 → P6	c	$\frac{c\pi}{3}$
3		P3 → P4	a	$\frac{a\pi}{3}$	6		P6 → P1	a	$\frac{a\pi}{3}$

Results discovered:

(1) The rotation locus graph created when any point in a triangle rolls upon a unit circle is presented as in figure 9.

(2) The limits are set on the ground that the rotation angle of a triangle is 60°, and the length of the locus $= \dfrac{2(a+b+c)\pi}{3}$.

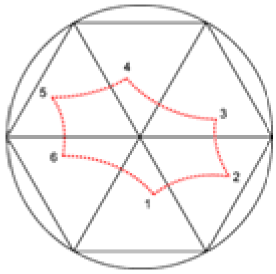


figure 9.

2. When any point inside a **square** (side=1) rolls upon a unit circle, the radius is not specified since the explanation includes any point inside. Therefore, we assume the radius of point P to A as “a”,

P to B as “b”,

P to C as “c”,

P to D as “d”(as shown in figure 10).

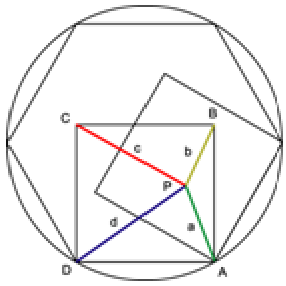
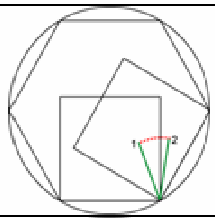
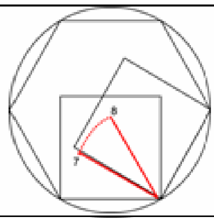
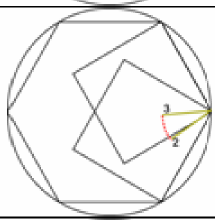
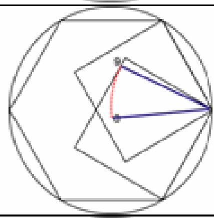
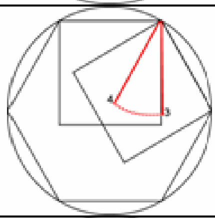
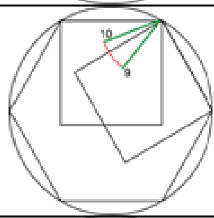
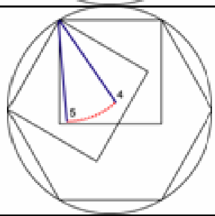
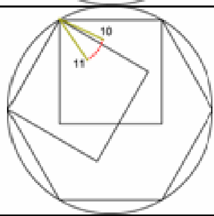
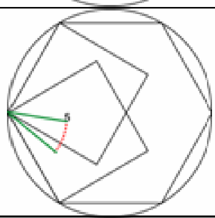
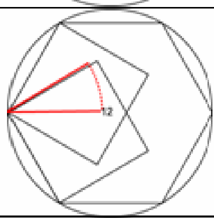
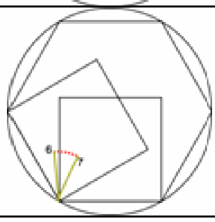
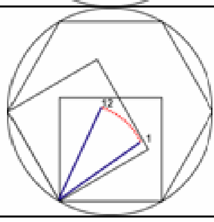


figure 10.

In the graphing table below, results are noted.

Step	Analytic graph	Rotation	Radius	Length	Step	Analytic graph	Rotation	Radius	Length
1		P1 → P2	a	$\frac{a\pi}{6}$	7		P7 → P8	c	$\frac{c\pi}{6}$
2		P2 → P3	b	$\frac{b\pi}{6}$	8		P8 → P9	d	$\frac{d\pi}{6}$
3		P3 → P4	c	$\frac{c\pi}{6}$	9		P9 → P10	a	$\frac{a\pi}{6}$
4		P4 → p5	d	$\frac{d\pi}{6}$	10		P10 → P11	b	$\frac{b\pi}{6}$
5		P5 → P6	a	$\frac{a\pi}{6}$	11		P11 → P12	c	$\frac{c\pi}{6}$
6		P6 → P7	b	$\frac{b\pi}{6}$	12		P12 → P1	d	$\frac{d\pi}{6}$

Results discovered:

(1) The rotation locus graph created when any point in a square (side=1) rolls upon unit circle is presented as in figure 11.

(2) The limits are set on the ground that the rotation angle of a square is 30°, and the length of the locus $=\frac{(a+b+c+d)\pi}{2}$.

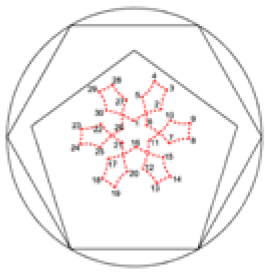
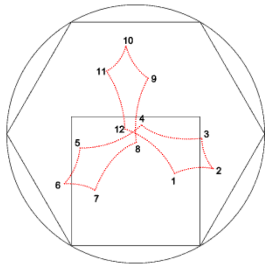


figure 11.

3. When any point inside a **regular pentagon** (side=1) rolls upon a unit circle. The following analytic graph has 20 steps, so we just show the overall figure on the right.

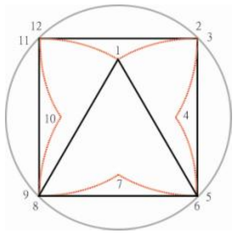


Conclusion:
Any point in a n sided polygon （n=3,4,5）rolling upon a circle will return to its original position and make a complete cycle for the locus. The cycle made by any point in the polygon can be formulated as $\frac{[n,6]}{6}$.

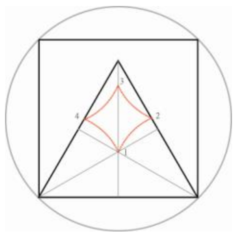
B. Probing into the locus and its length when a point in a regular m sided polygon (side=1) rolls upon another regular n sided polygon (m<n).

a. To probe into the rotation locus and its length when a point in a regular triangle (side=1 unit) rolls upon a regular n sided polygon (side=1).

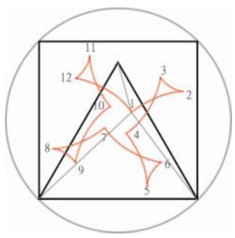
1. When a "**vertex**" of a regular triangle rolls upon a square, the length of the locus is $\frac{4}{3}\pi$ units.



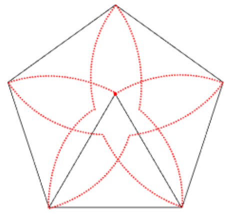
2. When a "**center**" of a regular triangle rolls upon a square, the length of the locus is $\frac{2\sqrt{3}}{9}\pi$ units.



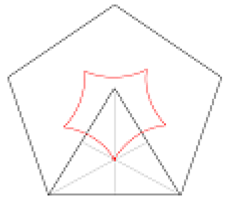
3. We assume that from **any point inside** a regular triangle to the three vertices respectively as "a", "b", and "c". When the triangle rolls upon a square, the length of the locus is $\frac{2(a+b+c)\pi}{3}$ units.



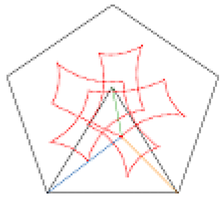
4. When a "**vertex**" of a regular triangle rolls upon a regular pentagon, the length of the locus is $\frac{8}{3}\pi$ units.



5. When a "**center**" of a regular triangle rolls upon a regular pentagon, the length of the locus is $\frac{4\sqrt{3}}{9}\pi$ units.

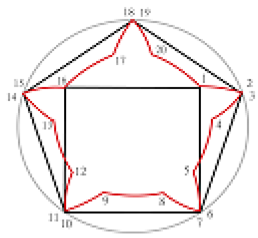


6. We assume that from **any point inside** a regular triangle to the three vertices respectively as "a", "b", and "c". When the triangle rolls upon a regular hexagon, its length of the locus is $\frac{2(a+b+c)\pi}{3}$ units.

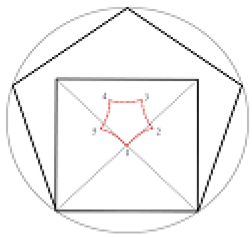


b. To probe into the rotation locus and its length when a point in a square (side=1) rolls upon a regular n sided polygon (side=1).

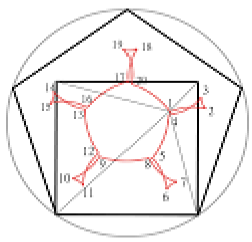
1. When a "**vertex**" of a square rolls upon a regular pentagon, the length of the locus is $\left(1+\frac{\sqrt{2}}{2}\right)\pi$ units.



2. When a "**center**" of a square rolls upon a regular pentagon, the length of the locus is $\frac{\sqrt{2}}{4}\pi$ units.



3. We assume that from **any point in a square** to the four vertices as "a", "b", "c", and "d". When the square rolls upon a regular pentagon, its length of the locus is $\frac{(a+b+c+d)\pi}{2}$ units.



Conclusion:

According to the results above, a point in a regular polygon（n sides）rolling upon another regular polygon（m sides）will make a locus figure with [m, n] number of sides (m, n as the lowest common multiple). The locus figure differs from one point (center, vertex, and any point inside) in the regular polygon to another.

We utilized the conclusion above and designed the "ruler". Please refer to section 5 Conclusions and applications for more information.

c. To probe into the parametric equation when a point in a regular n sided polygon (side=1) rolls upon a regular m sided polygon (side=1).

Take the parametric equation of the locus formed when any point in a "square" (side=1) rolls upon a "regular hexagon" (side=1).

Assume A_1 as a vertex of the regular hexagon, and B_1 as any point in the square.

$$A_1 = (a_1, b_1) = (r\cos\frac{2(1-1)\pi}{6}, r\sin\frac{2(1-1)\pi}{6})$$

$$B_1 = (c_1, d_1)$$

$$\alpha = \tan^{-1}\frac{b_1 - d_1}{a_1 - c_1} = \tan^{-1}\frac{0 - d_1}{a_1 - c_1} \ (90^0 < \alpha < 180^0)$$

$$\overline{A_1B_1} = \sqrt{(c_1 - a_1)^2 + (d_1 - b_1)^2}$$

Now take A_1 as the rotation center, $\overline{A_1B_1}$ as the rotation radius, and rotate B_1 clockwise to B_2.

Assume the coordinate of B_2 as (c_2, d_2).

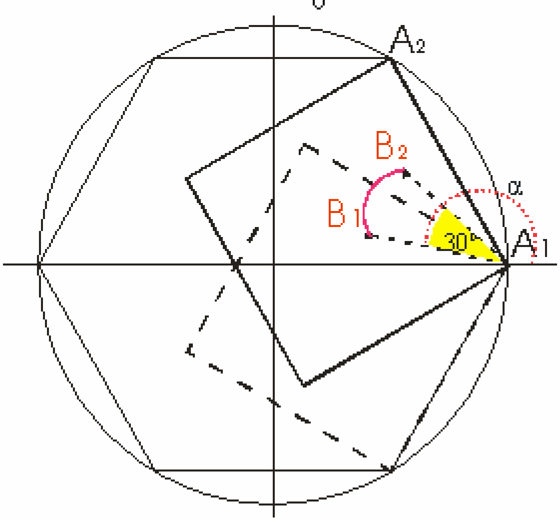
$$c_2 = \overline{A_1B_1}\cos(\alpha - 30^0) + r\cos\frac{2\times 0\times\pi}{6} = \overline{A_1B_1}\cos(\alpha - 30^0) + a_1$$

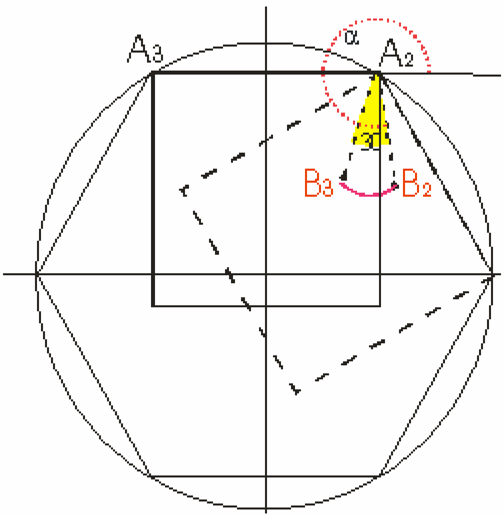
$$d_2 = \overline{A_1B_1}\sin(\alpha - 30^0) + r\sin\frac{2\times 0\times\pi}{6} = \overline{A_1B_1}\sin(\alpha - 30^0) + b_1$$

The parametric equation of $\overset{\frown}{B_1B_2}$ can be expressed as :

$$x = \overline{A_1B_1}\cos\theta + r\cos\frac{2\times 0\times\pi}{6} = \overline{A_1B_1}\cos\theta + a_1$$

$$y = \overline{A_1B_1}\sin\theta + r\sin\frac{2\times 0\times\pi}{6} = \overline{A_1B_1}\sin\theta + b_1, \quad \alpha - 30^0 \le \theta \le \alpha$$





As above, assume A_2 as the point in the hexagon after the point A_1, $B_2 = (c_2, d_2)$.

$$A_2 = (a_2, b_2) = (r\cos\frac{2(2-1)\pi}{6}, r\sin\frac{2(2-1)\pi}{6})$$

$$\alpha = \tan^{-1}\frac{b_2 - d_2}{a_2 - c_2}$$

$$\overline{A_2B_2} = \sqrt{(c_2 - a_2)^2 + (d_2 - b_2)^2}$$

Now take A_2 as the rotation center, $\overline{A_2B_2}$ as the rotation radius, and rotate B_2 clockwise to B_3.

Assume the coordinate of B_3 as (c_3, d_3).

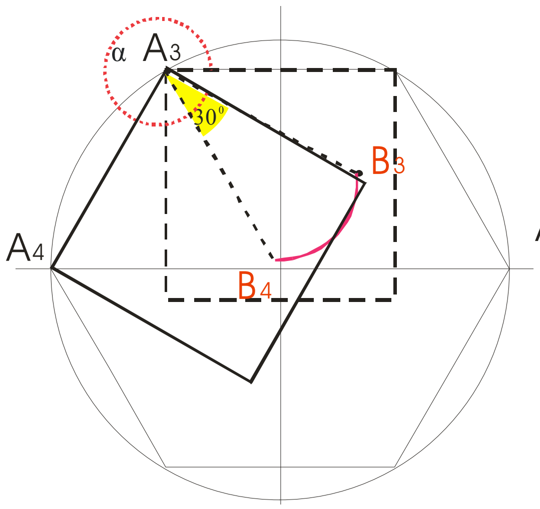
$$c_3 = \overline{A_2B_2}\cos(\alpha - 30^0) + r\cos\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\cos(\alpha - 30^0) + a_2$$

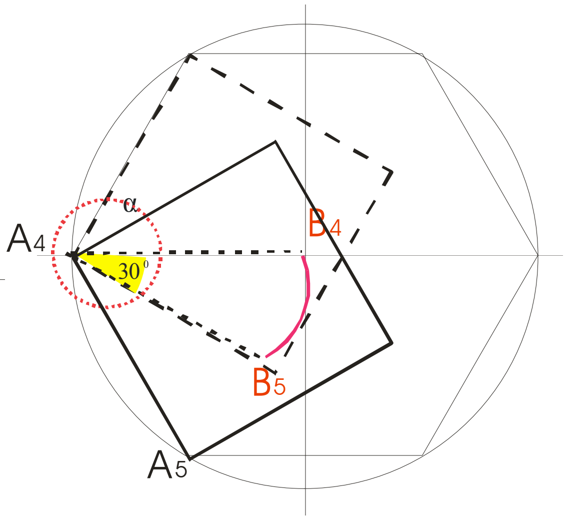
$$d_3 = \overline{A_2B_2}\sin(\alpha - 30^0) + r\sin\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\sin(\alpha - 30^0) + b_2$$

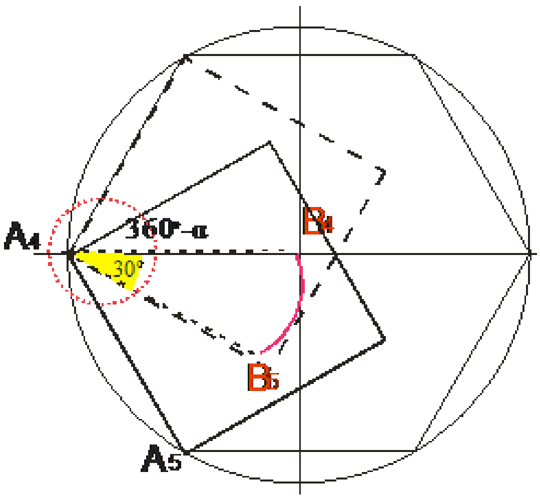
The parametric equation of (c_3, d_3) can be expressed as:

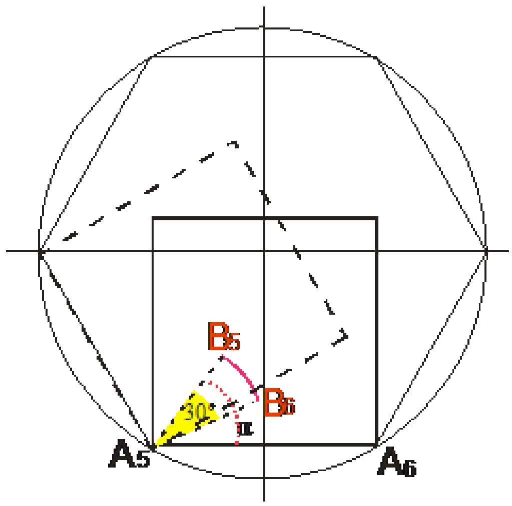
$$x = \overline{A_2B_2}\cos\theta + r\cos\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\cos\theta + a_2$$

$$y = \overline{A_2B_2}\sin\theta + r\sin\frac{2\times1\times\pi}{6} = \overline{A_2B_2}\sin\theta + b_2,\quad \alpha - 30^0 \le \theta \le \alpha$$









Assume Aj as a vertex of the regular hexagon, and Bj as any point in the square.

$r=\overline{OA_j}$ $\quad A_j=(a_j,b_j)=(r\cos\frac{2(j-1)\pi}{6},r\sin\frac{2(j-1)\pi}{6})$, $\quad j=1,2,...,12$

$B_j=(c_j,d_j)$, $\quad c_j=\overline{A_{j-1}B_{j-1}}\cos(\theta_i-30^0)+r\cos\frac{2(j-1)\pi}{6}$, $\quad i=1,2,3,4$

$d_j==\overline{A_{j-1}B_{j-1}}\sin(\theta_i-30^0)+r\sin\frac{2(j-1)\pi}{6}$, $\quad j=2,3,...,12$

$\overline{A_jB_j}=\sqrt{(c_j-a_j)^2+(d_j-b_j)^2}$, $\quad j=1,2,...,12$

Now take A_j as the center of a circle, $\overline{A_jB_j}$ as the rotation radius, B_j rolls 30 degrees clockwise to B_{j+1}.

The parametric equation of $\overline{B_jB_{j+1}}$ is

$$\begin{cases} x_j=\overline{A_jB_j}\cos\theta+r\cos\frac{2(j-1)\pi}{6} \\ y_j=\overline{A_jB_j}\sin\theta+r\sin\frac{2(j-1)\pi}{6} \end{cases} \quad \theta_i-30^0\le\theta\le\theta_i$$

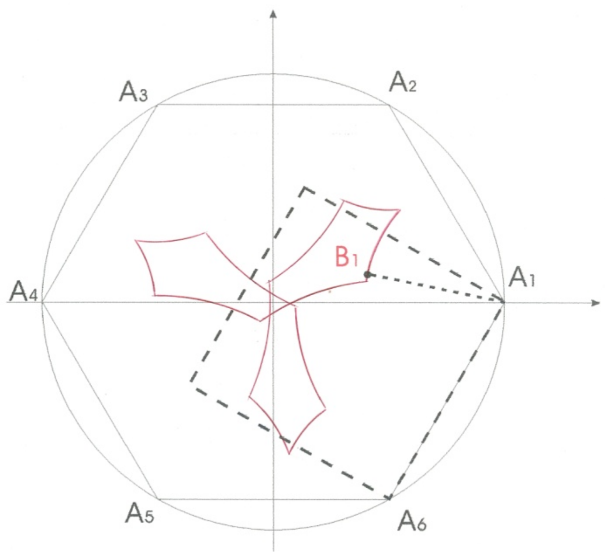
Determination of θ_i : $\beta=\tan^{-1}\left|\frac{c_j-a_j}{d_j-b_j}\right|$ is the angular relation when Aj, Bj correspond.

Case 1: If $c_j-a_j<0, d_j-b_j<0 \Rightarrow \theta_i=270^0-\beta$, and $\beta=\tan^{-1}\left|\frac{c_j-a_j}{d_j-b_j}\right|$

Case 2: If $c_j-a_j>0, d_j-b_j<0 \Rightarrow \theta_i=360^0-\beta$, and $\beta=\tan^{-1}\left|\frac{d_j-b_j}{c_j-a_j}\right|$

Case 3: If $c_j-a_j>0, d_j-b_j>0 \Rightarrow \theta_i=90^0-\beta$, and $\beta=\tan^{-1}\left|\frac{c_j-a_j}{d_j-b_j}\right|$

Case 4: If $c_j-a_j<0, d_j-b_j>0 \Rightarrow \theta_i=180^0-\beta$, and $\beta=\tan^{-1}\left|\frac{d_j-b_j}{c_j-a_j}\right|$



We extended to the parametric equation of the trace formed when any point in a regular n sided polygon rotates in a regular m sided polygon.

Aj is a vertex of a regular m sided polygon, Bj is any point inside a regular n sided polygon.

And $r = \overline{OA_j}$.

$$A_j = (a_j, b_j) = (r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m}) \text{ , } j = 1, 2, \ldots, [m, n]$$

$$B_j = (c_j, d_j) \text{ , } c_j = \overline{A_{j-1}B_{j-1}}\cos[\theta_i - (\frac{360^0}{n} - \frac{360^0}{m})] + r\cos\frac{2(j-1)\pi}{m} \text{ , } i = 1, 2, 3, 4$$

$$d_j = \overline{A_{j-1}B_{j-1}}\sin[\theta_i - (\frac{360^0}{n} - \frac{360^0}{m})] + r\sin\frac{2(j-1)\pi}{m} \text{ , } j = 2, 3, \ldots, [m, n]$$

$$\overline{A_jB_j} = \sqrt{(c_j - a_j)^2 + (d_j - b_j)^2} \text{ , } j = 1, 2, \ldots, [m, n]$$

Now take Aj as the rotation center, $\overline{A_jB_j}$ as the rotation radius, and rotate B_j $(\frac{360^0}{n} - \frac{360^0}{m})$ degrees to B_{j+1}.

The parametric equation of $\overset{\frown}{B_jB_{j+1}}$ can be expressed as:

$$\begin{cases} x_j = \overline{A_jB_j}\cos\theta + r\cos\frac{2(j-1)\pi}{m} \\ y_j = \overline{A_jB_j}\sin\theta + r\sin\frac{2(j-1)\pi}{m} \end{cases} \quad \theta_i - (\frac{360^0}{n} - \frac{360^0}{m}) \le \theta \le \theta_i$$

Determination of θ_i : $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$ is the angular relation when Aj , Bj correspond.

Case 1: If $c_j - a_j < 0, d_j - b_j < 0 \Rightarrow \theta_i = 270^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$

Case 2: If $c_j - a_j > 0, d_j - b_j < 0 \Rightarrow \theta_i = 360^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{d_j - b_j}{c_j - a_j}\right|$

Case 3: If $c_j - a_j > 0, d_j - b_j > 0 \Rightarrow \theta_i = 90^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$

Case 4: If $c_j - a_j < 0, d_j - b_j > 0 \Rightarrow \theta_i = 180^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{d_j - b_j}{c_j - a_j}\right|$

(5) Conclusions and applications:

1. Any point in a square (n=3, 4, 5) rolling upon a circle will return to its original position and make a complete cycle for the locus. The cycle made by any point in the square can be formulated as $\frac{[n,6]}{6}$.

2. A point in a regular polygon (n sides) rolling upon another regular polygon (m sides) will trace out a locus with [m, n] number of sides (m, n as the lowest common multiple). If m< n, the parametric equation can be expressed as :

Aj is a vertex of a regular m sided polygon, Bj is any point inside a regular n sided polygon.

And $r = \overline{OA_j}$.

$$A_j = (a_j, b_j) = (r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m}) \text{ , } j = 1,2,\ldots,[m,n]$$

$$B_j = (c_j, d_j) \text{ , } c_j = \overline{A_{j-1}B_{j-1}}\cos[\theta_i - (\frac{360^0}{n} - \frac{360^0}{m})] + r\cos\frac{2(j-1)\pi}{m} \text{ , } i = 1,2,3,4$$

$$d_j = \overline{A_{j-1}B_{j-1}}\sin[\theta_i - (\frac{360^0}{n} - \frac{360^0}{m})] + r\sin\frac{2(j-1)\pi}{m} \text{ , } j = 2,3,\ldots,[m,n]$$

$$\overline{A_jB_j} = \sqrt{(c_j - a_j)^2 + (d_j - b_j)^2} \text{ , } j = 1,2,\ldots,[m,n]$$

Now take Aj as the rotation center, $\overline{A_jB_j}$ as the rotation radius, and rotate B_j $(\frac{360^0}{n} - \frac{360^0}{m})$ degrees to B_{j+1}.

The parametric equation of $\overset{\frown}{B_jB_{j+1}}$ can be expressed as:

$$\begin{cases} x_j = \overline{A_jB_j}\cos\theta + r\cos\frac{2(j-1)\pi}{m} \\ y_j = \overline{A_jB_j}\sin\theta + r\sin\frac{2(j-1)\pi}{m} \end{cases} \quad \theta_i - (\frac{360^0}{n} - \frac{360^0}{m}) \le \theta \le \theta_i$$

Determination of θ_i : $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$ is the angular relation when Aj , Bj correspond.

Case 1: If $c_j - a_j < 0, d_j - b_j < 0 \Rightarrow \theta_i = 270^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$

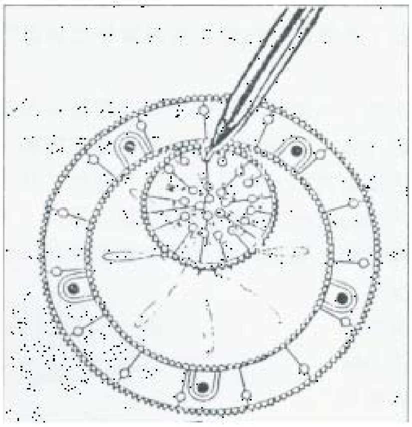
Case 2: If $c_j - a_j > 0, d_j - b_j < 0 \Rightarrow \theta_i = 360^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{d_j - b_j}{c_j - a_j}\right|$

Case 3: If $c_j - a_j > 0, d_j - b_j > 0 \Rightarrow \theta_i = 90^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{c_j - a_j}{d_j - b_j}\right|$

Case 4: If $c_j - a_j < 0, d_j - b_j > 0 \Rightarrow \theta_i = 180^0 - \beta$, and $\beta = \tan^{-1}\left|\frac{d_j - b_j}{c_j - a_j}\right|$

Applications

*The original version cycloid graphing kit:



A cycloid graphing kit has prevailed on the market before. The kit includes a smaller circular plate with a serrated edge, and a large plate with circles which have the sawteeth on the inside. The small plate can fit inside the large plate in which the sawteeth will bind perfectly when rotating. Insert a pencil into one of the holes on a small plate, a curve will be created by rolling the small plate.

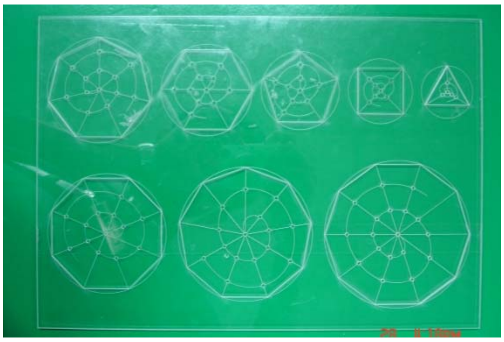
*A newly designed commercial game:

Purpose of the design：

It’s a hassle to draw regular polygonal curves using compasses and protractors even on a computer. Perhaps by combining the result of further research with the original cycloid graphing kit, we could design an “easy-to-use” curve graphing tool.

Theory of the design：

According to further research, this gadget includes various sizes of regular polygonal holes of equal side length, and different kinds of regular polygonal plates with side lengths equal to the holes. After several experiments, we have found that due to the angles of each regular polygon, we can’t indent the sides since they don’t bind well plus it slips off easily. Through many trials and errors in developing an anti-slip plate, we finally designed a regular polygonal graphing kit which doesn’t slip off that easily and is able to create a perfect curve in a short time. It deserves being mentioned because we already applied for a patent on it from the Intellectual Property Office, Ministry of Economic Affairs, R.O.C.（M305759）



（The original graphing kit failure that slips off easily）

(The final “Plate” we created)

(6) References:

Reference websites:

1. Amazing cycloids
   http://www.qhsms.com.cn/baixian/index.htm
2. Basic graphing
   http://elearning.emath.pu.edu.tw/mkuo2003/2003sdmgsp/cgsp/1_all.htm
3. Cycloid
   http://www-groups.dcs.st-and.ac.uk/~history/Curves/Cycloid.html
4. Cabri Java Applets
   http://www.math.ntnu.edu.tw/~jcchuan/demo/cabrijava/index.htm

評語

作者的表達能力很強，研究在應用上亦有可實現的地方，推廣方面可待加強。國中學生能有如此表現，展示出教師之用心及努力。