臺灣二００七年國際科學展覽會

科　　　別：數學

作 品 名 稱：能量環

得 獎 獎 項：第二名
美國正選代表：美國第 58 屆國際科技展覽會

學校 / 作者：臺北市立建國高級中學　　　鍾承道

作者簡介

大家好，我是鍾承道，目前就讀於臺北市立建國高級中學二年級。平日的嗜好是逛網站與用電腦，小時候學過英語和跆拳道，高中課程中我對物理與數學比較有興趣。我從這篇作品創作過程中學習到對數學研究工作嚴謹的態度，也學習到許多解決問題的方法。本篇作品只作了一點初淺的摸索，希望激起大家對這個問題的興趣。初次參加國際科展，很多方面仍很生疏，請大家不吝指導。

Quantum Rings

Quantum Rings are defined to be polygons with sides all of the same unit length that are connected with a fixed positive or negative angle. In the research, the number of Quantum Rings corresponding to a given number of sides and a fixed angle will be discussed.

Quantum Rings could be expressed by many sequences, based on its characteristics such as angles, vectors, and partitions. Converting Quantum Rings into mathematical sequences simplifies the problem.

A rough upper and lower bound was first obtained by analyzing the sequences via methods of permutation. In order the make the bounds more concise, the many-to-one nature of the sequences have been eliminated. That is to say, a Quantum Ring could be expressed by more than one sequence, due to symmetry caused by a ring.

Ways to create more Quantum Rings out of given ones will also be mentioned, this will provide us with useful information. Although not quantitative, the qualitative information will give us a better idea of the actual number of Quantum Rings that exist.

The operations were defined to be “flip” and “insert”. When a Quantum Ring has been flipped, a partial region of the whole mathematical sequence has been rearranged; when a quantum ring has been inserted, fragments of little sequences have been inserted into the original one.

In the future, we hope to find other applications to more general cases of some of the methods that were developed here in this research and to estimate better boundaries for the number of Quantum Rings corresponding to a given number of sides and a fixed angle.

摘要

Quantum Rings are defined to be polygons with sides all of the same unit length that are connected with a fixed positive or negative angle. In the research, the number of Quantum Rings corresponding to a given number of sides and a fixed angle will be discussed. Quantum Rings could be expressed by many sequences which would involve the theory of partitions and ways to eliminate the many to one nature of the sequences in order to evaluate the upper and lower bound. Besides estimating the upper and lower bound, a lot of the qualities of the Quantum Rings under certain circumstances will be mentioned.

「能量環」爲許多單位長度的線段以定角首尾相接，並且最後接回原點的多邊形。本研究將要探討對於給定邊長個數與相接角度的「能量環」的個數。「能量環」可以被表示成許多種數列的形式。在數列的運算中會牽涉到許多數字分割的理論與排列組合的排除重複以求得能量環個數的上下界。除了定量的求算出上下界以外，報告中也定性的歸納出許多給予特殊條件的能量環的性質。

壹、前言

一、研究動機

在試著將一個立體化學積木組成的環展開成平面上的環時，發現了有不只一種展開方式，因而想進一步了解其中展開的方法數。

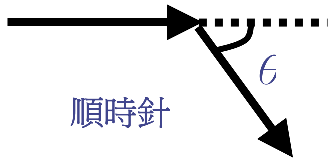
二、目的

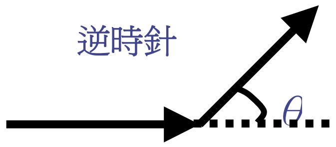
在給定的邊長與角度下，算出能夠構成的能量環數，與研究其性質

貳、研究過程

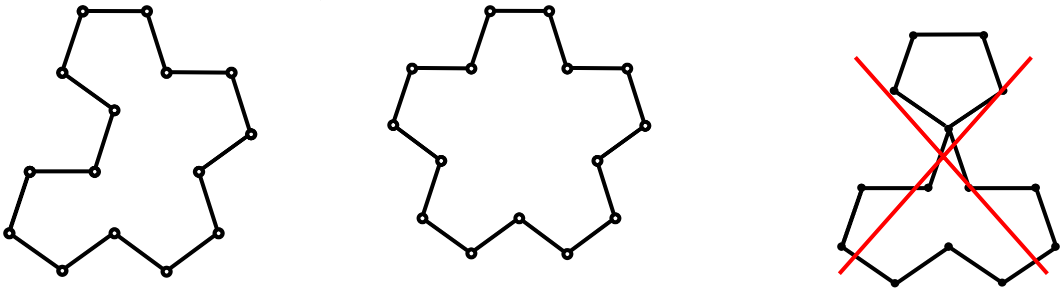
一、能量環的定義

定義：在許多長度爲單位長度的線段兩兩以固定夾角 θ (有順逆時針兩種接法)，首尾相接而經過有限段線段 r 以後，回到原出發點完成的多邊形





例如以 $\theta=72^{\circ}$，r=15 完成的兩種能量環



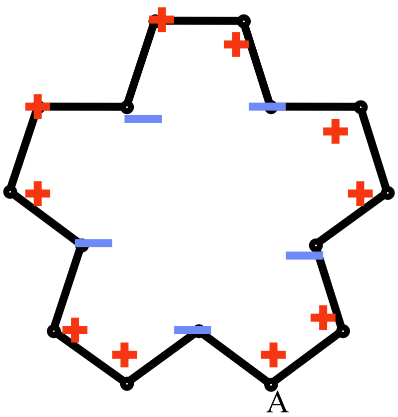
不過我們所討論的角度僅限於 $\theta=2\pi/n$ 的狀況。(n≥4,n 是質數)
而我們規定在能量環中不能出現正 n 邊形

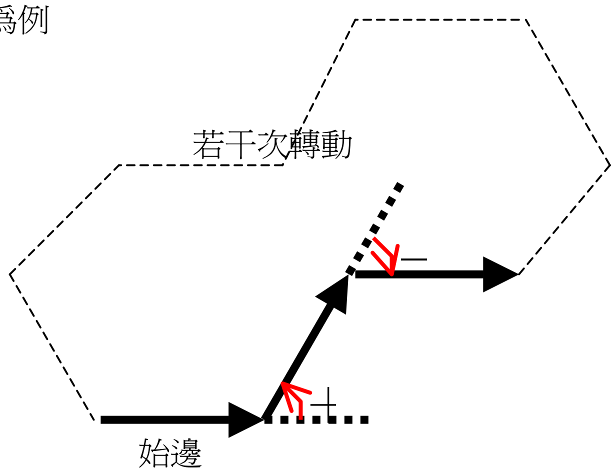
二、能量環的表式法

1、角度數列

由於一個能量環必定是由一個邊開始轉動一個順時針或逆時針角度連接下一個邊。這樣經過若干次轉動後繞一個完整的 2π 後回到原點。所以沿著環依逆時針方向繞可以將每一次的轉動是順時針或是逆時針記錄下來。這樣在不考慮起點是否相同及旋轉對稱或線對稱的狀況下；一個能量環可對應到一個環上所有角度順或逆的記錄。我們記錄逆時針爲「＋」、順時針爲「－」。

例如以 n=5, r=15 的一種能量環爲例





從 A 點以逆時針方向開始記錄的話可得到一組記錄
＋＋－＋＋－＋＋－＋＋－＋＋－
此種記錄稱爲角度數列
要求其必爲由「＋」開頭而以「－」結尾

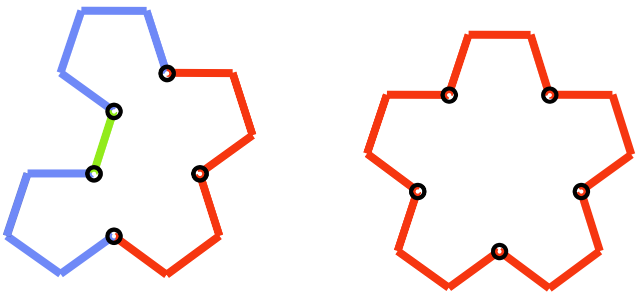
2、分割數列

但由於角度數列過爲繁瑣，所以要將其簡化。
連續 h-1 個「＋」轉後接一個「－」的一小串記錄我們將其簡化爲 h。由於我們在形成能量環時並不允許連 n 個「＋」轉或「－」轉。所以 h 必小於 n。h 可能爲 n-1,n-2,n-3……1 共 n-1 種。
設「－」轉的個數爲 y 個，每一個 h 中只有一個「－」轉，而每個「－」轉都被含入了 h。可以得知 h 的個數亦等於 y。

因爲各個 h 之和必爲邊長數 r，爲了方便起見，我們把這幾個 h 寫成類似於 r 的分割。

$r=h_1+h_2+h_3\ldots\ldots h_y$

例如以 n=5, r=15 的兩種能量環爲例



(+ + + −) (−) (+ + + −) (+ + −) (+ + −)

15=4 + 1 + 4 + 3 + 3

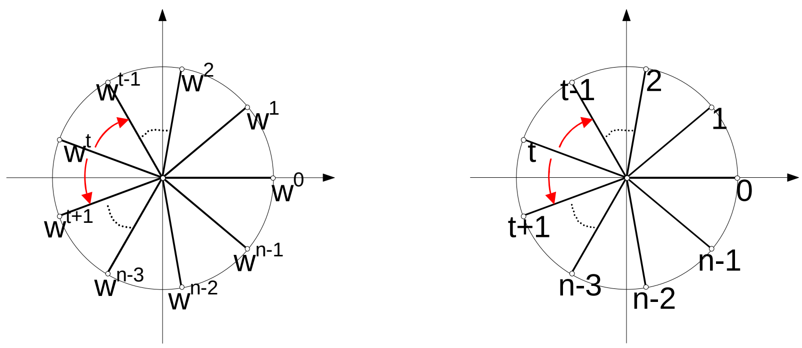
(+ + −) (+ + −) (+ + −) (+ + −) (+ + −)

15=3 + 3 + 3 + 3 + 3

這種類型的分割稱爲分割數列

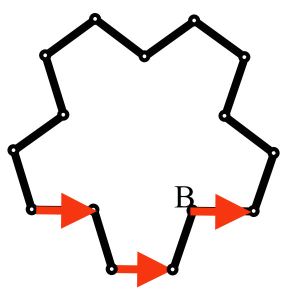
3、向量數列

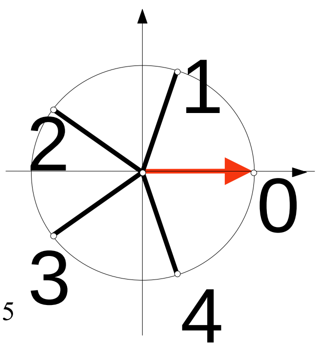
除了用角度的觀點以外，亦可把能量環想像成是許多首尾相接的向量。每一次在接點的所旋轉的每一個角度±2π/n，則相當於在複數平面上乘上 $w^{\pm 1}$(w 爲方程式 $x^n=1$ 的解)，所以在把環上其中一個邊定爲延 x 軸方向的單位向量後，可把每一個線段看成在單位圓上 n 種向量之一。



因爲每一種向量可以對應到一個解，而爲了方便起見，把每一個解 w^t 省略成一個向量編號 t。編號 t 的向量只能接編號(t+1)mod n 或編號(t-1)mod n 兩種向量。如此一來便可選定一個邊然後依逆時針方向延著環繞一圈並把所經過的向量記錄下來。這樣在不考慮起點是否相同及旋轉對稱或線對稱的狀況下；一個能量環可對應到一個環上所有邊長向量的記錄。

以 n=5 爲例





若將一個環上的邊一一以 0,1,2,3,4 標出，則可對應到一個數列，以 n=5 爲例若以 B 點右側的紅色箭頭爲起邊逆時針標出所有的邊則得到 012 123 234 340 401

此種數列稱爲向量數列

不過需要檢查第一個向量和最後一個向量能不能相接

綜合以上角度和向量觀點，將角度逆時針旋轉「＋」相當於將向量由邊號 t 轉到(t+1)mod n；將角度順時針旋轉「－」相當於將向量由邊號 t 轉到(t-1)mod n。於是可以在角度與角度記錄中加入向量邊號。形成一個完整的對應。

以 n=5 的一種能量環爲例

角度數列：＋＋＋－－＋＋＋－＋＋－＋＋－

向量數列：012321234340401

合併 0＋1＋2＋3－2－1＋2＋3＋4－3＋4＋0－4＋0＋1－

而前面所提過將角度數列簡化得方法是將連續正轉接一個負轉合併，在向量的觀點相當於是將連續遞增(包含由 n-1 到 0)的向量邊號合併在一起；而連接的下一邊號是一個遞減 1 的邊號(包含由 0 到 n-1)。這給與了之前的分割數列另一種展開後的型態。

分割數列：15=4 + 1 + 4 + 3 + 3

角度數列：=>(＋＋＋－) (－) (＋＋＋－) (＋＋－) (＋＋－)

向量數列：=>(0123) (2) (1234) (340) (401)

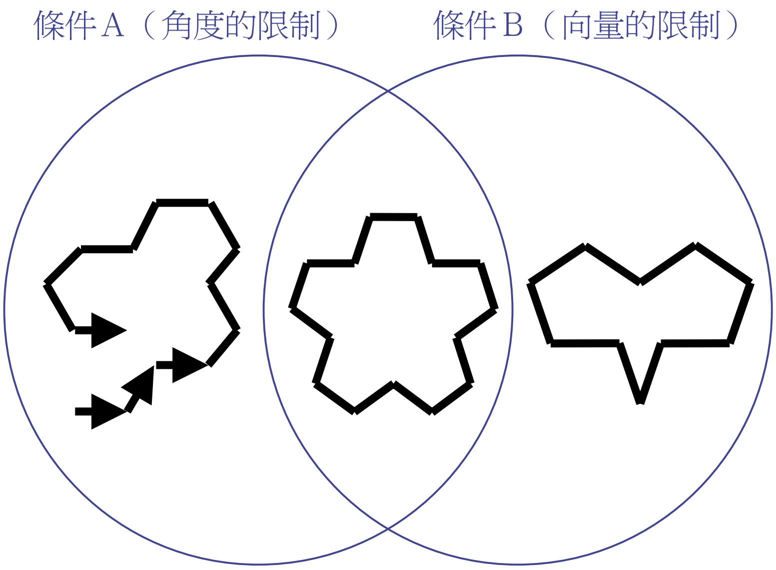
三、能量環的基本條件

形成一個能量環的充要條件：

條件 A 任意兩相鄰向量的編號的差的絕對值要等於 1 且恰旋轉一圈

且條件 B 向量總和要等於 0

條件 A 乃角度的限制，條件 B 乃向量的限制



1、角度的限制

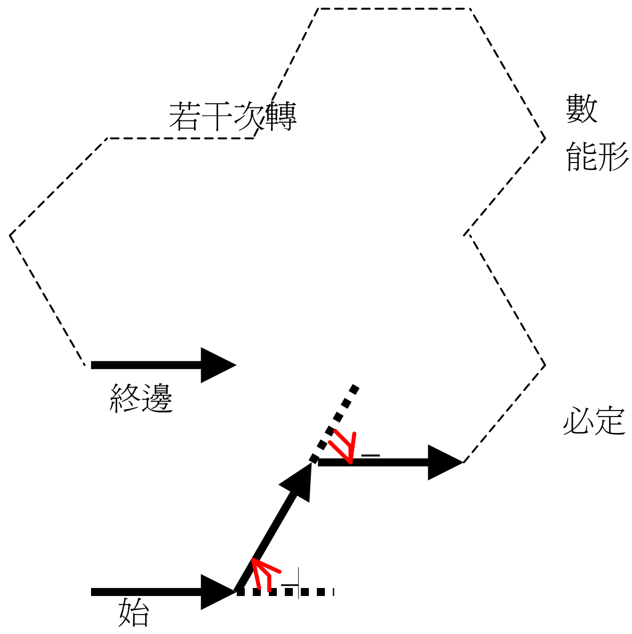
由於角度最後要會到原點所以正的角度和負的角度的和必為 2π 。在每次所旋轉的角度固定為 $\pm 2\pi/n$ ($n \geq 4$, n 是質數)的狀況下，「＋」要比「－」多 n 個，加起來才會等於 2π 。若能形成一個環，角度的個數要等於邊的個數，「＋」和「－」的個數合要等於 r。

設「＋」角的個數為 x、「－」角的個數為 y

$$\begin{cases} x-y=n \\ x+y=r \end{cases} \Rightarrow \begin{cases} x=(r+n)/2 \\ y=(r-n)/2 \end{cases}$$

在後面介紹向量的限制時會證明：當 n 為質數時，邊的個數 r 必需要是 n 的整數倍才有可能形成能量環。

即 r=nt (t 為正整數)



代入前面的式子

由於 x、y 都要是整數，又因為大於 3 的質數必定為奇數，t 必為奇數。

故令 t=2k+1 再代入

$x=n(k+1)$

$y=nk$

$r=x+y=n(2k+1)$

也就是說要形成能量環的話，其邊長的個數必定要符合這個條件

所以尋找能量環的個數，成為了尋找把 r 寫成(r－n)/2 個小於 n 的正整數分割的方法數。但是由於我們從剛開始就只有用到要形成一個環角度必需要旋轉 2π 的這個條件，並非每種邊長分割對應到的角度和邊長形成的圖形都能構一個環。

有可能最後邊長在經過旋轉之後形成了與起始邊平行的邊，但卻接不起來。所以還要將這種狀況排除。所以還需要考慮到向量的限制

2、向量的限制

如果把一個能量環的每一條邊都看成是首尾相接的向量，而因為能量環必然會回到原點，所以這些首尾相接的向量必然也會形成一個環，而以向量的觀點來說明的話就代表向量和為零。

因為每一次在接點的所旋轉的每一個角度 $2\pi/n$，則相當於在複數平面上乘上 $e^{\pm i(2\pi/n)}$，所以在把環上其中一個邊定為沿 x 軸正向方向的單位向量後，可把每一個線段看成在單位圓上 n-1 種向量之一。

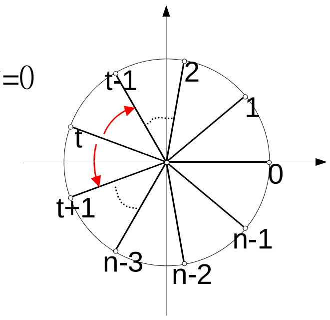
於是可以把這些向量定到複數平面上來表示。
向量邊號 0,1,2,3,,,n-2,n-1 的向量可以對應到方程式 $x^n-1=0$ 的 n 個根。
於是邊號 0,1,2,3,,,n-2,n-1 的向量對應到 $1,w,w^2,w^3,,,w^{n-2},w^{n-1}$，而分別有 $a_0,a_1,a_2,a_3,,,a_{n-2},a_{n-1}$ 個 ($a_0,a_1,a_2,a_3,,,a_{n-2},a_{n-1}$ 屬於正整數)。

假設在 n 是質數的能量環的狀況下 $a_0=a_1=a_2=a_3=,,,=a_{n-2}=a_{n-1}$ 這個式子不成立

則因爲能量環的首尾相連，向量和爲零
$a_0+a_1w+a_2w^2+a_3w^3+,,, a_{n-2}w^{n-2}+a_{n-1}w^{n-1}=0$

$x^n-1=(x-1)(1+x+x^2+x^3+,,,+x^{n-2} +x^{n-1})=0$
而因爲 w 是 x^n-1 之根而且又非實數，所以 $1+w+w^2+w^3+,,,+w^{n-2}+w^{n-1}=0$

則可把向量和的式子寫成
$a_{n-1} (1+w+w^2+w^3+,,,+w^{n-2}+w^{n-1})+ b_0+b_1w+b_2w^2+b_3w^3+,,, b_{n-2}w^{n-2}+b_{n-1}w^{n-1}=0$
$b_0,b_1,b_2,b_3,,, b_{n-2},,,b_{n-1}$ 有可是正負整數或是零而 $b_{n-1}=0$
所以 $b_0+b_1w+b_2w^2+b_3w^3+,,, b_{n-2}w^{n-2}=0$



而因爲 $a_0=a_1=a_2=a_3=,,,=a_{n-2}=,,,=a_{n-1}$ 這個式子不成立
$b_0,b_1,b_2,b_3,,, b_{n-2}$ 至少有一個不等於零
所以 w 是 $b_0+b_1x+b_2x^2+b_3x^3+,,, b_{n-2}x^{n-2}=0$ 這個實係數方程式的根
同時也是 $1+x+x^2+x^3+,,,+x^{n-2} +x^{n-1}=0$ 這個實係數方程式的根
而且 $1+x+x^2+x^3+,,,+x^{n-2} +x^{n-1}=0$ 在實數上不可分解
故必有 $b_0+b_1x+b_2x^2+b_3x^3+,,, b_{n-2}x^{n-2}$ 可以 $1+x+x^2+x^3+,,,+x^{n-2} +x^{n-1}$ 整除
然而前者的 degree 小於後者的 degree→←

故得證 $a_0=a_1=a_2=a_3=,,,=a_{n-2}=a_{n-1}$
而能量環的總邊數 $r=a_0+a_1+a_2+a_3+,,, a_{n-2}+a_{n-1}=na_0=nt$
所以當 n 爲質數時，邊的個數 r 必需要是 n 的整數倍才有可能形成能量環。
即 r=nt(t 爲正整數)

要形成能量環的話，向量和必爲零。

結論：當 n 爲質數時，邊數必爲 n 之正整數倍。要讓向量合爲零的唯一方法是讓每一種向量都要用過一樣多次。

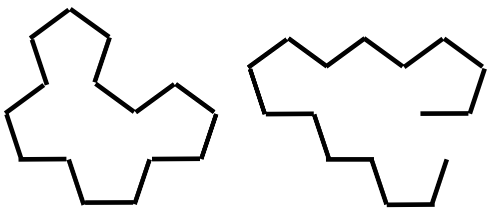
由以上的討論我們可以知道
從所有符合條件(a)中的向量數列中排除不符合條件(b)的向量數列要遠比從所有符合條件(b)中的向量數列中排除不符合條件(a)的向量數列來的容易，所以

四、能量環的刪減式求法

只要將 r 個邊分成(r-n)/2 個分割，由於是從角度出發去思考的，自然會符合條件 A。爲了排除之前提過的角度雖然旋轉了 2π 卻還是接不起來的狀況(不符條件 B)，將分割數列展開成向量形式的向量數列，並將向量和不爲零的狀況踢除。

當我們代入 r=n(2k+1)時，可以變成：
將 n(2k+1)個邊分成 nk 個分割，再將分割數列展開成向量形式的向量數列，並檢查每一種向量的個數是否會一樣多，將不合者踢除。

例如
r=15 n=5 (k=1)
之前的限制告訴我們要有(r-n)/2 個負角，也就是要有 5 個分割
15=4+1+4+3+3
(0123, 2, 1234, 340, 401)O
=4+2+4+2+3
(0123, 23, 2340, 40, 401)X
=……



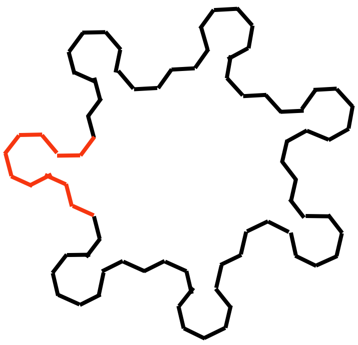
……

五、能量環個數的上下界

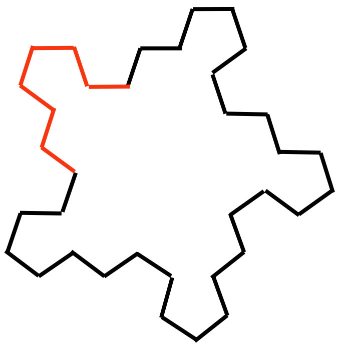
1、基態能量環

之前提過如何將一個能量環以分割的型式來表達；就是如何將 n(2k+1)個邊長(角度)分割成大於 0 且小於 n 的 nk 堆。但是其中當此 nk 堆中的 k 個分割是以同樣的方式重覆出現 n 遍的能量環；這類的能量環一共有 n 個一模一樣的部份，每個部份有 2k+1 個邊(角)、k 個分割。我們稱之爲基態能量環。

例如 322 322 322 322 322 就是一個 n=5 ,k=3 的 基態能量環。再者如圖



r=77 n=7 (k=5)時
77=(1+1+6+1+2)x7



r=35 n=5 (k=3)時
35=(1+4+2)x5

定理：每一種能夠將 2k+1 分割成 k 個大於 0 且小於 n 的方法，將其重覆 n 遍，都能夠形成一個基態能量環。

證明：由於每一個部份一共有 k+1 個「＋」角，k 個「－」角，故在每一個部份「＋」角會比「－」角多一個。共 n 個部份，一個角 2π/n，故共繞一圈，角度的條件符合形成一個能量環的條件。

接下來看向量，定第一個部份的第一個分割的第一個向量邊號為 0。由能量環的性質(請參考(七)-1 首向)，第 t 個部份的第一個分割的第一個向量邊號=第 tk+1 個分割的第一個向量邊號=$\Sigma(r=1\sim tk)h_r-2(t-1)k=(t-1)(2k+1)-2(t-1)k=t-1$

如此一來，每一部份的第一個分割的第一個向量邊號會從 0 每次遞增 1 一直循環到 n-1，使得向量邊號 0~n-1 各出現一次。由於每一部份的分割方法(即角度數列)完全相同，故每一部份的第二個向量邊號也會每次遞增 1 一直循環，亦使得向量邊號 0~n-1 各出現一次。同理每一部份不管第幾個向量邊號都會互相對應，使得向量邊號 0~n-1 各出現一次。由此可知，各種向量邊號的個數都相同=2k+1，故向量和為零。角度符合恰繞一圈的條件，向量和為零又告訴我們其首尾接的起來，故得證。

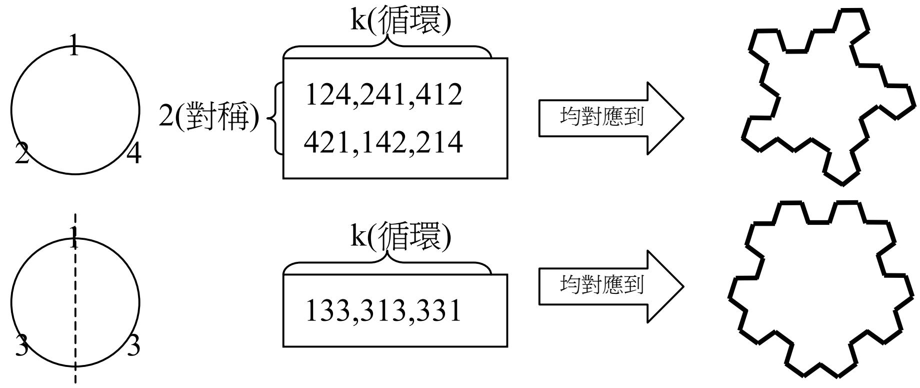
2、下界

對於任何邊數為 n(2k+1)的能量環，均有基態能量環。所以這對能量環而言是一個下界。而由之前的定理知道，基態能量環寫成分割數列的個數相當於是將 2k+1 分給 k 個不為零且小於 n 的堆的方法數，即為 N(2k+1,k,n)。(關於函數 N(a,b,n),請參考(七)-2 函數證明)由於這些分堆法全部都能一一對應到能量環，不需要檢查向量和。但是分割數列僅是在直線上做排列，而能量環則是在環上做排列，所以中間還需要再排除掉循環對稱與線對稱的重覆性。

由於(k,2k+1)=1，所以任何 k 之因數也不會整除 2k+1，所以知道所有的將 2k+1 分給 k 的方法均不會在內部出現小單位重覆的現象，所以所有能量環的分割數列均需除以 k，即可消除循環對稱。

再來考慮如何消除線對稱。有些能量環的分割數列放到環上時會有對稱軸，令這些分割數列的子集合為 b；有些則不會有對稱軸，令這些分割數列的子集合為 a。由於這兩個子集合互斥，N(2k+1,k,n)=｜b｜+｜a｜。其中｜a｜需要除以 2 以消除線對稱。
所以能量環數即為｜b｜/k +｜a｜/2k。

以 n=5, r=35, k=3 (r=n(2k+1))的兩種基態能量環爲例



而由排除重覆(請參考其他性質)的推導我們又知道｜b｜/k=B

當 k 爲奇數時

$$B=\sum_{l=1}^{n-1}N(\frac{2k+1-l}{2},\frac{k-1}{2},n)$$

當 k 爲偶數時

$$B=\sum_{l=1}^{n-1}N(\frac{2k+1-l}{2},\frac{k-2}{2},n)\left[\frac{N(l,2,n)+1}{2}\right]$$

能量環下界數=｜a｜/2k+｜b｜/k

= [N(2k+1,k,n)-｜b｜]/2k+｜b｜/k

= N(2k+1,k,n)/2k-｜b｜/2k

= N(2k+1,k,n)/2k- B/2

3、上界

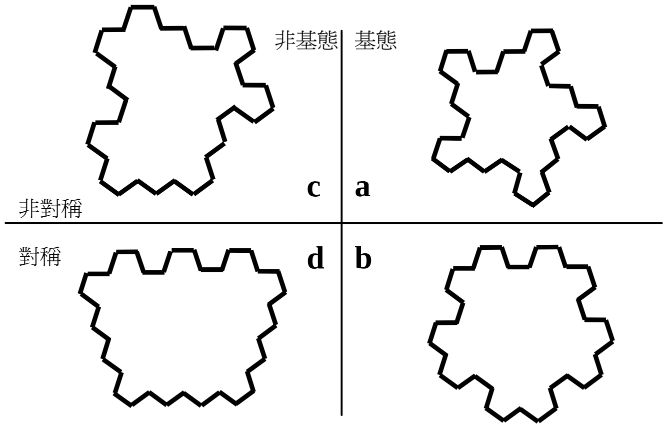
對於能量環的上界相當於是將 n(2k+1)分給 nk 個不爲零且小於 n 的堆。由於這些分堆法並非全部都能形成能量環，還需要檢查向量和。所以能量環的上界即爲分割數列的個數 N(n(2k+1),nk,n)。但是考慮到旋轉與對稱等會重覆算的問題，可將上界壓低。

但由於(n(2k+1), nk)=n，知道要消除旋轉對稱的話，須考慮到有一些分割須除以 nk，有一些分割僅須除以 k。再考慮旋轉對稱，有一些分割須除以 2，有一些分割不用除以 2。

所以將 N(n(2k+1),nk,k)分成四個互斥的子集合 a,b,c,d。

N(nk,n(2k+1),k)= ｜a｜+｜b｜+｜c｜+｜d｜

a 爲非對稱的基態能量環的分割數列，｜a｜須除以 2k；b 爲對稱的基態能量環的分割數列，｜b｜須除以 k；c 爲基態能量環的分割數列以外的非對稱分割數列，｜c｜須除以 2nk；d 爲基態能量環的分割數列以外的對稱分割數列，｜d｜須除以 nk。



其中｜b｜/k 即是對稱基態能量環的個數，即爲 B

其中｜a｜/2k 即是不對稱基態能量環的個數，即爲 N(2k+1,k,n)/2k-B/2

令將 n(2k+1)分給 nk 個不爲零且小於 n 在環上的堆，恰行成線對稱形式的環的個數有 T 個。

當 k 爲奇數時

$$T = \sum_{l=1}^{n-1} N\left(\frac{n(2k+1)-l}{2}, \frac{nk-1}{2}, n\right)$$

當 k 爲偶數時

$$T = \sum_{l=1}^{n-1} N\left(\frac{n(2k+1)-l}{2}, \frac{nk-2}{2}, n\right)\left[\frac{N(l,2,n)+1}{2}\right]$$

｜d｜/nk=T-B

能量環上界數

=｜a｜/2k+｜b｜/k+｜c｜/2nk+｜d｜/nk

=｜a｜/2k+｜b｜/k+[N(n(2k+1),nk,k)-｜b｜-｜a｜-｜d｜]/2nk+｜d｜/nk

=n｜a｜/2nk+2n｜b｜/2nk+[N(n(2k+1),nk,k)-｜b｜-｜a｜-｜d｜]/2nk+2｜d｜/2nk

=(n-1)｜a｜/2nk+(2n-1)｜b｜/2nk+N(n(2k+1),nk,k)/2nk+｜d｜/2nk

= [N(2k+1,k,n)/2k-B/2] (n-1)/n+B(2n-1)/2n+N(n(2k+1),nk,k)/2nk+(T-B)/2

= N(2k+1,k,n)(n-1)/2nk-B(n-1)/2n+B(2n-1)/2n+N(n(2k+1),nk,k)/2nk+T/2-B/2

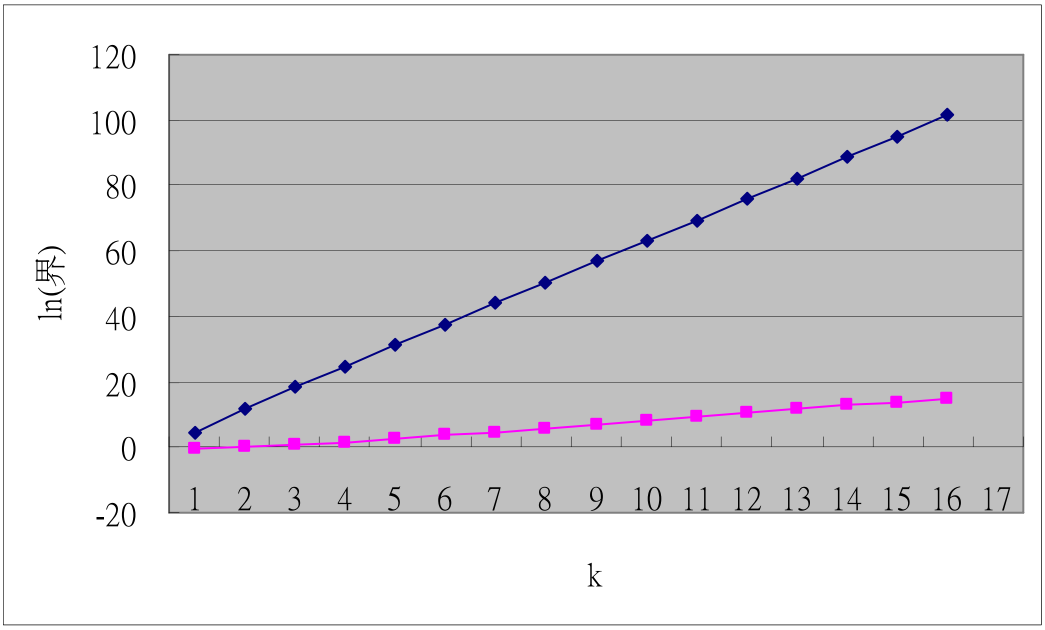
= N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2+B(-(n-1)/2n+(2n-1)/2n-n/2n)

= N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2

實際上的數值

n=5

k	上界	下界	ln(上界)	ln(下界)	上界斜率	下界斜率
1	101	0.5	4.6151	-0.6931	7.0488	0.6931
2	116304	1	11.6640	0.0000	6.5749	0.6931
3	83372562	2	18.2388	0.6931	6.4643	0.9163
4	5.35E+10	5	24.7031	1.6094	6.4247	0.9933
5	3.3E+13	13.5	31.1278	2.6027	6.4079	1.0349
6	2E+16	38	37.5357	3.6376	6.4001	1.0719
7	1.21E+19	111	43.9358	4.7095	6.3965	1.0986
8	7.23E+21	333	50.3323	5.8081	6.3950	1.1189
9	4.33E+24	1019.5	56.7272	6.9271	6.3945	1.1354
10	2.59E+27	3173	63.1217	8.0624	6.3946	1.1488
11	1.55E+30	10009	69.5164	9.2112	6.3951	1.1600
12	9.29E+32	31928	75.9115	10.3712	6.3957	1.1695
13	5.57E+35	102818	82.3072	11.5407	6.3964	1.1776
14	3.34E+38	333808	88.7036	12.7183	6.3971	1.1847
15	2E+41	1091403	95.1007	13.9030	6.3979	1.1908
16	1.2E+44	3590485	101.4986	15.0938		



此處之上下界取用的是 上界=N(n(2k+1),nk,n) 與 下界=N(2k+1,k,n)/2k

六、能量環的追加式求法

1、翻

定義〔翻〕

在一個分割數列，設爲…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…

當其中 $h_i=3, h_{i-1}\leqq n-2$, $h_{i+1}\leqq n-2$

將…h_{i-2}, h_{i-1}, 3, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}+1$, 1,$h_{i+1}+1$,h_{i+2}…

的動作稱之爲「翻 3」

或當其中 $h_i=1, h_{i-1}\geqq 2$, $h_{i+1}\geqq 2$

將…h_{i-2}, h_{i-1}, 1, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}-1$, 3, $h_{i+1}-1$,h_{i+2}…

的動作稱之爲「翻 1」

定理：

在一個分割數列，設爲…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…

其中 $h_{i-1}\leqq n-2$, $h_{i+1}\leqq n-2$

將…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}+1$, h_i-2,$h_{i+1}+1$,h_{i+2}…

只有當 $h_i=3$ 時才不會影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割的向量數列

其中 $h_{i-1}\geqq 2$, $h_{i+1}\geqq 2$

將…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}-1$, h_i+2,$h_{i+1}-1$,h_{i+2}…

只有當 $h_i=1$ 時才不會影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割的向量數列

証明：

假設存在 $h_i=t$，使得經過題目的轉換以後

不會影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割

由於我們假設此操作不影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割的向量數列，h_{i-1} 的首向量與 h_i+1 的尾向量不能受到影響，故當 h_{i-1} 變成 $h_{i-1}+1$ 影響的是 h_{i-1} 的尾向量、h_{i+1} 變成 $h_{i+1}+1$ 影響的是 h_{i+1} 的首向量。所以 $h_{i-1}+1$ 相當於是維持與 h_{i-1} 相同的向量數列只不過在尾部多增加一個向量邊號；$h_{i+1}+1$ 相當於是維持與 h_{i+1} 相同的向量數列只不過在首部多增加一個向量邊號

要檢查其向量數列,先將 h_i 的向量數列列出

設其爲 a+1,a+2,a+3….a+t

則 h_i-1 的尾向量爲 a+1,a+2；h_i+1 的首向量爲 a+t-1,a+t

故 h_{i-1}, (h_i), h_{i+1}=…a+1, a+2, (a+1,a+2,a+3….a+t), a+t-1, a+t…

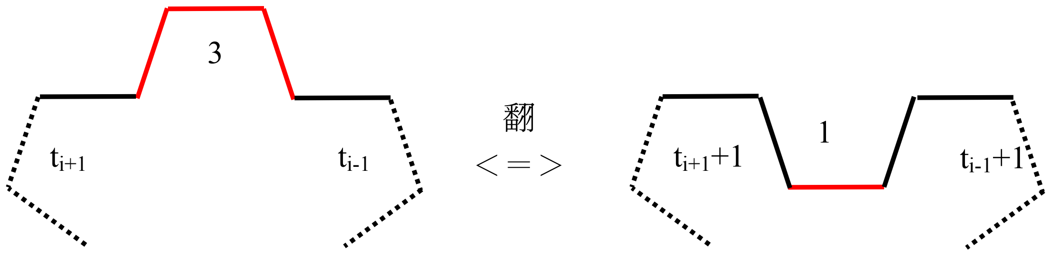
將轉換成爲 $h_{i-1}+1$, h_i-2, $h_{i+1}+1$ 以後

$h_{i-1}+1$, (h_i-2), $h_{i+1}+1$=…a+1,a+2, a+3, (a+2,a+3,a+4….a+t-1),a+t-2, a+t-1, a+t…

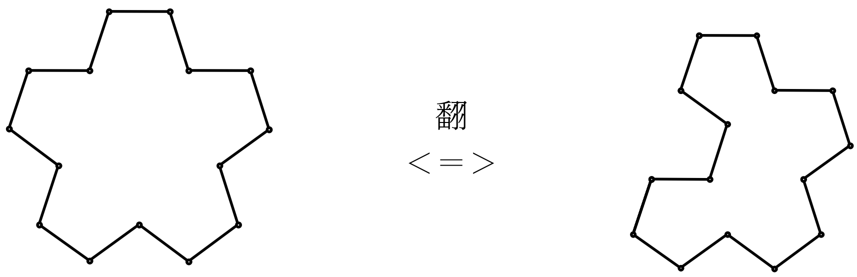
由於不影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割，故其向量和前後不變，經過比較、a+1 和 a+t 的向量和等於 a+3 和 a+t-2 的向量和、又知 $a+t \neq a+t-2$, $a+3 \neq a+1$
故知 a+t=a+3 , a+t-2=a+1 兩式都可得出 t=3 故得證

同理可証
將…$h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}$…轉換成…$h_{i-2}, h_{i-1}-1, h_i+2, h_{i+1}-1, h_{i+2}$…
只有當 $h_i=1$ 時才不會影響除了 h_{i-1}, h_i, h_{i+1} 以外的分割的向量數列

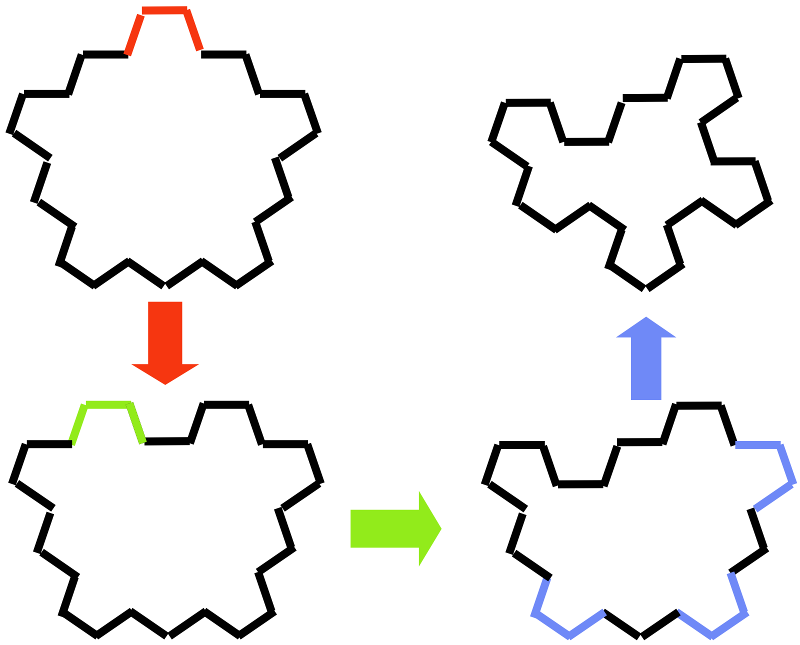
由已上定理可以知道，在一個能量環上做翻的動作不會影響到其他部份，故還是可以行成另一個能量環。



事實上有另一種更直觀的看法來說明翻，就是從圖形上去想，如上圖
「翻 3」與「翻 1」其實是互為反操作



翻的動作有利於在已得到一個能量環的狀況下，快速求出更多的能量環。如上圖



能量環的邊長數更大的時候，翻的動作可以連續做許多次，而透過此種操作所能得出的能量環數將會隨著邊長數的增加而越來越可觀。如上圖

2、插

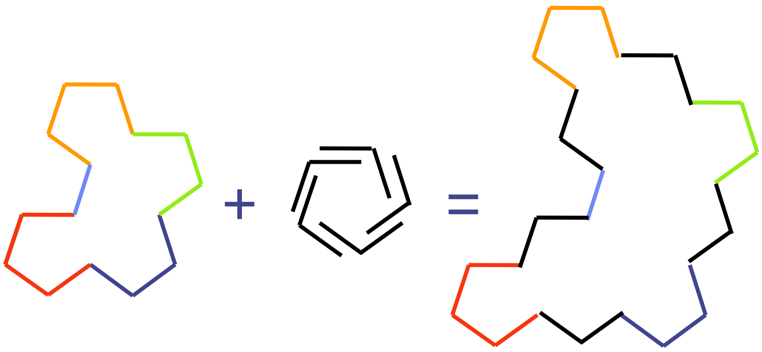
定義「插」:

製造出 n 個 2 的分割，使其向量數列分別爲(0,1),(1,2),(2,3),…,(n-2,n-1),(n-1,0)。

令向量編號爲 t 的那個分割 t,t+1 選擇一個向量編號爲 t+1 結尾的分割後插入 …t+1,t…=>…t+1,(t,t+1),t…對每一個 2 的分割都進行如此操作

可知此操作會符合角度的限制條件，即＋角要比－角多 n 個，而 2 的分割內＋－角恰各一個。又知此我們所插入的 n 個 2 的分割向量和爲零，如果我們所插入的圖形爲能量環的話，能量環的向量和是零，所以總和亦爲零。如此可知對一個能量環進行插，插完後還是一個能量環。

例如下圖即是對 n=5 r=15 的一個能量環進行此操作。



能量環的追加式求法即是進行許多次翻與插的交叉操作而得到許多能量環。翻是在同樣的邊長數下製造出不同的能量環；插是在每次增加邊長數 2n 製造出不同的能量環。

定理：任何能量環均能進行「插」

這個命題的充要條件相當於是每一種向量編號均在首向量中出現至少一遍，因爲有一個分割的首向量編號爲 t 時，分割(t,t+1)即可插在其前面。

證明：

對於一個能量環，假設有一種向量編號 t 沒有出現在首向量

則所有含有 t 的分割必長成(…t-1,t…)的樣子或是說 t 的左側必定要接 t-1，而非 t+1，否則 t 就會成爲一個分割的首向量。

故對每一個向量 t 都如此，因爲要形成能量環的話，每一種向量要一樣多，所以對每一個向量 t-1 都接在 t 的左側。所以向量數列中…,t-1,t,…必成對出現

如此每一個向量編號爲 t-2 的向量，其左測必爲 t-3，而不可能是 t-1。因爲每一至種向量邊號均一樣多個，故…t-3,t-2…亦必成對出現。

依此類推(t-5,t-4),(t-7,t-6)…(t+2,t+3)亦必成對出現。由於有 n 種首向量而 n 又是質數，除了 2 以外的質數爲奇數。故必除了向量編號 t+1 以外其餘的向量均會成對出現。

向量編號爲 t+1 之向量之左測必爲 t，而不可能是 t+2，而 t+1 有幾個，t 就要有幾個，故會成(t-1,t,t+1)一起出現。再考慮 t+1 右側，由於我們已將所有的 t 都配到左側去了，所以右側僅可能是 t+2，所以會成(t-1,t,t+1,t+2,t+3)一起出現。

依此類推到了最後(t-1,t,t+1,t+2,t+3,…t-3,t-2)會一起出現，這是一個個數爲 n 的分割，但這已經違反了我們限定每一個分割均必需小於 n 的規定。→←

(其實此圖形就是一個正 n 邊形)

故知對於任何能量環，每一種向量編號在首向量中至少會出現一次，所以每一種 2 的分割均能找到位置插入能量環。

故得證

七、能量環的其他性質

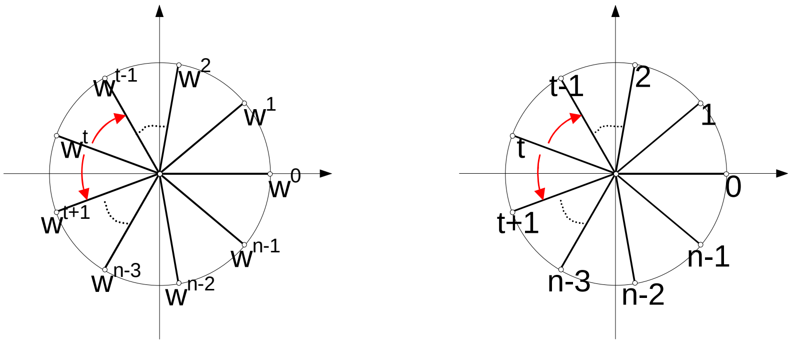
1、首向

定理：對一個邊長分割 $r=h_1+h_2+h_3\ldots\ldots h_{(r-n)/2}$ 第 t 個分割開頭的第一個向量邊號

$$a_t \equiv \left(\sum_{k=1}^{t-1} h_k - 2(t-1)\right) \bmod n$$

証明：由於向量編號增加 1 就代表向量逆時針旋轉一個角度 2π/n；減少 1 就代表向量順時針

旋轉一個角度 2 π /n。相當於在複數平面上乘上 $w^{\pm1}$，則為了方便起見，我們將第 1 個分割開頭的第一個向量邊號 a_1 定為 0。如此，每一個向量邊號 t 均可對應到 $x^n=1$ 的根 w^t。



根據分割的定義，我們將一長串的角度數列從第 t-1 個「－」之前到第 t 個「－」之後斷成一個分割，此分割內部的角度數(邊長數)為 h_t。所對第 t 個分割開頭的第一個向量邊號 a_t 而言，它的前面已有 $\sum_{k=1}^{t-1} h_k$ 個角度(邊長)。

其中有 t-1 個「－」，Σ(k=1~t-1) h_k-(t-1)個「＋」；如此一來，w^0 就要乘上

$w^{\sum_{k=1}^{t-1} h_k-(t-1)} \times w^{-(t-1)} = w^{\sum_{k=1}^{t-1} h_k-2(t-1)} = w^{(\sum_{k=1}^{t-1} h_k-2(t-1)) \bmod n}$。由於 $x^n=1$ 的方根 w^t 可對應到向量編號 t，

$a_t \equiv (\sum_{k=1}^{t-1} h_k - 2(t-1)) \bmod n$。故得證

得到了這個定理之後，若要一一把每一個分割的首向量求出來，會需要重複計算許多數字，於是我們可以把求其每一個分割的首向量的方法簡化成一個運算法則。

以 n=5, r=25, 分割數列為 3412333141 的能量環為例。

先寫出其分割數列

3 4 1 2 3 3 3 1 4 1

再將第一個分割下面寫一個 0，並加上其上面的數字，寫到其右方。重覆此步驟一直寫到最後一項

3	4	1	2	3	3	3	1	4	1
0	3	7	8	10	13	16	19	20	24

再將第一欄的下面寫一個 0，以等差為 2 的等差數列一直寫到最後一項

3	4	1	2	3	3	3	1	4	1
0	3	7	8	10	13	16	19	20	24
0	2	4	6	8	10	12	14	16	18

然後將第二列與第三列相減

	3	4	1	2	3	3	3	1	4	1
	0	3	7	8	10	13	16	19	20	24
−)	0	2	4	6	8	10	12	14	16	18
	0	1	3	2	2	3	4	5	4	6

將最後相減得出的數列取除以 5 的餘數便是每一個分割的首向量

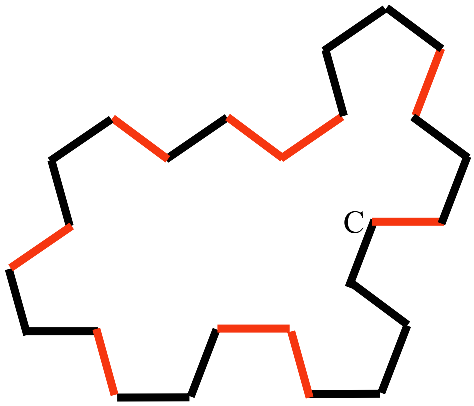
	3	4	1	2	3	3	3	1	4	1
	0	3	7	8	10	13	16	19	20	24
−)	0	2	4	6	8	10	12	14	16	18
	0	1	3	2	2	3	4	0	4	1

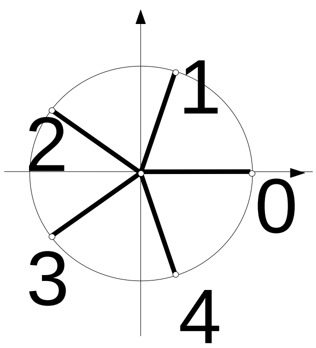
我們將其分割數列完整的展開成向量數列與結果比較

3412333141(從點 C 開始逆時鐘)

=>(012)(1234)(3)(23)(234)(340)(401)(0)(4012)(1)

發現兩者相同





預備定理：所有的基態能量環，所有分割的首向量中，每種向量編號會出現一樣多遍。

証明：

之前一提過一個基態能量環的定義即是在形成分割數列時，有 k 個相連的分割重覆出現 n 遍的能量環。由於這每個部份的分割數列相同，所以角度數列也相同。而從之前的證明中知道第 r 個部份的第一個分割的首向量編號 a_r，會以每次增加一的方式遞增。所以每個第 r+1 個部份的向量編號，等於在第 r 個部份對應位置上的向量編號增加 1。

在第 1 個部份的第 t 個向量編號假設為 i

則第 2 個部份的第 t 個向量編號為 i+1

…

則第 u 個部份的第 t 個向量編號為 i+u-1

…

則第 n 個部份的第 t 個向量編號為 i+n-1

於是可以知道在第一個部分上面的每一個向量編號都會在整個能量環中出現 n 遍，每一種向量編號恰一遍。

而之前也介紹過每一個分割的首向量的求法。於是我們知道在一個基態能量環在第一個部分的首向量會有 k 個，每一個部份的首向量一共有 nk 個。這 nk 個向量編號中每種向量邊號會出現一樣多遍。但是，這個性質可以被推廣

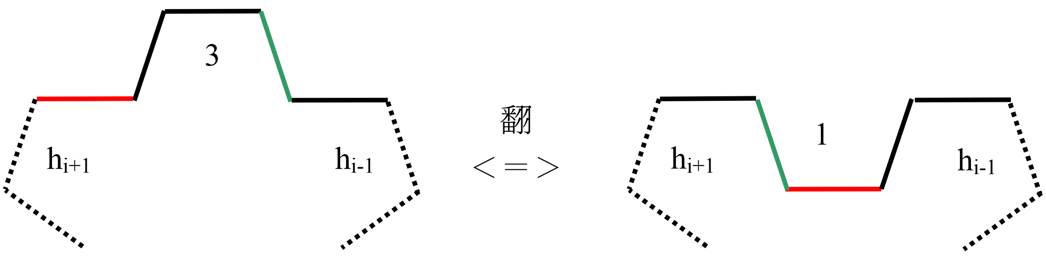
定理：所有的基態能量環經過翻的操作有限次以後，所有分割的首向量中，每種向量編號會出現一樣多遍。

証明：
由於經過翻已後，原先基態能量環分割的對稱性已被破壞，雖然有可能最後翻成了另一種基態能量環，不過有更大的可能已經不再是基態能量環。所以要從「所有的基態能量環，所有分割的首向量中，每種向量編號會出現一樣多遍。」這個定理出發而不是一味的尋求對稱性。

對於「翻 3」：
在一個分割數列，設為…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…
當其中 $h_i=3, h_{i-1} \leqq n-2, h_{i+1} \leqq n-2$
將…h_{i-2}, h_{i-1}, 3, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}+1$, 1, $h_{i+1}+1$, h_{i+2}…

我們一樣將 h_i 的向量數列列出
設其為 a+1,a+2,a+3
則 h_i-1 的尾向量為 a+2；h_i+1 的首向量為 a+2，h_i 的首向量為 a+1
故 h_{i-1}, (h_i), h_{i+1}=…a+1, a+2, (a+1,a+2,a+3), a+2, a+3…
翻完後 $h_{i-1}+1$, (h_i-2), $h_{i+1}+1$=…a+1, a+2, a+3, (a+2), a+1, a+2, a+3…
由於我們可以知道，這個操作不影響 h_i-1 的首向量
而 h_i+1 的首向量為 a+1，h_i 的首向量為 a+2

由於在基態能量環時每一種向量的種類在首向量中會出現一樣多遍，而翻完 h_i 已後，只是將 h_i 與 h_{i+1} 的首向量位置交換，故不影響向量種類的分部，各種向量還是出現一樣多遍。



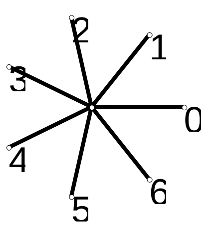
對於「翻 1」:
在一個分割數列，設爲…$h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}$…
當其中 $h_i=1,h_{i-1}\geqq 2, h_{i+1}\geqq 2$
將…$h_{i-2}, h_{i-1}, 1, h_{i+1}, h_{i+2}$…轉換成…$h_{i-2}, h_{i-1}-1, 3, h_{i+1}-1,h_{i+2}$…

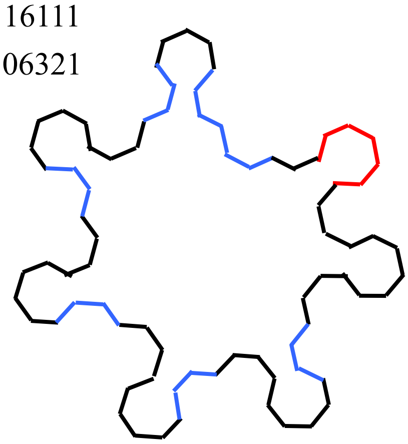
我們一樣將 h_i 的向量數列列出
設其爲 a+2
則 h_i-1 的尾向量爲 a+3；h_i+1 的首向量爲 a+1，h_i 的首向量爲 a+2
故 $h_{i-1}, (h_i), h_{i+1}$=…a+2, a+3, (a+2), a+1, a+2…
翻完後 $h_{i-1}+1, (h_i-2), h_{i+1}+1$=…a+1, a+2, (a+1, a+2, a+3,) a+2, a+3…
由於我們可以知道，這個操作不影響 h_i-1 的首向量
而 h_i+1 的首向量爲 a+2，h_i 的首向量爲 a+1

由於在基態能量環時每一種向量的種類在首向量中會出現一樣多遍，而翻完 h_i 已後，只是將 h_i 與 h_{i+1} 的首向量位置交換，故不影響向量種類的分佈，各種向量還是出現一樣多遍。
故得証
注意到雖然質數基態能量環與基態能量環翻出的能量環均有首向量中每一種向量出現一樣多遍，但這並非代表只有基態能量環與基態能量環翻出的能量環才具有這種性質

以 n=7,r=77, 分割數列爲 61212 16121 16121 16121 16121 16121 16111(從紅色的那個分割開始逆時針數)的能量環爲例。

很明顯其非基態能量環，而其所能翻的分割(被標成了藍色)也僅能在原處進行翻，翻完後無法製造出更多可以翻的分割。所以不可能是基態能量環所翻成的能量環(類基態能量環)。其分割數列中每一個分割對應到的首向量分別爲
分割 61212 16121 16121 16121 16121 16121 16111
首向 04332 21544 32655 43066 54100 65211 06321
可發現首向中每一種向量出現一樣多遍





所以在首向量中，若每一種向量沒有出現一樣多遍，則一定不是基態能量環或是類基態能量環。但如果首向量中，每一種向量都出現同樣多遍，則未必是基態能量環或是類基態能量環。

2、函數證明

定義函數 N(a,b,n)=把 a 物分給 b 個人每一堆大於等於 1，小於等於 n-1 的方法數

定義函數 K(a,b,n)=把 a 物分給 b 個人每一堆大於等於 0，小於等於 n-1 的方法數

定義函數 K(a,b)=把 a 物分給 b 個人每一堆大於等於 0 的方法數

定義當 n≦$\left\lceil \frac{a}{b} \right\rceil$，K (a,b,n)=0

定義當 n≦$\left\lceil \frac{a}{b} \right\rceil$，N (a,b,n)=0

定義當 0≧$\left\lfloor \frac{a}{b} \right\rfloor$，N (a,b,n)=0

定義當 a,b,n 不是正整數時，N (a,b,n)=0

易知 N(a,b,n)=K(a-b,b,n-1) <=> K(a,b,n)=N(a+b,b,n+1)

定理： $K(a,b)=\binom{a+b-1}{a}$

證明：

K(a,b)相當於方程 $x_1+x_2+x_3+x_4+x_5+\ldots+x_b=a$ 大於等於零的整數解的個數。

於是可把等號右側想像成 a 個 1 與 b-1 個加號的排列。

b-1 個加號相當於將 a 分割成 b 堆，與左側的 $x_1+x_2+x_3+x_4+x_5+\ldots+x_b$ 一對一對應

於是每一組不為負的整數解與等號右測的一種排列一對一對應。

K(a,b)

=方程 $x_1+x_2+x_3+x_4+x_5+\ldots+x_b=a$ 大於等於零的整數解的個數

=等號右側的排列方法數

$$=\frac{(a+b-1)!}{a!(b-1)!}=\binom{a+b-1}{a}$$

定理： $K(a,b,n)=\sum_{t=0}^{b}(-1)^t\binom{b}{t}K(a-nt,b)=\sum_{k=0}^{b}(-1)^n\binom{b}{t}\binom{a-nt+b-1}{a-nt}$

證明：

*K(a,b,n)*相當於方程 $x_1+x_2+x_3+x_4+x_5+\ldots+x_b=a$ 大於等於零，小於等於 n-1 的整數解的個數。

則 *K(a,b,n)*相當於從大於等於零整數解中，扣除所有有大於等於 n 的解。

要求其中有 1 堆大於等於 n 的解的個數

可把 n 先分給這 1 堆，再將剩下的 a-n 分給 b 堆

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+x_b=a-n$

所以當把 n 分給這 1 堆時的解的有 $K(a-n,b)$
這 1 堆有 $\binom{b}{1}$ 種組合
所以有 1 堆大於等於 n 的解再還沒考慮到重複算的狀況有 $\binom{b}{1}K(a-n,b)$
要求其中有 2 堆大於等於 n 的解的個數
可把 n 先分給這 2 堆，再將剩下的 a-2n 分給 b 堆

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+(x_{t_2}-n)+\ldots+x_b=a-2n$

所以當把 n 分給這 2 堆時的解的有 $K(a-2n,b)$
這 2 堆有 $\binom{b}{2}$ 種可能
所以有 2 堆大於等於 n 的解再還沒考慮到重複算的狀況有 $\binom{b}{2}K(a-2n,b)$

同理
要求其中有 i 堆大於等於 n 的解的個數
可把 n 先分給這 i 堆，再將剩下的 a-in 分給 b 堆

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+(x_{t_2}-n)+\ldots+(x_{t_i}-n)+\ldots+x_b=a-in$

所以當把 n 分給這 i 堆時的解的有 $K(a-in,b)$
這 i 堆有 $\binom{b}{i}$ 種組合
所以有 i 堆大於等於 n 的解再還沒考慮到重複算的狀況有 $\binom{b}{i}K(a-in,b)$

由容斥原理
至少有一堆大於等於 n 的解
=(有 1 堆大於大於等於 n 的解-有 2 堆大於大於等於 n 的解+有 3 堆大於大於等於 n 的解…$(-1)^{i+1}$
有 i 堆大於大於等於 n 的解…有 b 堆大於大於等於 n 的解)

$$=\binom{b}{1}K(a-n,b)-\binom{b}{2}K(a-2n,b)+\ldots+(-1)^{i-1}\binom{b}{i}K(a-in,b)+\ldots+(-1)^{b-1}\binom{b}{b}K(a-bn,b)$$

K(a,b,n)
=大於等於零，小於等於 n-1 的整數解的個數
=大於等於零整數解-至少有一堆大於等於 n 的解

$$=\binom{b}{0}K(a,b)-[\binom{b}{1}K(a-n,b)-\binom{b}{2}K(a-2n,b)+\ldots+(-1)^{i-1}\binom{b}{i}K(a-in,b)+\ldots+(-1)^{b-1}\binom{b}{b}K(a-bn,b)]$$

$$=\sum_{t=0}^{b}(-1)^t\binom{b}{t}K(a-nt,b)=\sum_{t=0}^{b}(-1)^t\binom{b}{t}\binom{a-nt+b-1}{a-nt}$$

故得證
易知 N(a,b,n)=K(a-b,b,n-1) <=> K(a,b,n)=N(a+b,b,n+1)

所以 $N(a,b,n)=\sum_{t=0}^{b}(-1)^t\binom{b}{t}\binom{a-(n-1)t-1}{a-b-(n-1)t}$

3、排除重覆

假設將 a 同物分給 b 個在環上大於零且小於 n 的堆，恰行成線對稱形式的環的個數有 G 個。

當 b 是奇數時

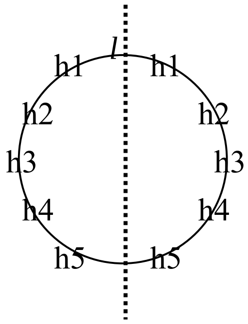
由於當 b 個堆會行成對稱的分割方式，唯一的可能就是恰有一堆在通過環中心的對稱軸上。設其爲 l

則剩下的每 b-1 堆，一共要分成 a-l 個大於零且小於 n 的堆。

由於對稱性，相當於是(a-l)/2 物，一共要分成(b-1)/2 個大於零且小於 n 的堆

=N((b-1)/2,(a-l)/2,n)

$$G=\sum_{l=1}^{n-1}N(\frac{a-l}{2},\frac{b-1}{2},n)$$



當 b 是偶數時

由於當 b 個堆會行成對稱的分割方式，唯二的可能就是恰有兩堆在對稱軸上，或對稱軸不通過任何堆。設其個數分別爲 G_a,G_b，則 $G=G_a+G_b$

當恰有兩堆在對稱軸上

設其爲 x,y 令 x+y=l

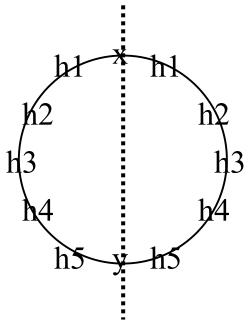
則剩下的每 b-2 堆，一共要分成 a-l 個大於零且小於 n 的堆。

由於對稱性，相當於是(a-l)/2 物，一共要分成 (b-2)/2 個大於零且小於 n 的堆

=N((b-1)/2,(a-l)/2,n)

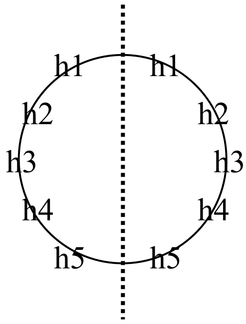
對於每一個 l,將 l 一共要分成 2 個大於零且小於 n 的堆，方法數即爲 N(l,2,n)，但爲了避免重複計算，我們必須規定 x≧y，所以方法數變爲[(N(l,2,n)+1)/2]

$$G_a=\sum_{l=1}^{n-1}N(\frac{a-l}{2},\frac{b-2}{2},n)\left[\frac{N(l,2,n)+1}{2}\right]$$



當對稱軸恰不通過任何一堆時
其有 b 堆，一共要分成 a 個大於零且小於 n 的堆。
由於對稱性，相當於是 a/2 物，一共要分成 b/2 個大於零且小於 n 的堆。

$$G_b = N(\frac{a}{2},\frac{b}{2},n)$$



$$G = G_a + G_b = \sum_{l=1}^{n-1} N(\frac{a-l}{2},\frac{b-2}{2},n)\left[\frac{N(l,2,n)+1}{2}\right] + N(\frac{a}{2},\frac{b}{2},n)$$

注意到由定義，當 a 或 b 爲奇數時 G_b=0，故 G=G_a

參、研究結果

形成能量環的條件

對於以定角 θ =2 π / n 的狀況。
(n≥4,n 是質數)

單位邊長的個數需爲 r=n(2k+1)
需要形成 nk 個分割(k 爲自然數)

若把各邊長看成是 n 種向量的話，各種向量的個數需要一樣多
才能使能量環接回原點

能量環的上下界

下界

能量環下界數= N(2k+1,k,n)/2k- B/2

當 k 爲奇數時

$$B = \sum_{l=1}^{n-1} N(\frac{2k+1-l}{2},\frac{k-1}{2},n)$$

當 k 爲偶數時

$$B = \sum_{l=1}^{n-1} N(\frac{2k+1-l}{2},\frac{k-2}{2},n)\left[\frac{N(l,2,n)+1}{2}\right]$$

上界

能量環上界數

= N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2

當 k 爲奇數時

$$T=\sum_{l=1}^{n-1}N(\frac{n(2k+1)-l}{2},\frac{nk-1}{2},n)$$

當 k 爲偶數時

$$T=\sum_{l=1}^{n-1}N(\frac{n(2k+1)-l}{2},\frac{nk-2}{2},n)\left\lfloor\frac{N(l,2,n)+1}{2}\right\rfloor$$

能量環的刪減式求法

只要將 r 個邊分成(r-n)/2 個分割，由於是從角度出發去思考的，自然會符合條件 A。爲了排除之前提過的角度雖然旋轉了 2π 卻還是接不起來的狀況(不符條件 B)，將分割數列展開成向量形式的向量數列，並將向量和不爲零的狀況踢除。

當我們代入 r=n(2k+1)時，可以變成：

將 n(2k+1)個邊分成 nk 個分割，再將分割數列展開成向量形式的向量數列，並檢查每一種向量的個數是否會一樣多，將不合者踢除。

能量環的追加式求法

翻

定義〔翻〕

在一個分割數列，設爲…h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}…

當其中 $h_i=3$,$h_{i-1}\leqq n-2$, $h_{i+1}\leqq n-2$

將…h_{i-2}, h_{i-1}, 3, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}+1$, 1,$h_{i+1}+1$,h_{i+2}…

的動作稱之爲「翻 3」

或當其中 $h_i=1$,$h_{i-1}\geqq 2$, $h_{i+1}\geqq 2$

將…h_{i-2}, h_{i-1}, 1, h_{i+1}, h_{i+2}…轉換成…h_{i-2}, $h_{i-1}-1$, 3, $h_{i+1}-1$,h_{i+2}…

的動作稱之爲「翻 1」

定理：「翻 3」和「翻 1」均不會影響除了 hi-1, hi, hi+1 以外的分割的向量數列

插

定義「插」:

製造出 n 個 2 的分割，使其向量數列分別爲(0,1),(1,2),(2,3),…,(n-2,n-1),(n-1,0)。

令向量編號爲 t 的那個分割 t,t+1 選擇一個向量編號爲 t+1 結尾的分割後插入

…t+1,t…=>…t+1,(t,t+1),t…對每一個 2 的分割都進行如此操作

定理：任何能量環均能進行「插」

其它性質

首向

對一個邊長分割 r=h1+h2+h3……h(r－n)/2 第 t 個分割開頭的第一個向量邊號

$$a_t \equiv (\sum_{k=1}^{t-1} h_k - 2(t-1)) \bmod n$$

定理：所有的基態能量環經過翻的操作有限次以後，所有分割的首向量中，每種向量編號會出現一樣多遍。

函數證明

定義函數 N(a,b,n)=把 a 物分給 b 個人每一堆大於等於 1，小於等於 n-1 的方法數

$$N(a,b,n) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \binom{a-(n-1)t-1}{a-b-(n-1)t}$$

排除重覆

假設將 a 同物分給 b 個在環上大於零且小於 n 的堆，恰行成線對稱形式的環的個數有 G 個。

當 b 爲奇數時

$$G = \sum_{l=1}^{n-1} N(\frac{a-l}{2}, \frac{b-1}{2}, n)$$

當 b 爲偶數時

$$G = \sum_{l=1}^{n-1} N(\frac{a-l}{2}, \frac{b-2}{2}, n) \left[\frac{N(l,2,n)+1}{2} \right] + N(\frac{a}{2}, \frac{b}{2}, n)$$

肆、結論與應用

本文首先研究給定角度與邊長數的能量環其各種數列表示法，並且說明能量環的形成條件，進一步求出能量環個數的上界與下界函數。但上下界之間的差距還是存在的，所以試著以兩種追加的方式從一個能量環推展出更多能量環，來補足這個縫，並且定性的看出一些能量環的性質。從上述的研究中，爲了解決許多算數上的問題，也連帶的發展出了一些可以運用於數列分割、排除重覆的定理，不只局限於此，還可以更加的拓展。

伍、參考文獻

[1] George E. Andrews,1975, A theorem on reciprocal polynomials with applications to permutations and compositions, American Mathethmatical Monthly, 1975 Oct, p830~ 833

[2] Gian-Carlo Rota, The Encyclopedia of Mathematics and Its Applications Series, first issue, New York, Addison-Wesley Publishing Company, chapter 1,1976

I. Abstract

EASE polygons are defined to be polygons with sides all of the same unit length that are connected with a fixed positive or negative angle. In the research, the number of EASE polygons corresponding to a given number of sides and a fixed angle will be discussed. EASE polygons could be expressed by many sequences which would involve the theory of partitions and ways to eliminate the many to one nature of the sequences in order to evaluate the upper and lower bound. Besides estimating the upper and lower bound, a lot of the qualities of the EASE polygons under certain circumstances will be mentioned.

II. Foreword

A. Motivation

When I tried to flatten a three dimensional ring structure constructed of blocks of chemical models, I found that there were more than one results. So I wanted to know more about the total number of ways to flatten the structure.

B. Goal

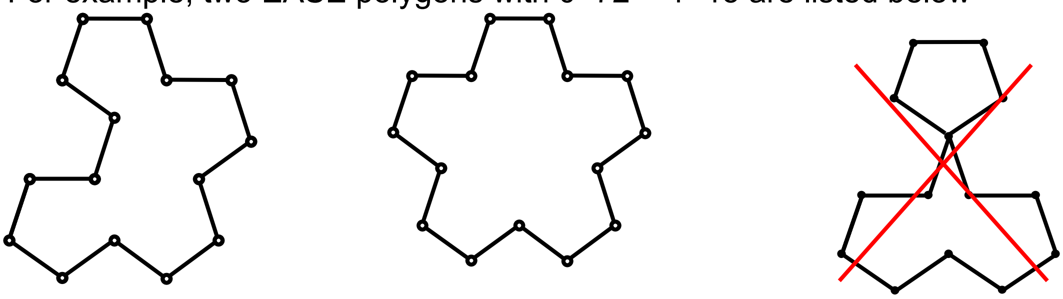
To compute the number EASE polygons corresponding to a given angle and number of sides.

III. Process of Research

A. The definition of EASE polygons

Definition: EASE polygons are defined to be polygons constructed of r segments all of the same unit length that are connected with a fixed positive or negative angle.

For example, two EASE polygons with θ=72°，r=15 are listed below



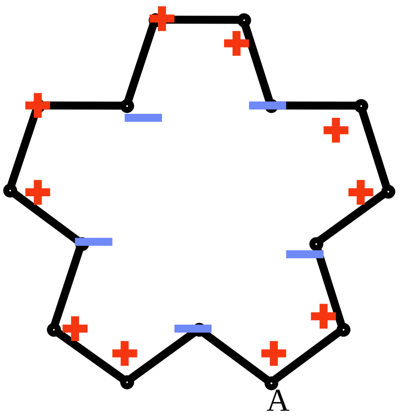
However, we only discuss angles with the confinement of θ=2π/n (n≥4,n is a prime number) in this research and regular polygons are not allowed to appear in EASE polygons.

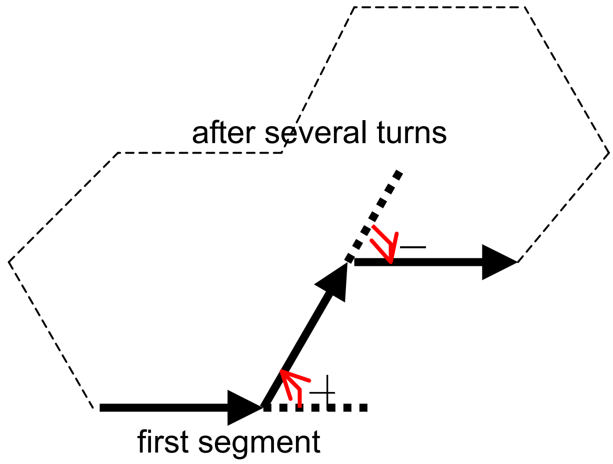
B. Ways to describe EASE polygons

1. Angle sequences

Starting with one point on an EASE polygon, we could traverse counter clockwise along the segments. Whenever a point is met, the direction in which we travel turns an angle +2π/n or -2π/n.

After traversing several segments, we ultimately return to the point we started from turning a total angle of 2π.After giving the symbol “+” for counter clockwise and “-” clockwise, we could then record every angle we have turned during the process.





++− ++− ++− ++− ++−

For instance, starting from point A and traveling counter clockwise, the angle sequence of an EASE polygon with n=5 and r=15 is listed above. Angle sequences are supposed to start with “+” and end with “-”.

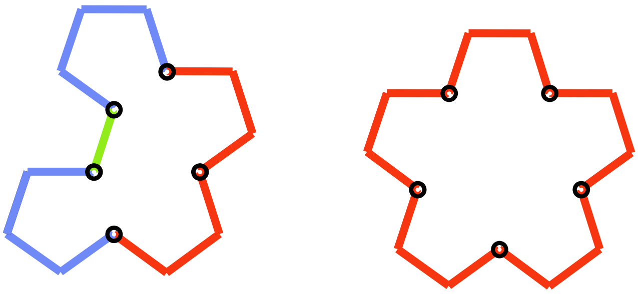
2. Partition sequences

In order to simplify angle sequences, we have derived another expression, partition sequences, for EASE polygons. Every continuous h-1 positive angles and 1 negative angle is represented with the partition h. Because regular polygons are not allowed to appear in EASE polygons, n continuous positive or negative angles are forbidden. There for h could only be positive integers smaller than n. h could be n-1,n-2,n-3……1 with a total of n-1 possibilities.

Because every partition includes only one negative angle, the number of negative angles in a partition sequence equals the total number of partitions.

Since the sum of all the partitions h equals the total number of sides in an EASE polygon r, for simplicities sake, we put partition sequences in the form: $r=h_1+h_2+h_3+\ldots+h_y$.

For example, two partition sequences of two EASE polygons with qualities of n=5 and r=15 are listed below



(+ + + −) (−) (+ + + −) (+ + −) (+ + −)

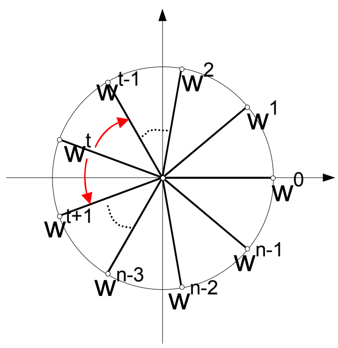
15=4 + 1 + 4 + 3 + 3

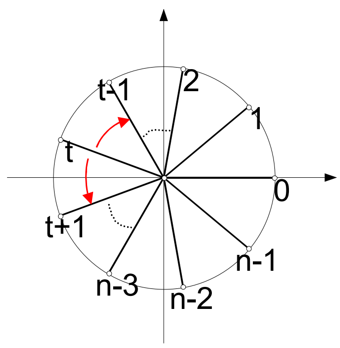
(+ + −) (+ + −) (+ + −) (+ + −) (+ + −)

15=3 + 3 + 3 + 3 + 3

3. Vector sequences

Besides describing EASE polygons from the point of view of angles, we could imagine the lines that we have traversed to be unit vectors. Each time an angle ±2π/n is turned, the unit vector is multiplied by $w^{\pm 1}$(w is the root of x^n=1) on the complex number plane. Therefore, each side of an EASE polygon could correspond to a root of the function x^n=1, giving us a total of n different kinds of unit vectors.

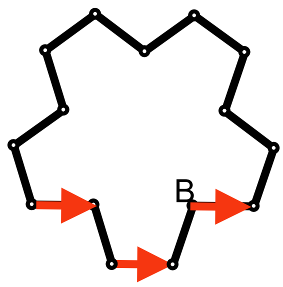


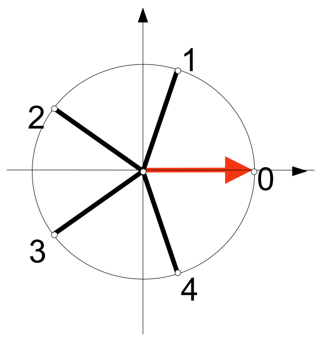


Since each kind of unit vector corresponds to only one root, for convenience's sake, the unit vector w^t is represented by the vector number t. Then after traversing along the sides of an EASE polygon counter clockwise and recording the vector number of each unit vector that we run into during the process, we end of with a number sequence which is defined to be the vector sequence.

From the definition of EASE polygons, vector number t could be followed and preceded only by the vector number (t+1) mod n and (t-1) mod n.

For example, the vector sequence of an EASE polygon with n=5, r=15 is demonstrated below.





Starting from point B and traveling counterclockwise along the ring, we end up with the sequence 012 123 234 340 401.

Notice that the first vector number and the last vector number also have to follow the rule of “vector number t could be followed and preceded only by the vector number (t+1) mod n and (t-1) mod n”.

Combining all the view points from above, the symbol “+” or “–” represents the process of turning an angle of 2π/n counter clockwise or clockwise, meaning the same thing as the vector number t followed by the vector number (t+1) mod n or (t-1) mod n. Therefore, we could insert the angle sequence into the vector sequence forming a more complete point of view.

For example, an EASE polygon with n=5 r=15:
Angle sequence :＋＋＋－－＋＋＋－＋＋－＋＋－
Vector sequence :012321234340401
Combined : 0＋1＋2＋3－2－1＋2＋3＋4－3＋4＋0－4＋0＋1－

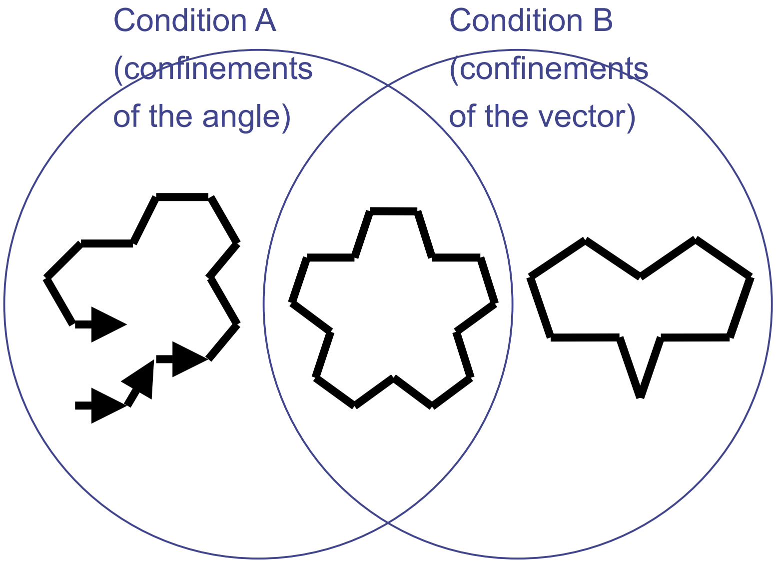
Partition sequences were described as a simplified form of angle sequences. Each partition of a partition sequence stands for several “+”s and one “-” of the angle sequence. However, it could also be expanded into a vector sequence.

Since positive turns would correspond to an increase and negative turns would correspond to a decrease in vector number, within each partition, the vector numbers would continuously increase, and between partitions, the vector number would decrease.

Partition sequences: 15=4 + 1 + 4 + 3 + 3
Angle Sequences: (＋＋＋－) (－) (＋＋＋－) (＋＋－) (＋＋－)
Vector sequences: (0123) (2) (1234) (340) (401)

C. Conditions of forming an EASE polygon

The sufficient and necessary conditions of forming an EASE polygon:
Condition A: the absolute of the difference between the vector number of any two continuous sides must be 1 and a total angle of 2π must be turned.
Condition B: the vector sum of all the sides must be zero
Condition A is the confinement of the angle and Condition B is the confinement of the vector.

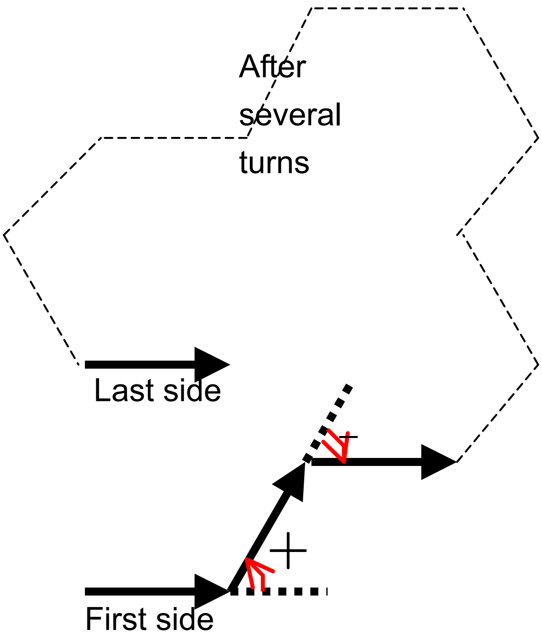


1. Confinements of the angle

In order to form EASE polygons, the sides must ultimately return to the point which it started from. The sum of the positive angles and negative angles must be 2π and every angle that is turned is either +2π/n or -2π/n. Therefore, the number of positive angles subtracted by the number of negative angles must be n. The sum of the number of positive angles and negative angles equals the number of sides r in an EASE polygon.

Assume that the number of positive angles is x and the number of negative angles is y.

$$\begin{cases} x - y = n \\ x + y = r \end{cases} \Rightarrow \begin{cases} x = (r+n)/2 \\ y = (r-n)/2 \end{cases}$$



It will be proved later that the number of sides r must be an integer multiplication of n. we substitute r with r=nt (t is a natural number).

$x=n(t+1)/2$
$y=n(t-1)/2$

Because both x and y are integers, and all prime numbers greater than 3 are odd numbers, t has to be an odd number. Therefore we substitute t with 2k+1(k is any natural number).

$x=n(k+1)$
$y=nk$
$r=x+y=n(2k+1)$
Which is another way to say: in order to form EASE polygons, the number of sides must be n(2k+1).

The search for EASE polygons is translated into the search of ways you could break down n(2k+1) in to a partition of nk parts. However, these partition sequences only satisfy confinements of the angle. There is exists a possibility that after a total angle of 2π is turned, the last side still does not return to the origin.
Therefore we need to consider the confinements of the vector.

2. Confinements of the vector

If we think of each side of the EASE polygon to be unit vectors connected head to tail and the last side must return to the origin, this will mean that the vector sum is zero.

Since each time an angle of ±2π/n is turned can be imagined to be a unit vector timed by $e^{\pm i(2\pi/n)}$ on the complex plane, each side of the EASE polygon will correspond to one of the n kinds of vector of the root of the equation $x^n-1=0$.

Theorem: the number r of sides has to be a multiple of n in order to form an EASE polygon when n is a prime number

After putting the n kinds of vectors on the complex plane, the vector numbers 0,1,2,3,,,n-2,n-1 correspond to the roots of the equation $x^n-1=0$ which are $1,w,w^2,w^3,,,w^{n-2},w^{n-1}$. Suppose that the number of vectors corresponding to each kind of vector number is $a_0,a_1,a_2,a_3,,,a_{n-2},a_{n-1}$.

We assume that $a_0=a_1=a_2=a_3=,,,=a_{n-2}=,,,=a_{n-1}$ is not true

Because we already know that EASE polygons form a closed cycle, the vector sum of the roots has to be zero.

$a_0+a_1w+a_2w^2+a_3w^3+,,, a_{n-2}w^{n-2}+a_{n-1}w^{n-1}=0\ldots(1)$

$x^n-1=(x-1)(1+x+x^2+x^3+,,,+x^{n-2} +x^{n-1})=0\ldots(2)$

We substitute equation (2) with one of its root w. Because w is not a real number, w-1 does not eqaul zero. Therefore

$1+w+w^2+w^3+,,,+w^{n-2}+w^{n-1}=0...(3)$

Then we could write equation (1) into

$a_{n-1}(1+w+w^2+w^3+,,,+w^{n-2}+w^{n-1})+b_0+b_1w+b_2w^2+b_3w^3+,,,b_{n-2}w^{n-2}+b_{n-1}w^{n-1}=0...(4)$

$b_0,b_1,b_2,b_3,,,b_{n-2}$, could be positive integers, negative integers or zero and $b_{n-1}=0$.

Therefore $b_0+b_1w+b_2w^2+b_3w^3+,,,b_{n-2}w^{n-2}=0$

Because $a_0=a_1=a_2=a_3=,,,=a_{n-2}=,,,=a_{n-1}$ is not true according to our assumption, At least one number of $b_0,b_1,b_2,b_3,,,b_{n-2}$ is not zero.

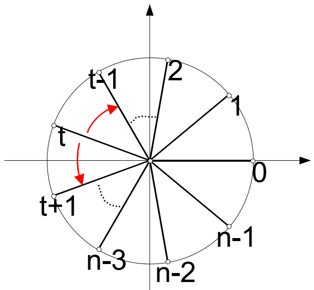
w is the root of both $b_0+b_1x+b_2x^2+b_3x^3+,,,b_{n-2}x^{n-2}=0$ and $1+x+x^2+x^3+,,,+x^{n-2}+x^{n-1}=0$.

$1+x+x^2+x^3+,,,+x^{n-2}+x^{n-1}=0$ could not be factorized.

Therefore $b_0+b_1x+b_2x^2+b_3x^3+,,,b_{n-2}x^{n-2}$ could be divided by $1+x+x^2+x^3+,,,+x^{n-2}+x^{n-1}$.

However the degree of $1+x+x^2+x^3+,,,+x^{n-2}+x^{n-1}$ is smaller than $b_0+b_1x+b_2x^2+b_3x^3+,,,b_{n-2}x^{n-2}$.

→←



Therefore we know that $a_0=a_1=a_2=a_3=,,,=a_{n-2}=a_{n-1}$.

And the total number of sides $r=a_0+a_1+a_2+a_3+,,,a_{n-2}+a_{n-1}=na_0=nt$

So when n is an integer, the number of sides r has to be a multiple of r in order to form an EASE polygon. r=nt (t is an integer)

From the discussion above, a conclusion could be drawn: in order for the vector sum of all the vectors to be zero, the number of vectors corresponding to each kind of vector has to be the same.

It will be easier to pick out the sequences that don't satisfy Condition B out of several sequences that already satisfy Condition A than to do the reverse. Therefore a method to find EASE polygons has been devised.

D. The destructive method of obtaining EASE polygons

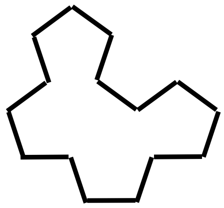
The only conditions that have to be met in order to satisfy the confinements of the angle are:

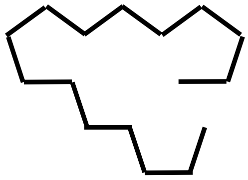
1) $r=n(2k+1)$

2) the number of parts in the partition sequence equals nk

So we write down all the partitions of n(2k+1) with nk parts and then expand the partition sequence into vector sequences. We then count the number of vectors corresponding to each kind of vector number and see if they are the same. If they are then that means the vector sum of this sequence is zero therefore. And we would have found an EASE polygon.

For example r=15 n=5 (k=1)
From the discussion mentioned before we know that there will be 5 partitions
15=4+1+4+3+3
(0123, 2, 1234, 340, 401)O
=4+2+4+2+3
(0123, 23, 2340, 40, 401)X
=……





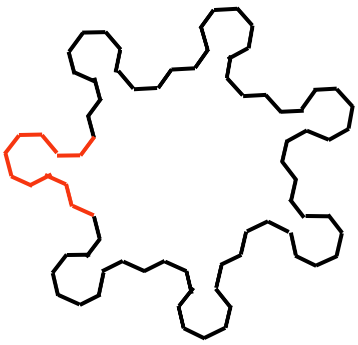
……

E. The lower and upper bound of the number of EASE polygons

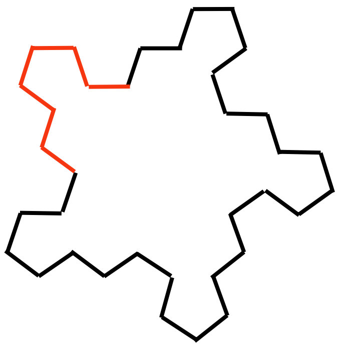
1. Basic EASE polygons

The partition sequences of EASE polygons are partitions of n(2k+1) with nk parts. However, if k consecutive parts of the partition repeat itself for n times, we call these EASE polygons basic EASE polygons. Basic EASE polygons have n regions that differ only in its angle orientation. Within each region there are 2k+1 sides and k parts.

For instance, two basic EASE polygons are listed below.



r=77 n=7 (k=5)
77=(1+1+6+1+2)x7



r=35 n=5 (k=3)
35=(1+4+2)x5

Theorem:
every partition of 2k+1 with k parts repeated for n times will form an EASE polygon.

Proof:
Since every partition of 2k+1 will have k+1 positive angles and k negative angles, so a total angle of 2π/n will be turned. There are n partitions of 2k+1, therefore a total angle of 2π is turned. The confinement of the angle is satisfied.

Then we check the vector sum. Assume that the first vector number of the first partition of 2k+1 is 0. According to qualities of the EASE polygon, (further explained in the section “other properties of the EASE polygon”) the first vector of the first part of t^{th} partition of 2k+1 is actually the first vector of the $tk+1^{th}$ part.

Which equals $(\sum_{k=1}^{tk} h_k - 2(t-1)k) \bmod n = (t-1)(2k+1) - 2(t-1)k = t-1$.

Therefore the first vector number of each region will begin from 0 and increase 1 each time a region is repeated until it reaches n-1. Every integer between 0 to n-1 will appear once in the vector number. Because the n partitions of 2k+1 are the same, it will also happen for the second vector of the n partitions of 2k+1.

Therefore, for every vector in a single partition of 2k+1, there exists another n-1 vectors that will correspond to it, making all the vector numbers from 0 to n-1 appear once. The vector sum of each corresponding system is zero, so the total sum is also zero. This satisfies the confinement of the vector. All partition sequences that satisfy both confinements of the angle and the vector form EASE polygons.

2. The lower bound

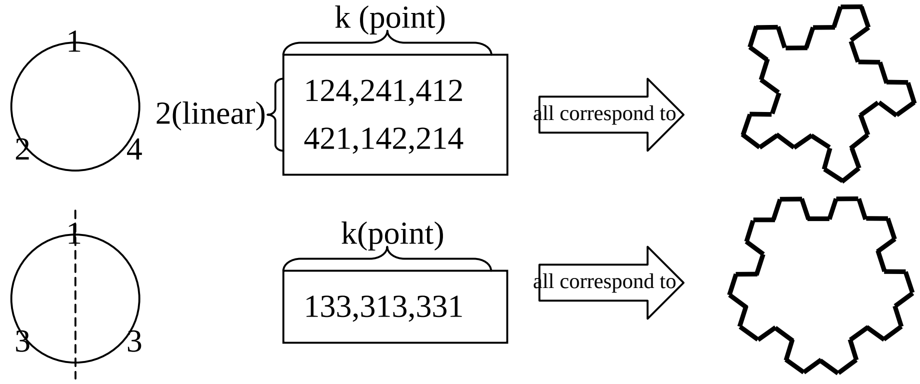
For any EASE polygons with the number r of sides n(2k+1), there exists Basic EASE polygons. Therefore, the number of Basic EASE polygons will actually be a lower bound. From the section above, the number of EASE polygons equals the number of partitions of 2k+1 with k parts. We define the function N(a,b,n) to be the number of ways to distribute a identical objects to b persons such that each person receives an amount m, of objects with 0<m<n. (for further explanation, please read the section “other properties of the EASE polygon”.) After substituting a with 2k+1 and b with k the number of partitions become N(2k+1,k,n). However, the problem of both linear and point symmetry has to be dealt with. Several sequences could correspond to the same EASE polygon.

Because (k,2k+1)=1, any factor of k will not divide 2k+1. So we know that for any way you can distribute 2k+1 to k parts there does not exist a possibility of a small sequence repeating itself. So the problem of point symmetry could be fixed by simply dividing N(2k+1,k,n) by k.

Then we consider the problem of linear symmetry. Some sequences of Basic EASE polygons will be linearly symmetric when put on a ring. Suppose that the subset of these sequences is b. Some will not be linearly symmetric. Suppose that the subset of these sequences is a. a and b are exclusive subsets. Therefore N(2k+1,k,n)= $|b|+|a|$.

｜a｜has to be divided by 2 to eliminate the linear symmetry.
So the number of Basic EASE polygons is ｜b｜/k +｜a｜/2k.

Take two EASE polygons with n=5, r=35, k=3 as an example



From the ways to eliminate the many to one nature(further explained in the section "other qualities of the EASE polygon") we know that｜b｜/k=B.

when k is odd

$$B = \sum_{l=1}^{n-1} N\left(\frac{2k+1-l}{2}, \frac{k-1}{2}, n\right)$$

when k is even

$$B = \sum_{l=1}^{n-1} N\left(\frac{2k+1-l}{2}, \frac{k-2}{2}, n\right)\left\lfloor \frac{N(l,2,n)+1}{2} \right\rfloor$$

the lower bound of the EASE polygon
=｜a｜/2k+｜b｜/k
= [N(2k+1,k,n)-｜b｜]/2k+｜b｜/k
= N(2k+1,k,n)/2k-｜b｜/2k
= N(2k+1,k,n)/2k- B/2

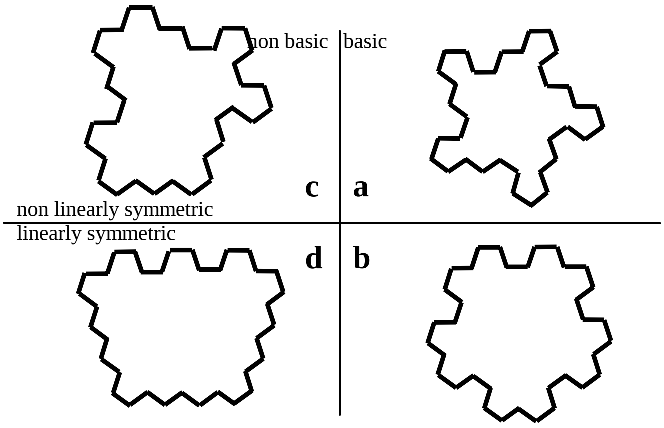
3. The upper bound

Not all partition sequences of n(2k+1) with nk parts can form an EASE polygon. The vector needs to be checked. So a rough upper bound would naturally be N(n(2k+1),nk,n). But after considering the redundancy of symmetry, the upper bound could be slightly lowered.

Because the highest common factor of n(2k+1) and nk is n, when we try to eliminate the problem point symmetry we have to consider two cases. Some series of partitions has to be divided by nk and some by k. When we take the problem of linear symmetry into consideration, some series of partitions have to be divided by 2 and some don't.

So we divide N(n(2k+1),nk,n) into four exclusive subsets a,b,c,d
with N(nk,n(2k+1),n)= ｜a｜+｜b｜+｜c｜+｜d｜.

a is the subset of the sequences of non linearly symmetric basic EASE polygons, ｜a｜has to be divided by 2k; b is the subset of the sequences of linearly symmetric basic EASE polygons, ｜b｜has to be divided by k; c is the subset of the sequences of non linearly symmetric non basic EASE polygons, ｜c｜has to be divided by 2nk; d is the subset of the sequences of linearly symmetric non basic EASE polygons, ｜d｜has to be divided by nk.



｜b｜/k is the number of linearly symmetric EASE polygons which is B .
｜a｜/2k is the number of non linearly symmetric EASE polygons which is N(2k+1,k,n)/2k-B/2.

Suppose that T is the number of ways to distribute n(2k+1) objects to nk people that will seem linearly symmetric on a ring. (with each person an amount of objects that's larger than 0 but smaller than n)

when k is odd

$$T = \sum_{l=1}^{n-1} N\left(\frac{n(2k+1)-l}{2}, \frac{nk-1}{2}, n\right)$$

when k is even

$$T = \sum_{l=1}^{n-1} N\left(\frac{n(2k+1)-l}{2}, \frac{nk-2}{2}, n\right)\left\lfloor \frac{N(l,2,n)+1}{2} \right\rfloor$$

｜d｜/nk=T-B

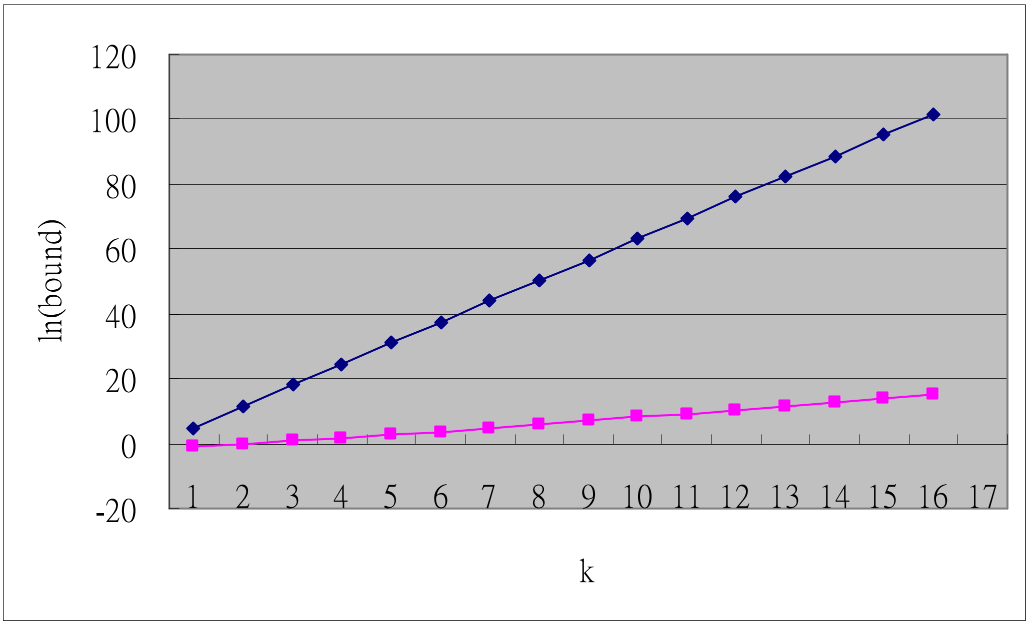
The upper bound of the EASE polygon
=｜a｜/2k+｜b｜/k+｜c｜/2nk+｜d｜/nk
=｜a｜/2k+｜b｜/k+[N(n(2k+1),nk,k)-｜b｜-｜a｜-｜d｜]/2nk+｜d｜/nk

=n｜a｜/2nk+2n｜b｜/2nk+[N(n(2k+1),nk,k)-｜b｜-｜a｜-｜d｜]/2nk+2｜d｜/2nk
=(n-1)｜a｜/2nk+(2n-1)｜b｜/2nk+N(n(2k+1),nk,k)/2nk+｜d｜/2nk
= [N(2k+1,k,n)/2k-B/2] (n-1)/n+B(2n-1)/2n+N(n(2k+1),nk,k)/2nk+(T-B)/2
= N(2k+1,k,n)(n-1)/2nk-B(n-1)/2n+B(2n-1)/2n+N(n(2k+1),nk,k)/2nk+T/2-B/2
= N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2+B(-(n-1)/2n+(2n-1)/2n-n/2n)
= N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2

Actual figures

n=5

k	upper bound	lower bound	Ln(upper bound)	Ln(lower bound)	slope of upper bound	slope of lower bound
1	101	0.5	4.6151	-0.6931	7.0488	0.6931
2	116304	1	11.6640	0.0000	6.5749	0.6931
3	83372562	2	18.2388	0.6931	6.4643	0.9163
4	5.35E+10	5	24.7031	1.6094	6.4247	0.9933
5	3.3E+13	13.5	31.1278	2.6027	6.4079	1.0349
6	2E+16	38	37.5357	3.6376	6.4001	1.0719
7	1.21E+19	111	43.9358	4.7095	6.3965	1.0986
8	7.23E+21	333	50.3323	5.8081	6.3950	1.1189
9	4.33E+24	1019.5	56.7272	6.9271	6.3945	1.1354
10	2.59E+27	3173	63.1217	8.0624	6.3946	1.1488
11	1.55E+30	10009	69.5164	9.2112	6.3951	1.1600
12	9.29E+32	31928	75.9115	10.3712	6.3957	1.1695
13	5.57E+35	102818	82.3072	11.5407	6.3964	1.1776
14	3.34E+38	333808	88.7036	12.7183	6.3971	1.1847
15	2E+41	1091403	95.1007	13.9030	6.3979	1.1908
16	1.2E+44	3590485	101.4986	15.0938		



The upper and lower bound presented are N(n(2k+1),nk,n) and N(2k+1,k,n)/2k respectively.

F. The constructive method of obtaining EASE polygons

1. The operation "flip"

Flip

[Flip defined]

Suppose ⋯hi-2, hi-1, hi, hi+1, hi+2… is a partition sequence.

case 1)

$h_i=3, h_{i-1} \leqq n-2,\ h_{i+1} \leqq n-2,$

$\ldots h_{i-2},\ h_{i-1},\ 3,\ h_{i+1},\ h_{i+2}\ldots$

is converted to

$\ldots h_{i-2},\ h_{i-1}+1,\ 1,\ h_{i+1}+1,\ h_{i+2}\ldots$

case 2)

$h_i=1, h_{i-1} \geqq 2,\ h_{i+1} \geqq 2,$

$\ldots h_{i-2},\ h_{i-1},\ 1,\ h_{i+1},\ h_{i+2}\ldots$

is converted to…

$h_{i-2},\ h_{i-1}-1,\ 3,\ h_{i+1}-1,\ h_{i+2}\ldots$

This method to generate new sequences is called flip.

The operation flip wound not change the vector sum of the EASE polygon, and will not defy the confinements of the angle. When the number r of sides of the EASE polygon is large enough, the operation can be performed several times.

Theorem:

Suppose ⋯hi-2, hi-1, hi, hi+1, hi+2… is a partition sequence.

case 1)
$h_{i-1} \leqq n-2$, $h_{i+1} \leqq n-2$,
$\ldots h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2} \ldots$
is converted to
$\ldots h_{i-2}, h_{i-1}+1, h_i-2, h_{i+1}+1, h_{i+2} \ldots$
The vector sum of the partition sequence will only remain the same when $h_i=3$

case 2)
$h_{i-1} \geqq 2$, $h_{i+1} \geqq 2$,
$\ldots h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2} \ldots$
is converted to…
$h_{i-2}, h_{i-1}-1, h_i+2, h_{i+1}-1, h_{i+2} \ldots$
The vector sum of the partition sequence will only remain the same when $h_i=1$

Proof:

Suppose that there exists a number $t=h_i$,
such that the vector sum will remain the same change after the operation mentioned above.

The only differences in the partition sequence before and after the operation are h_{i-1}, h_i, h_{i+1}. We have assumed the sum of the vector sequence to remain the same. Therefore the operation would only modify the vector numbers within these three parts in the partition sequence. The first vector of the part h_{i-1} and the last vector of the part h_{i+1} should remain the same.

First we have to examine the effect of this operation on the vector sequence.
So we list out the vectors numbers of the part h_i are a+1,a+2,a+3….a+t.
Therefore the last vectors of the part h_{i-1} would be a+1,a+2;
the first vectors of the part h_{i+1} would be a+t-1,a+t.

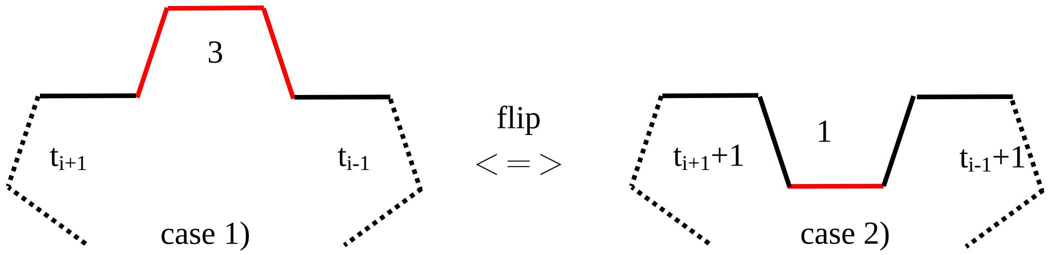
case 1)
So the area of our vector sequence that needs to be examined would be
h_{i-1}, (h_i), h_{i+1}=…a+1, a+2, (a+1,a+2,a+3….a+t), a+t-1, a+t…

After being converted into $h_{i-1}+1$, $(h_i)-2$, $h_{i+1}+1$
$h_{i-1}+1$, $(h_i)-2$, $h_{i+1}+1$=…a+1,a+2, a+3, (a+2,a+3,a+4….a+t-1),a+t-2, a+t-1, a+t…

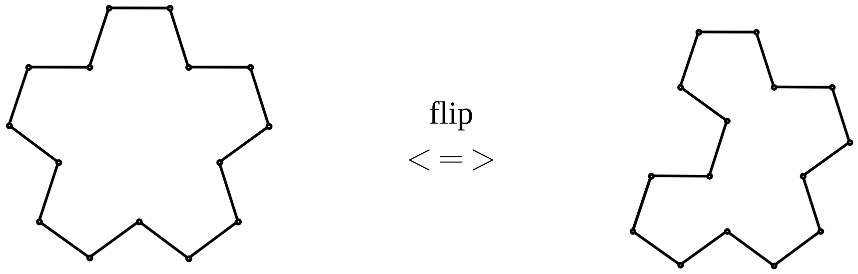
Because the vector sum should remain the same, after comparing the two sequences, the vector sum of the vector numbers a+1 and a+ t has to be the same as the sum of the vector numbers a+3 and a+t-2. We also know that a+t≠a+t-2 and a+3≠a+1. Therefore a+t=a+3 , a+t-2=a+1. From both we could withdraw the conclusion that $h_i=3$.

After a similar discussion,another conclusion could be withdrawn that $h_i=1$ for case 2).

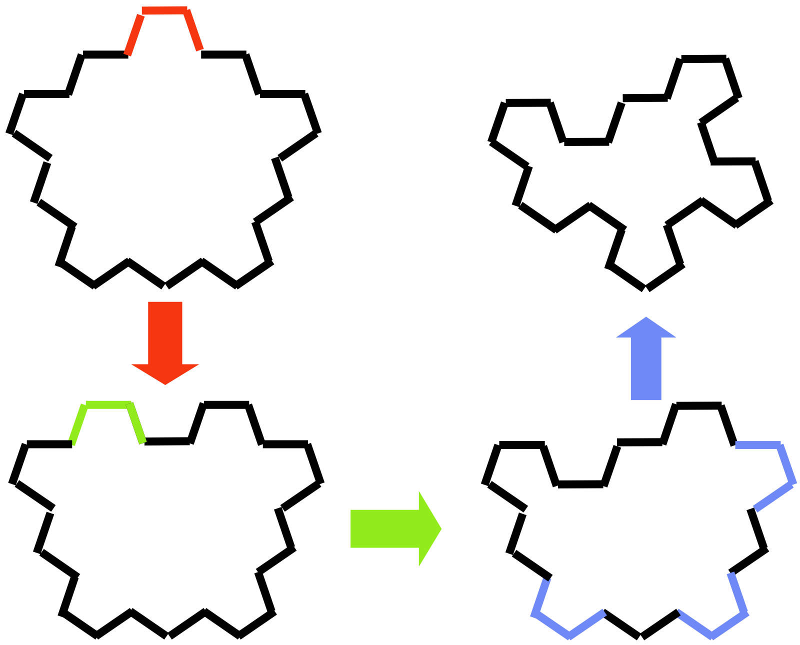
From the theorem above, we know that the opperation flip would not effect the vector sum of the entire EASE polygon. An EASE polygon will still be an EASE polygon regardless of how many times its flipped.



Actually, the two cases of flip are actually inverse operations of each other. The picture above explains the fact in a simple way.



The operation flip is quite useful when it comes to creating a lot of EASE polygons in a short amount of time.



The number of EASE polygons that can be flipped into from an original EASE polygon will become exceedingly larger with the growth of r, as listed in the examples above.

2. The operation “insert”

Insert

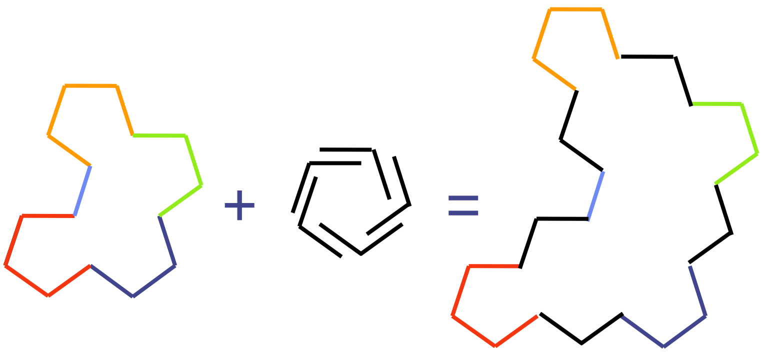
[Insert defined]

Create n partitions of 2 with 0 to n-1 being the first vector numbers

Insert the partition with the first vector number t into the original partition sequence after any partition with the first vector number being t+1

Each partition of 2 includes a negative and positive angle, which means it will not affect the confinements of the angles. n partitions of 2 were created, each starting with a different first vector, which will mean the vector sum of the added components is zero thus not affecting the vector sum.

For instance, the an EASE polygon with n=5 and r=15 is inserted below



The constructive method of creating EASE polygons is performing several operations of flip and insert on an EASE polygon to obtain more EASE polygons. “Flip” is a method to create more EASE polygons with the same number r of sides; “Insert” is a method to create more EASE polygons with an increase of 2n to the number r of sides.

The theorem of expandability:

The operation insert can be performed on any EASE polygon.

The necessary and sufficient conditions for the theorem of expandability:

All kinds of vector numbers should appear at least once in the first vectors of the parts.

Because when the first vector number is t, the part (t,t+1) could be inserted before it.

Proof:

Suppose that a kind vector number t, did not appear amongst the first vector numbers in an EASE polygon, then all parts containing the vector number t should look like (…t-1,t…). Which is another way of saying the left side of t has got to be t-1.If the left side of t had been t+1, t would have become the first vector of a part.

So that holds for every vector number t. In order to form EASE polygons, every kind of vector number has to appear the same amount of time. Therefore, every vector number t-1 would only appear on the left side of t. That is to say that t and t-1 would always appear together.

For every the vector number t-3 should appear on the left of every vector number t-2, because the right side of all vector numbers t-1 has been used. Therefore, t-3 and t-2 will also appear together in the form(…t-3,t-2…).

Likewise, (t-5,t-4), (t-7,t-6)... (t+2,t+3) are the only forms the vector numbers could appear in. Because n is a prime number and any prime number except for 2 is odd, all vector numbers except for t+1 would always appear in pairs.

The left side of the vector number t+1 is left with the only choice t and not t+2. There would be the same amount of (t+1)s and ts therefore the t-1, t and t+1 would always appear together in the form (t-1, t, t+1).

Then we consider what could be connected to the right of t+1. Because we have already distributed all t+1s to the left of t+1, the only choice left would be t+2. So we would see the form (t-1,t,t+1,t+2,t+3) appear together.

Likewise, (t-1,t,t+1,t+2,t+3,…t-3,t-2) will also appear together. This is a single part of an EASE polygon with n sides which already is in conflict with our definition “parts”. For each part there should be less than n sides. →←

This proves that any EASE polygon could be inserted. (Actually, the vector sequence of the part(t-1,t,t+1,t+2,t+3,…t-3,t-2) is a regular polygon.)

G. Other properties of the EASE polygon

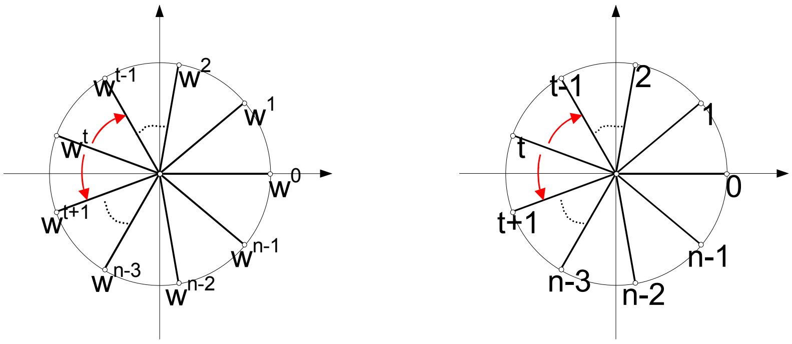
1. First vectors

Theorem:

For the first vector number of the tth part in a partition sequence r=h_1+h_2+h_3……$h_{n(2k+1)}$ is $a_t \equiv (\sum_{k=1}^{t-1} h_k - 2(t-1)) \bmod n$

Proof:

Because every increase of 1 between consecutive vector numbers would mean turning an angle of 2π/n; decrease of 1 between consecutive vector numbers would mean turning an angle of -2π/n. Therefore the operation of turning an angle of positive or negative 2π/n could be imagined to be multiplying the original vector by $w^{\pm 1}$ on the complex plain. For convenience's sake, we define the first vector number of the first part to be zero. Therefore, every vector number t could correspond to w^t, a root of the function $x^n=1$.



According to the definition of partition sequences, the part h_t is consisted of all the angles between the (t-1)[th] negative angle to the t[th] negative angle. So for the first vector number a_t of the part h_t there are a total of $\sum_{k=1}^{t-1} h_k$ angles (or sides) before it.

Amongst those angles, there should be t-1 negative angles and $\sum_{k=1}^{t-1} h_k$ -(t-1) positive angles.

Therefore w^0 has to be timed by $w^{\sum_{k=1}^{t-1} h_k-(t-1)} \times w^{-(t-1)} = w^{\sum_{k=1}^{t-1} h_k-2(t-1)} = w^{(\sum_{k=1}^{t-1} h_k-2(t-1)) \bmod n}$. Because w^t corresponds to the vector number t, the first vector number $a_t \equiv (\sum_{k=1}^{t-1} h_k - 2(t-1)) \bmod n$.

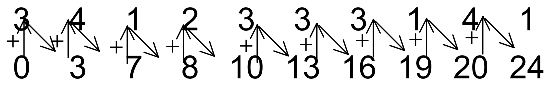
If we want to obtain the first vector number of each part in a partition sequence with the formula above, there would be a lot of calculating to do. Therefore, a simple algorism to calculate the first vectors numbers has been developed. This method becomes quite handy under situations with the absence of calculators.

Take an EASE polygon with n=5 and r=25 as an example.
Its partition sequence is 25=3+4+1+2+3+3+3+1+4+1.

First, we write down the partition sequence.

3 4 1 2 3 3 3 1 4 1

Then we create a second row with a zero directly under the first part. Then starting from the zero we just wrote in the second row, we add it with the number in the first row that's in the same column with it. Finally we put the sum to one column right in the second row. We repeat the process until the last part number in the first row is reached.



Then starting with zero, we create an arithmetic sequence with the difference being 2 in the third row. We write this sequence until we reach the end of the first row.

3	4	1	2	3	3	3	1	4	1
0	3	7	8	10	13	16	19	20	24
0	2	4	6	8	10	12	14	16	18

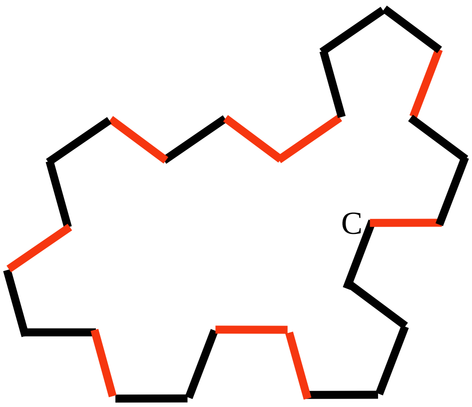
Finally we subtract row 2 with row 3 and write the result in row 4.

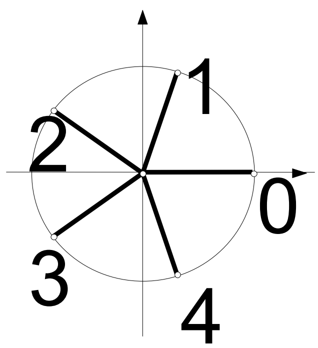
	3	4	1	2	3	3	3	1	4	1
	0	3	7	8	10	13	16	19	20	24
−)	0	2	4	6	8	10	12	14	16	18
	0	1	3	2	2	3	4	5	4	6

After taking the modulus of 5 for all the numbers in row 4, the first vector number corresponding to parts in a partition sequence is obtained respectively.

	3	4	1	2	3	3	3	1	4	1
	0	3	7	8	10	13	16	19	20	24
−)	0	2	4	6	8	10	12	14	16	18
	0	1	3	2	2	3	4	0	4	1

We compare our results with the vector sequence and find that they match.
3412333141(starting counter-clockwise from point C)
=>(012)(1234)(3)(23)(234)(340)(401)(0)(4012)(1)





Lemma: For all basic EASE polygons, every kind of vector number will appear the same amount of times amongst the first vectors.

Proof:

It had been mentioned before that in the partition sequence of a basic EASE polygon, k consecutive parts appear n times forming a total of nk parts. From what was proved before, we also know that the first vector of the first part of the r^{th} region a_r will increase 1 with every consecutive region that follows. So the every vector number of the $r+1^{th}$ region is actually the corresponding vector number in the r^{th} region +1.

Suppose that the t^{th} vector in the 1^{st} region is i, then
the t^{th} vector in the 2^{nd} region is i+1;
the t^{th} vector in the 3^{rd} region is i+2;
…
the t^{th} vector in the u^{th} region is i+u-1;
…
the t^{th} vector in the n^{th} region is i+n-1.

So we know that every vector number in the first region will appear n times in the whole EASE polygon, with every kind of vector number appearing once.

Methods to obtain the first vector numbers have been discussed before. We know that there will be k first vectors of parts in the first region and a total of nk first vectors of parts in the whole EASE polygon. Amongst these nk first vectors, every kind of vector would appear the same amount of times. However, this lemma could be expanded into a theorem.

Theorem: For all basic EASE polygons, every kind of vector number will appear the same amount of times amongst the first vectors no matter how many times an EASE polygon is flipped.

Proof:

Because after basic EASE polygons are flipped several times, the original symmetry has been ruined, there exists a great possibility that it's no longer a basic EASE polygon. We shall consider the problem from the view point of all the parts, instead of looking for symmetric properties as was done in the lemma.

case 1)
Suppose …hi-2, hi-1, hi, hi+1, hi+2… is a partition sequence.
$h_i=3, h_{i-1} \leqq n-2$, $h_{i+1} \leqq n-2$, $\ldots h_{i-2}, h_{i-1}, 3, h_{i+1}, h_{i+2} \ldots$
is converted to $\ldots h_{i-2}, h_{i-1}+1, 1, h_{i+1}+1, h_{i+2} \ldots$

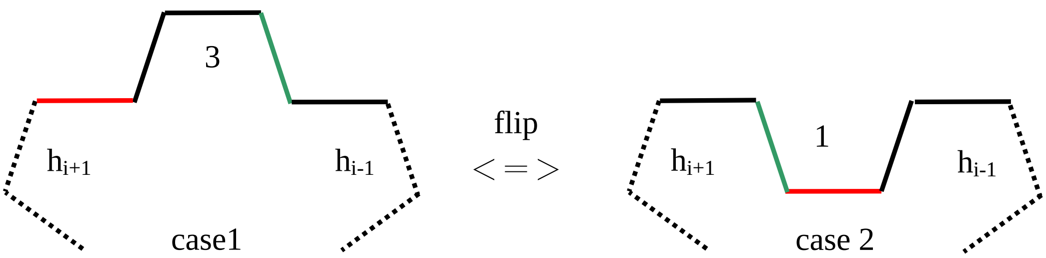
We list out the vector numbers that h_i is composed of.

Suppose it to be a+1,a+2,a+3, then
the last vector of h_{i-1} is a+2;
the first vector of h_{i+1} is a+2;
the first vector of h_i is a+1.

Originally h_{i-1}, (h_i), h_{i+1}=…a+1, a+2, (a+1,a+2,a+3), a+2, a+3…
After it's flipped, h_{i-1}+1, (h_i-2), h_{i+1}+1=…a+1, a+2, a+3, (a+2), a+1, a+2, a+3…

The operation
does not affect the first vector of h_{i-1};
the first vector of h_{i+1} becomes a+1;
the first vector of h_i becomes a+2.

We could see that the only effect the operation had was to switch the first vectors of h_{i+1} and h_i.



case 2)
Suppose ···hi-2, hi-1, hi, hi+1, hi+2… is a partition sequence.
h_i=1,$h_{i-1} \geqq 2$, $h_{i+1} \geqq 2$, …h_{i-2}, h_{i-1}, 1, h_{i+1}, h_{i+2}…
is converted to …h_{i-2}, h_{i-1}-1, 3, h_{i+1}-1, h_{i+2}…

We list out the vector numbers that h_i is composed of.

Suppose it to be a+2 then
the last vector of h_{i-1} is a+3;
the first vector of h_{i+1} is a+1;
the first vector of h_i is a+2.

Originally h_{i-1}, (h_i), h_{i+1}=…a+1, a+2, a+3, (a+2), a+1, a+2, a+3…
After it's flipped, h_{i-1}+1, (h_i-2), h_{i+1}+1=…a+1, a+2, (a+1,a+2,a+3), a+2, a+3…

The operation

does not affect the first vector of h_{i-1};
the first vector of h_{i+1} becomes a+2;
the first vector of h_i becomes a+1.

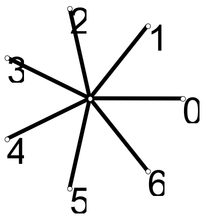
We could see that from both cases, the only effect the operation had was to switch the first vectors of h_{i+1} and h_i.

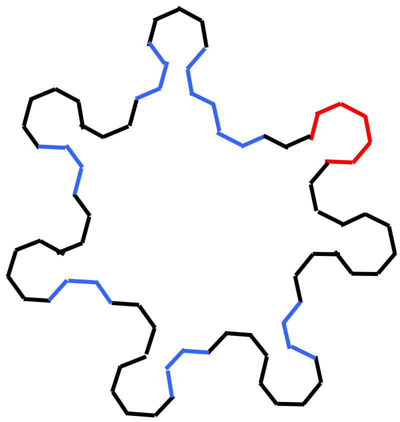
Because the vector distribution among the first vectors was originally the same, it will remain the same after the EASE polygon is flipped. Thus the theorem is proved.

Notice that although both basic EASE polygons and EASE polygons obtained by flipping basic EASE polygons have the property of "the vector distribution amongst the first vectors is the same for every kind of vector" this does not mean that only they have such properties.

For instance, a partition sequence and figure of an EASE polygon with n=7 and r=77 is given below. 61212 16121 16121 16121 16121 16121 16111(starting counterclockwise from the red part.)

Obviously, this is not a basic EASE polygon, and all the areas (marked in blue) where the operation flip could be performed are locked in their original position. No further operations could be performed after the first few flips. So this is definitely not an EASE polygon that was obtained by flipping basic EASE polygons. We examine its first vectors and find
61212 16121 16121 16121 16121 16121 16111 partition sequence
04332 21544 32655 43066 54100 65211 06321 corresponding first vectors
"the vector distribution amongst the first vectors is the same for every kind of vector".





Finally we could conclude that if the EASE polygon does not have the quality of "the vector distribution amongst the first vectors is the same for every kind of vector", then it is definitely not an EASE polygon, nor an EASE polygon obtained by flipping EASE polygons.

However, we could not say that if the EASE polygon has the quality of
“the vector distribution amongst the first vectors is the same for every kind of vector”,
it is an EASE polygon, or an EASE polygon obtained by flipping EASE polygons.

2. Proof of functions

We define N(a,b,n)=ways to distribute a identical objects amongst b people with each person receiving an amount of m objects with 0<m<n

We define K(a,b,n)=ways to distribute a identical objects amongst b people with each person receiving an amount of m objects with 0≦m<n

We define K(a,b)=ways to distribute a identical objects amongst b people with each person receiving an amount of m objects with 0≦m

We define that when n≦$\left\lceil \frac{a}{b} \right\rceil$，K (a,b,n)=0

We define that when n≦$\left\lceil \frac{a}{b} \right\rceil$，N (a,b,n)=0

We define that when 0≧$\left\lfloor \frac{a}{b} \right\rfloor$，N (a,b,n)=0

We define that when a,b,n are not positive integers, N(a,b,n)=0

It’s obvious that N(a,b,n)=K(a-b,b,n-1) <=> K(a,b,n)=N(a+b,b,n+1)

Lemma: $K(a,b) = \binom{a+b-1}{a}$

This lemma is pretty well known, therefore it is stated with out proof.

Theorem: $K(a,b,n) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} K(a-nt,b) = \sum_{k=0}^{b} (-1)^n \binom{b}{t} \binom{a-nt+b-1}{a-nt}$

Proof

K(a,b,n) is the number of solutions for the function $x_1+x_2+x_3+x_4+x_5+\ldots+x_b=a$ with roots 0≦ and ≦n-1. Then, *K(a,b,n)* is subtracting the number of solutions with roots≧0 by the number of solutions with roots >n.

To calculate the condition where at least 1 person receives an amount greater than n, we could first give him n objects. We then distribute the remaining a-n to b people.

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+x_b=a-n$

So when n objects were distributed to person, there would be $K(a-n,b)$ kinds of roots.
There would be $\binom{b}{1}$ ways to pick out this person
So under this condition, there would be $\binom{b}{1}K(a-n,b)$ possibilities.

To calculate the condition where at least 2 people receive an amount greater than n, we could first give them n objects. We then distribute the remaining a-2n to b people.

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+(x_{t_2}-n)+\ldots+x_b=a-2n$

So when n objects were distributed to person, there would be $K(a-2n,b)$ kinds of roots.
There would be $\binom{b}{2}$ ways to pick out these people
So under this condition, there would be $\binom{b}{2}K(a-2n,b)$ possibilities.

Likewise, to calculate the condition where at least i people receive an amount greater than n, we could first give them n objects. We then distribute the remaining a-in to b people.

$x_1+x_2+x_3+x_4+x_5+\ldots+(x_{t_1}-n)+\ldots+(x_{t_2}-n)+\ldots+(x_{t_i}-n)+\ldots+x_b=a-in$

So when n objects were distributed to person, there would be $K(a-in,b)$ kinds of roots.
There would be $\binom{b}{i}$ ways to pick out these people
So under this condition, there would be $\binom{b}{i}K(a-in,b)$ possibilities.

From the inclusion-exclusion principal

K(a,b,n)

$$=\binom{b}{0}K(a,b)-[\binom{b}{1}K(a-n,b)-\binom{b}{2}K(a-2n,b)+\ldots+(-1)^{i-1}\binom{b}{i}K(a-in,b)+\ldots+(-1)^{b-1}\binom{b}{b}K(a-bn,b)]$$

$$=\sum_{t=0}^{b}(-1)^t\binom{b}{t}K(a-nt,b)=\sum_{t=0}^{b}(-1)^t\binom{b}{t}\binom{a-nt+b-1}{a-nt}$$

It’s obvious that N(a,b,n)=K(a-b,b,n-1) <=> K(a,b,n)=N(a+b,b,n+1)

So $N(a,b,n)=\sum_{t=0}^{b}(-1)^t\binom{b}{t}\binom{a-(n-1)t-1}{a-b-(n-1)t}$

3. Eliminating the many-to-one nature

Suppose that the number of ways one can distribute a identical objects to b different boxes linearly symmetrically on a ring with each person receiving an amount of objects that’s greater than zero yet smaller than n is G.

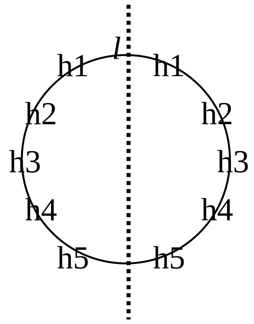
<u>When b is odd</u>

The only way for the distribution of objects to look symmetric are when the axis of symmetry passes one of the boxes. Suppose that the box receives an amount of *l* objects

Then the remaining a-*l* objects will be divided into b-1 boxes with each receiving an amount greater than 0 and smaller than n.

=>From symmetry, the remaining (a-*l*)/2 objects will be divided into (b-1)/2 boxes with each receiving an amount greater than 0 and smaller than n; which in turn equals N((b-1)/2,(a-*l*)/2,n).

$$G = \sum_{l=1}^{n-1} N\left(\frac{a-l}{2}, \frac{b-1}{2}, n\right)$$



When b is even

b boxes are put on a ring. The only two ways for the distribution of objects to look symmetric are when the axis of symmetry passes

1) two boxes (suppose that there are G_a ways)

2) nothing (suppose that there are G_b ways)

then the total number of ways is $G=G_a+G_b$

When the axis of symmetry passes two boxes

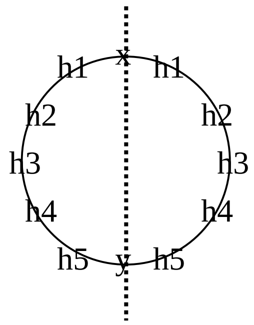
Suppose that the boxes on the axis receives amounts of objects x and y , x+y=*l*

Then the remaining a- *l* objects have to be divided into b-2 boxes with each receiving an amount greater than 0 and smaller than n.

=>From symmetry, (a- *l*)/2 objects have to be divided into (b-2)/2 boxes with each receiving an amount greater than 0 and smaller than n; which in turn equals N((b-1)/2,(a-*l*)/2,n).

For every different *l*, ways to distribute *l* objects into two boxes with each receiving an amount greater than 0 and smaller than n is N(*l*,2,n). But to avoid the problem of double-counting, we make x≧y so that the number of ways become [(N(*l*,2,n)+1)/2].

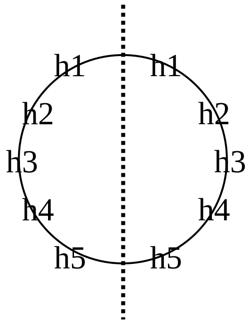
$$G_a = \sum_{l=1}^{n-1} N\left(\frac{a-l}{2}, \frac{b-2}{2}, n\right)\left[\frac{N(l,2,n)+1}{2}\right]$$



When the axis of symmetry does not pass any box

A total of a objects has to be divided into b boxes with each receiving an amount greater than 0 and smaller than n.

=>From symmetry, a/2 objects have to be divided into b/2 boxes with each receiving an amount greater than 0 and smaller than n.



$$G_b = N(\frac{a}{2}, \frac{b}{2}, n)$$

$$G = G_a + G_b = \sum_{l=1}^{n-1} N(\frac{a-l}{2}, \frac{b-2}{2}, n)\left\lceil \frac{N(l,2,n)+1}{2} \right\rceil + N(\frac{a}{2}, \frac{b}{2}, n)$$

Notice that from our definition, when a or b is odd G_b=0, so G=G_a.

IV. Results

A. Conditions of forming an EASE polygon

For EASE polygons the fixed angle θ=2π/ n,

(n≥4, n is a prime number)

the number of sides has to be r=n(2k+1),

and a total of nk parts have got to be formed in the partition sequence.

In the vector point of view, all the kinds of vectors have to appear the same amount of times in order for the vector sum to be zero, thus returning to the origin.

B. The upper and lower bound of the number of EASE polygons

1. The lower bound

N(2k+1,k,n)/2k- B/2

When k is odd

$$B = \sum_{l=1}^{n-1} N(\frac{2k+1-l}{2}, \frac{k-1}{2}, n)$$

When k is even

$$B = \sum_{l=1}^{n-1} N(\frac{2k+1-l}{2}, \frac{k-2}{2}, n)\left\lceil \frac{N(l,2,n)+1}{2} \right\rceil$$

A rougher lower bound: N(2k+1,k,n)/2k

2. The upper bound

N(2k+1,k,n)(n-1)/2nk+N(n(2k+1),nk,k)/2nk+T/2

When k is odd

$$T=\sum_{l=1}^{n-1}N(\frac{n(2k+1)-l}{2},\frac{nk-1}{2},n)$$

When k is even

$$T=\sum_{l=1}^{n-1}N(\frac{n(2k+1)-l}{2},\frac{nk-2}{2},n)\left[\frac{N(l,2,n)+1}{2}\right]$$

A rougher upper bound: N(n(2k+1),nk,n)

C. The destructive method of obtaining EASE polygons

So we write down all the partitions of n(2k+1) with nk parts and then expand the partition sequence into vector sequences. We then count the number of vectors corresponding to each kind of vector number and see if they are the same. If they are then that means the vector sum of this sequence is zero therefore. And we would have found an EASE polygon.

D. The constructive method of obtaining EASE polygons

1. The operation “flip”

Suppose ⋯hi-2, hi-1, hi, hi+1, hi+2… is a partition sequence.

case 1)
$h_i=3, h_{i-1}\leqq n-2$, $h_{i+1}\leqq n-2$,
…h_{i-2}, h_{i-1}, 3, h_{i+1}, h_{i+2}…
is converted to
…h_{i-2}, $h_{i-1}+1$, 1, $h_{i+1}+1$, h_{i+2}…

case 2)
$h_i=1, h_{i-1}\geqq 2$, $h_{i+1}\geqq 2$,
…h_{i-2}, h_{i-1}, 1, h_{i+1}, h_{i+2}…
is converted to…
h_{i-2}, $h_{i-1}-1$, 3, $h_{i+1}-1$, h_{i+2}…

This method to generate new sequences is called flip.

Theorem: The effect of flipping will only effect the vector arrangement of the parts h_{i-1}, h_i, h_{i+1} and not affect the total vector sum of the original sequence.

2. The operation "insert"

[Insert defined]

Create n partitions of 2 with 0 to n-1 being the first vector numbers

Insert the partition with the first vector number t into the original partition sequence after any partition with the first vector number being t+1

Theorem: Any EASE polygon could be inserted.

E. Other properties of the EASE polygon

1. First vectors

For a partition sequence r=h1+h2+h3……h(r－n)/2, the first vector of the t[th] partition

$$a_t \equiv (\sum_{k=1}^{t-1} h_k - 2(t-1)) \bmod n$$

.

Theorem: For all basic EASE polygons, every kind of vector number will appear the same amount of times amongst the first vectors no matter how many times an EASE polygon is flipped.

2. Proof of functions

We define the function N(a,b,n) to be the number of ways to distribute a identical objects to b persons such that each person receives an amount m, of objects with 0<m<n.

$$N(a,b,n) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \binom{a-(n-1)t-1}{a-b-(n-1)t}$$

3. Eliminating the many to one nature

The number of ways one can distribute a identical objects to b different boxes linearly symmetrically on a ring with each person receiving an amount of objects that's greater than zero yet smaller than n is G.

When b is odd

$$G = \sum_{l=1}^{n-1} N(\frac{a-l}{2}, \frac{b-1}{2}, n)$$

When b is even

$$G = \sum_{l=1}^{n-1} N(\frac{a-l}{2}, \frac{b-2}{2}, n) \left[\frac{N(l,2,n)+1}{2} \right] + N(\frac{a}{2}, \frac{b}{2}, n)$$

V. Conclusion and applications

In this research, ways to express EASE polygons with given fixed angles and fixed number of sides were first discussed. Then the conditions of forming EASE polygons were discussed. An upper and lower bound to the number of EASE polygons corresponding to a given number of sides and a fixed angle was obtained. However, the difference between the upper and lower bound remains. Therefore, algorisms to create new EASE polygons out of given ones were introduced to fill in the gap. Some qualitative results were obtained through the operations. The methods developed here in this research to solve problems of symmetric redundancy and partition sequences, could also be applied to more general problems.

VI. Reference

[1]George E. Andrews, A theorem on reciprocal polynomials with applications to permutations and compositions, American Mathematical Monthly 82 (1975), p830~833.

[2]George E. Andrews, Theory of Partitions ,The Encyclopedia of Mathematics and Its Applications Series, volume 2 , Addison-Wesley Publishing Company, New York, 1976.

評語

問題和想法新奇有趣，也能清楚生動的表達，整個理論的完整性則有加強的空間。