台灣二〇〇二年國際科學展覽會

科　　別：數學科

作品名稱：棋盤的費伯那契

得獎獎項：第三名
　　　　　美國第五十三屆國際科展團隊正選代表

學　　校：國立彰化高級中學

作　　者：吳韋達　林俊男　梁益昌

棋盤的費伯那契

壹・研究動機

在學校科研營的教材中，有一個題目，其內容相當於:「在一列格子中，放入黑棋與白棋。規定白棋不可連續放置，而黑棋不受此限，請問共有幾種可能的排列方式？」此題之答案，基本上就是鼎鼎大名的費伯那契數列(Fibonacci Sequence)，那麼，在此規則下，若將格子推廣爲 m 列 n 行的棋盤，那又如何呢？我們對此好奇不已。

貳・研究目的

(一).定義一:「相鄰」的兩格即爲有共同的邊(不包括共同角)

(二)規定：在長方形棋盤中，每一格皆放入黑棋或白棋，規定白棋不可和白棋相鄰，而黑棋沒有限制

例：

A	B
C	白

A：可放入白棋或黑棋

B、C：只可放入黑棋

(三).定義二：我們用 F (m , n) 表示在 m 列 n 行的棋盤，遵照上述規定放入白棋與黑棋,所有可能的排法的總數

(四).我們的目的：探討 F (m , n) 的性質，另外，我們也想探討不同形狀棋盤的方法數，例如：圓環狀棋盤。

參・研究設備與器材

紙、筆、計算機、個人電腦

肆．研究方法及過程

(一).F (m , n)的求法：

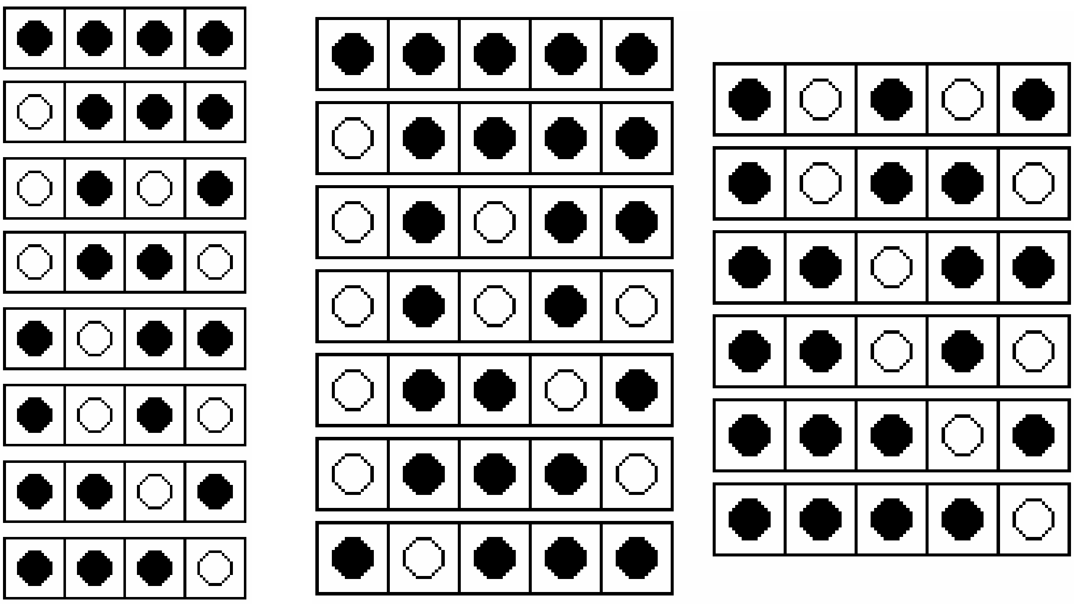
(甲).實際排列：

定義：顧名思義就是把每一種可能的排列組合一一列出，再清點所有個數.

優點；想法簡單

缺點；作業繁雜，當行數及列數增加，則可能排列方法數據增

例；如圖一，為一列四行及一列五行所有可能的排列方法



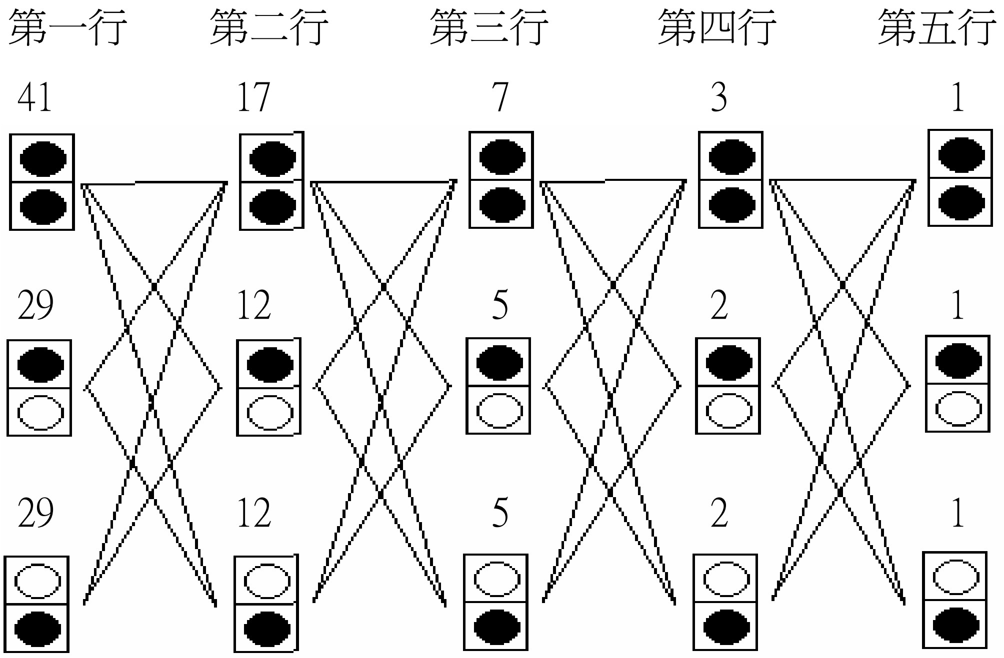
(圖一)

(乙).導向排列：

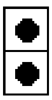
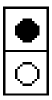
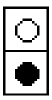
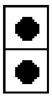
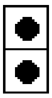
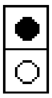
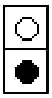
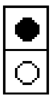
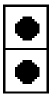
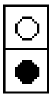
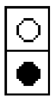
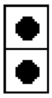
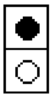
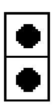
定義：見圖二為導向法的樹狀結構圖，圖為二列 5 行可能的排列個數

優點：計算速度加快，可處理較大的行列數

缺點：仍需要要大量紙張畫圖歸納



(圖二)

圖二中，為二列五行方法數的推導。我們先在第五行放入棋子，而這有 、、 等三種放法。再由第五行往回推至第四行，第四行中的 可和第五行中的 、、相鄰，而第四行中 ，只可與第五行中的 、相鄰，又第四行中的 ，只可與第五行中的 、相鄰，因此，到第四行的總方法數(即為到第三行  的方法數)為 3+2+2=7

種，依照此原理，即可快速推導至第一行的方法數(第一行方法數總合即為一列五行的總方法數)，按照同樣的方法，一定可以快速求出 F (m , n)。

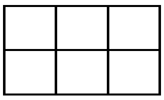
(丙).導向排列符號化：

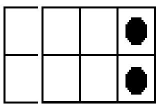
為方便計算，我們將棋盤符號化。

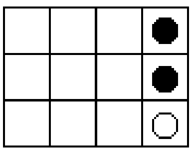
優點：(a)節省紙張上的繁雜圖示

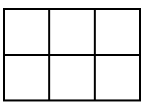
(b)標示清楚易懂

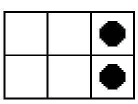
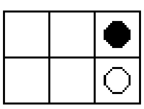
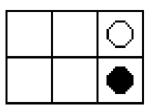
首先，我們將已放入黑棋的格子以 b 表示，而放入白棋的格子以 w 表示。

例： F() = F (2 , 3)

F() $= F\left(4,\ \begin{matrix} b \\ b \end{matrix}\right)$

F() $= F\left(4,\ \begin{matrix} b \\ b \\ w \end{matrix}\right)$

F (2 , 3) = F ()

= F() + F () + F ()

$= F\left(3,\ \begin{matrix} b \\ b \end{matrix}\right) + F\left(3,\ \begin{matrix} b \\ w \end{matrix}\right) + F\left(3,\ \begin{matrix} w \\ b \end{matrix}\right)$

$= \left[F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \bullet \\ \hline\end{array}\right) + F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \circ \\ \hline\end{array}\right) + F\left(\begin{array}{|c|c|}\hline & \circ \\ \hline & \bullet \\ \hline\end{array}\right) \right] + \left[F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \bullet \\ \hline\end{array}\right)\right.$

$\left. + F\left(\begin{array}{|c|c|}\hline & \circ \\ \hline & \bullet \\ \hline\end{array}\right) \right] + \left[F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \bullet \\ \hline\end{array}\right) + F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \circ \\ \hline\end{array}\right) \right]$

$= \left[F\left(2,\ \begin{matrix} b \\ b \end{matrix}\right) + F\left(2,\ \begin{matrix} b \\ w \end{matrix}\right) + F\left(2,\ \begin{matrix} w \\ b \end{matrix}\right) \right] + \left[F\left(2,\ \begin{matrix} b \\ b \end{matrix}\right)\right.$

$\left. + F\left(2,\ \begin{matrix} w \\ b \end{matrix}\right) \right] + \left[F\left(2,\ \begin{matrix} b \\ b \end{matrix}\right) + F\left(2,\ \begin{matrix} b \\ w \end{matrix}\right) \right]$

$= 3F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \bullet \\ \hline\end{array}\right) + 2F\left(\begin{array}{|c|c|}\hline & \bullet \\ \hline & \circ \\ \hline\end{array}\right) + 2F\left(\begin{array}{|c|c|}\hline & \circ \\ \hline & \bullet \\ \hline\end{array}\right)$

$= 3F\left(2,\ \begin{matrix} b \\ b \end{matrix}\right) + 2F\left(2,\ \begin{matrix} b \\ w \end{matrix}\right) + 2F\left(2,\ \begin{matrix} w \\ b \end{matrix}\right)$

$= 3\left[F\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}\right) + F\left(\begin{array}{|c|}\hline \bullet \\ \hline \circ \\ \hline\end{array}\right) + F\left(\begin{array}{|c|}\hline \circ \\ \hline \bullet \\ \hline\end{array}\right) \right] + 2\left[F\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}\right) + F\left(\begin{array}{|c|}\hline \circ \\ \hline \bullet \\ \hline\end{array}\right) \right]$

$+ 2\left[F\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}\right) + F\left(\begin{array}{|c|}\hline \bullet \\ \hline \circ \\ \hline\end{array}\right) \right]$

$= 3\left[F\left(1,\ \begin{matrix} b \\ b \end{matrix}\right) + F\left(1,\ \begin{matrix} b \\ w \end{matrix}\right) + F\left(1,\ \begin{matrix} w \\ b \end{matrix}\right) \right] + 2\left[F\left(1,\ \begin{matrix} b \\ b \end{matrix}\right)\right.$

$\left. + F\left(1,\ \begin{matrix} w \\ b \end{matrix}\right) \right] + 2\left[F\left(1,\ \begin{matrix} b \\ b \end{matrix}\right) + F\left(1,\ \begin{matrix} b \\ w \end{matrix}\right) \right]$

$= 7F\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}\right) + 5F\left(\begin{array}{|c|}\hline \bullet \\ \hline \circ \\ \hline\end{array}\right) + 5F\left(\begin{array}{|c|}\hline \circ \\ \hline \bullet \\ \hline\end{array}\right)$

$= 17$

另外，我們定義 $F(1, 0) = F\left(\begin{array}{|c|}\hline \bullet \\ \hline\end{array}\right) = 1$

$F(2, 0) = F\left(\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline\end{array}\right) = 1$

乃至 $F(m, 0) = 1$ ， $\forall m \in N$.

(丁).程式計算：

定義：延續第二、三點之邏輯，利用電腦進行其繁雜的運算

優點：計算級數數據方面的處理速度大增

例如：例如由美國微軟公司(MicroSoft)研發的 QuickBasic 軟體，進行程式編輯，以下爲三行 n 列的原始程式碼(三、四、五、六行 n 列的原始程式碼，見附表一)

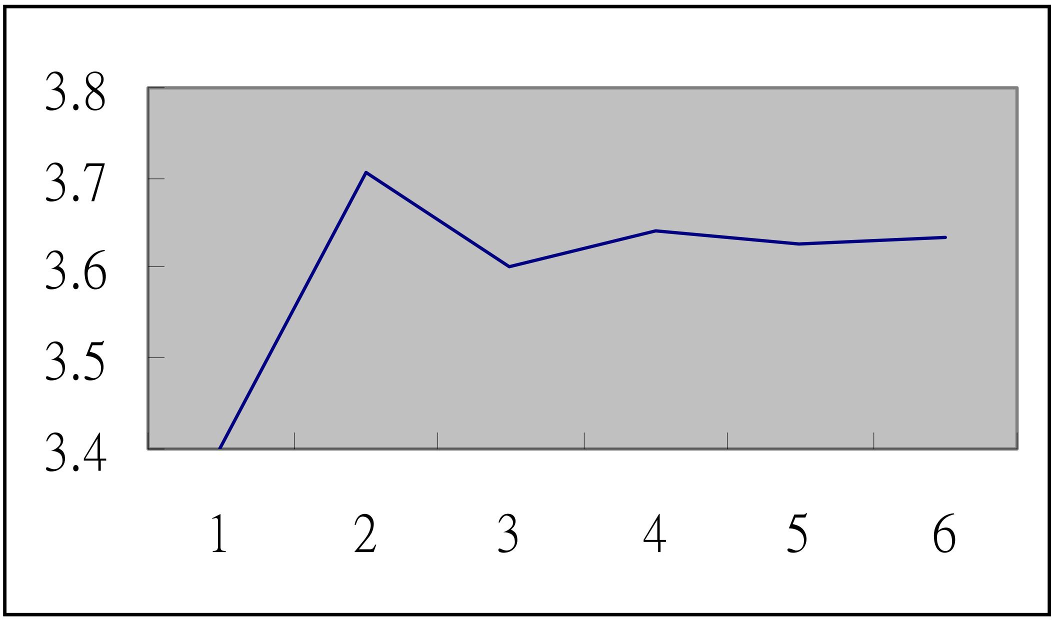
10 CLS	清除螢幕
20 DIM A#(5),B#(5)	宣告設定五組變數(因爲三列一行有五種合法的放法)
30 FOR I=1TO 5 40 A#(I)=1 50 B#(I)=1 60 NEXT I	設定所有 AB 起始值皆爲 1
70 INPUT N%	輸入行數
80 FOR m=1 TO N% 90 A#(1)=B#(1)+B#(2)+B#(3)+B#(4)+B#(5) 100 A#(2)=B#(1)+B#(3)+B#(4) 110 A#(3)=B#(1)+B#(2)+B#(4)+B#(5) 120 A#(4)=B#(1)+B#(2)+B#(3) 130 A#(5)=B#(1)+B#(3)	利用導向排列方法中的 F(2,m)及所有殘缺項彼此間的關係，求出 F(2,m+1)
140 FOR J=1 TO 5 150 SWAP A#(J),B#(J) 160 NEXT J	A、B 變數內容交換，B 變數之空間留給(m+2)行
170 PRINT B#(1),m	列印計算結果
180 NEXT m	進行下一次運算

以下爲利用電腦所計算之數據表(表一)

行\列	1	2	3	4	5	6	7	8
一	2	3	5	8	13	21	34	55
二	3	7	17	41	99	239	577	1393
三	5	17	63	227	827	2999	10897	39561
四	8	41	227	1234	6743	36787	200798	1095851
五	13	99	827	6743	55447	454385	3729091	30584687
六	21	239	2999	36787	454385	5598861	69050253	851302029

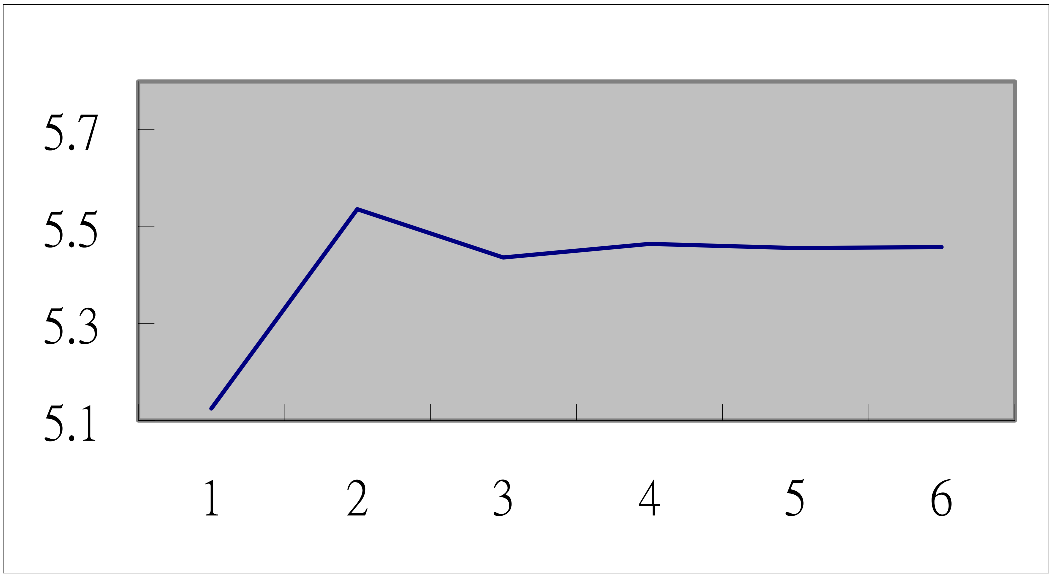
由數據觀察分析，可知 $\forall m \in N$, 當$n \to \infty$時，$\dfrac{F(m,n)}{F(m,n-1)}$會趨近一個定值(如圖三、四、五、六)。

下圖爲 F(3,n+1)/F(3,n)之圖形變化



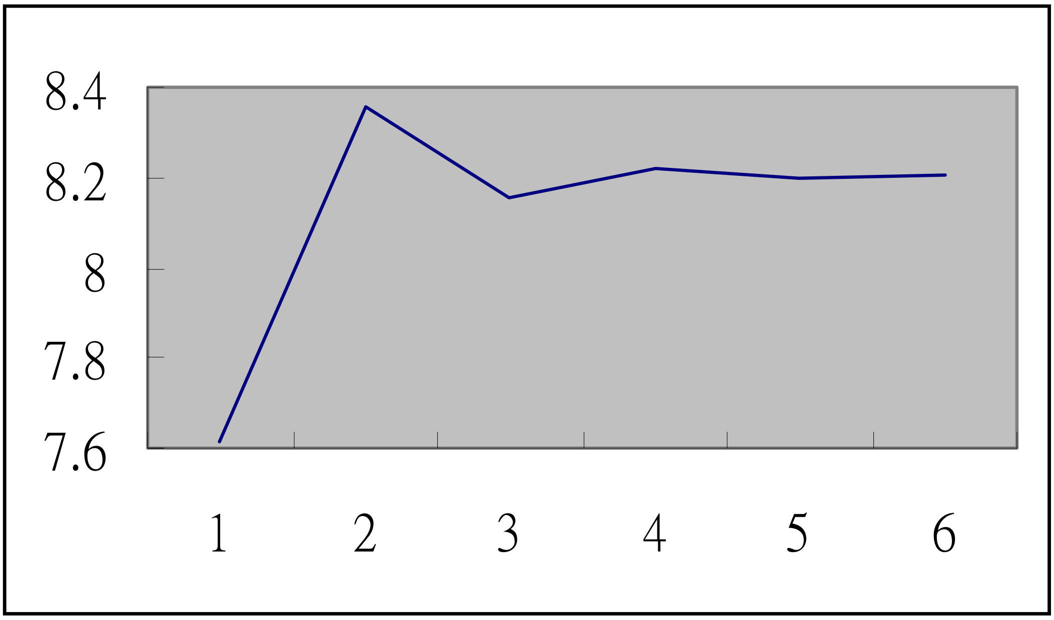
(圖三)

下圖為 F(4,n+1)/F(4,n)之圖形變化



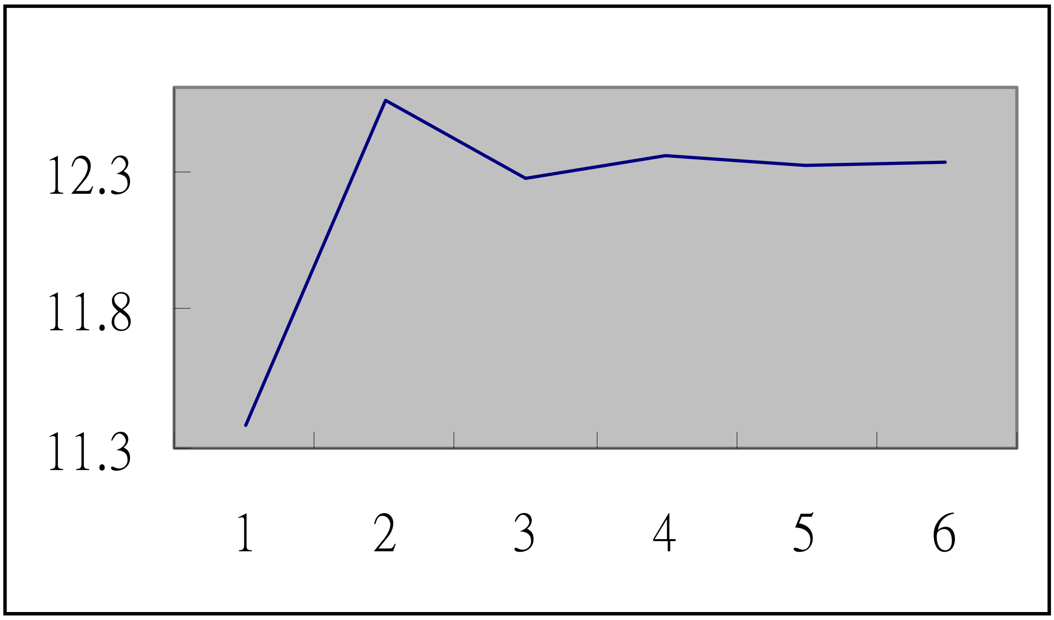
(圖四)

下圖為 F(5,n+1)/F(5,n)之圖形變化



(圖五)

下圖爲 F(6,n+1)/F(6,n)之圖形變化



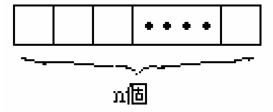
(圖六)

(二).F (m , n)的性質：

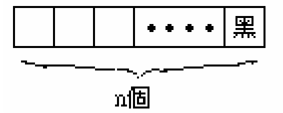
首先我們知道 F (m , n) = F (n , m)，底下我們將固定列數 m，而探討數列 <F(m,n)>, $n \in N$ 之性質。

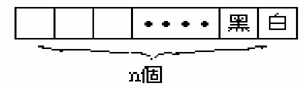
(甲) .一列性質之探討：

F (1 , n)基本上就是費伯那契數列，其證明如下：

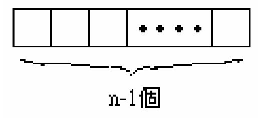
F (1 , n) = F(

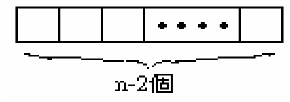
)

= F(

) + F(

)

= F(

) + F(

)

所以，我們有

<命題一> $F(1, n) = F(1, n-1) + F(1, n-2)$

又 $F(1, 0) = 1$ ， $F(1, 1) = 2$

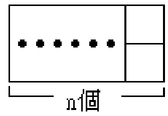
所以 $F(1, n)$ 就形成了費伯那契數列 1,2,3,5,8,13,21……，只比平常的費伯納契數列少一個”1”。

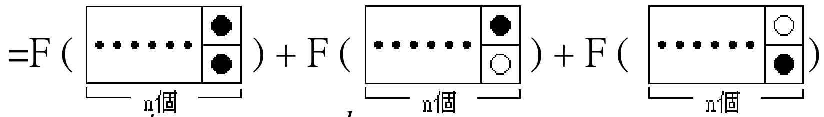
(乙) .二列性質之探討：

由表(一)直接觀察可猜測 2 列的遞迴公式爲：

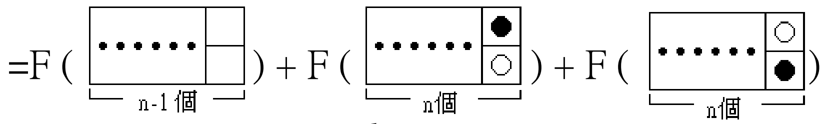
<命題二> $F(2, n) = 2F(2, n-1) + F(2, n-2) \quad \forall n \geq 3$

證明：

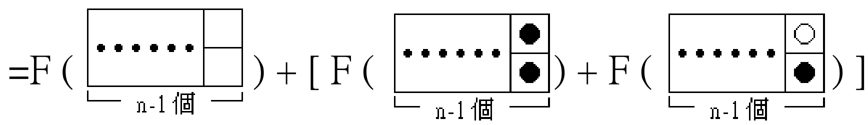
F (2 , n) =F ()

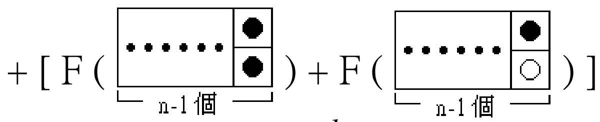


$$=F\left(n, {b \atop b}\right)+F\left(n, {b \atop w}\right)+F\left(n, {w \atop b}\right)$$

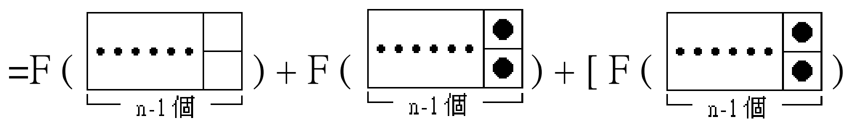


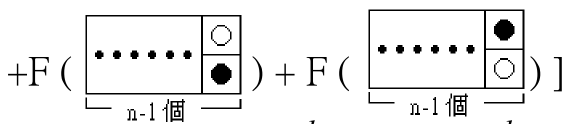
$$=F(2, n-1)+F\left(n, {b \atop w}\right)+F\left(n, {w \atop b}\right)$$



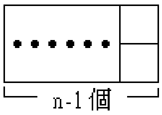
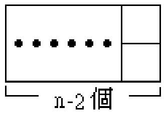
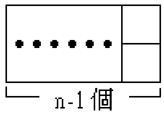


$$=F(2, n-1)+\left[F\left(n-1, {b \atop b}\right)+F\left(n-1, {w \atop b}\right)\right]+\left[F\left(n-1, {b \atop b}\right)+F\left(n-1, {b \atop w}\right)\right]$$

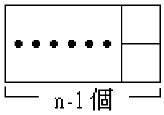
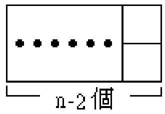




$$=F(2, n-1)+F\left(n-1, {b \atop b}\right)+\left[F\left(n-1, {b \atop b}\right)+F\left(n-1, {w \atop b}\right)+F\left(n-1, {b \atop w}\right)\right]$$

= F () + F () + F ()

= F (2 , n-1) + F (2 , n-2) + F (2 , n-1)

= 2 F () + F ()

= 2 F (2 , n-1) + F (2 , n-2)，證畢。

接下來我們想求$\frac{F(2,n)}{F(2,n-1)}$之極限(因為在參考書籍中，利用極限值可求出比內(binet)公式，即為一列的遞迴公式，因此我們沿用同一方法)

令 $\frac{F(2,n)}{F(2,n-1)} = b_n$ ， $\frac{F(2,n-1)}{F(2,n-2)} = b_{n-1}$.

由 $\frac{F(2,n)}{F(2,n-1)} = 2 + \frac{F(2,n-2)}{F(2,n-1)}$

可得 $b_n = 2 + \frac{1}{b_{n-1}}$

設有極限存在，且為 X

則 $\lim_{n \to \infty} b_n = \lim_{n \to \infty} (2 + \frac{1}{b_{n-1}})$

因此 $X = 2 + \frac{1}{X}$

所以 $X^2 - 2X - 1 = 0$

解得兩根為 $1+\sqrt{2}$ 及 $1-\sqrt{2}$

另外，我們知道對 F (1 , n) 而言，有一個比內公式:

$$F(1,n) = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{(n+1)} - (\frac{1-\sqrt{5}}{2})^{(n+1)}]$$

(請注意，我們的數列 < F (1 , n) > 是 1,2,3,5,8,13……)

因此，我們推測 F(2,n) 應該有一個類似的公式

首先我們假設

$$F(2,n)=\alpha(1+\sqrt{2})^n+\beta(1-\sqrt{2})^n$$

因爲 $F(2,1)=3$ 、 $F(2,2)=7$ 所以

$$\begin{cases} 3=\alpha(1+\sqrt{2})+\beta(1-\sqrt{2}) \\ 7=\alpha(1+\sqrt{2})^2+\beta(1-\sqrt{2})^2 \end{cases}$$

由此可得 $\alpha=\frac{1+\sqrt{2}}{2}$ 、 $\beta=\frac{1-\sqrt{2}}{2}$ 因此我們推得公式爲

$$F(2,n)=\frac{1}{2}\left[\ (1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\ \right]$$

我們再用 $F(2,3)$ 、 $F(2,4)$ 、 $F(2,5)$ 來檢查都符合

所以我們相信公式正確，也就是說，我們相信有性質如下：

<命題三> $F(2,n)=\frac{1}{2}\left[\ (1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\ \right]\ ,\forall n\geq 0$

證明：

1. 當 n=1 時，F(2,1) $=\frac{1}{2}[(1+\sqrt{2})^{1+1}+(1-\sqrt{2})^{1+1}]$

$=\frac{1}{2}[(3+2\sqrt{2})+(3-2\sqrt{2})]$

$=3$

n= 2 時 , $F(2,2) = \frac{1}{2}[(1+\sqrt{2})^{2+1} + (1-\sqrt{2})^{2+1}]$

$= \frac{1}{2}[(7+5\sqrt{2}) + (7-5\sqrt{2})]$

$=7$

2. 設 $1 \le n \le k$ 時 $(k \ge 2)$

$F(2,n) = [(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}]$ 皆成立

則 F(2,k+1)=2F(2,k)+F(2,k−1)

$= 2\cdot\frac{1}{2}[(1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1}] + \frac{1}{2}[(1+\sqrt{2})^{k} + (1-\sqrt{2})^{k}]$

$= \frac{1}{2}[2(1+\sqrt{2})^{k+1} + 2(1-\sqrt{2})^{k+1}] + \frac{1}{2}[(1+\sqrt{2})^{k} + (1-\sqrt{2})^{k}]$

$= \frac{1}{2}[(1+\sqrt{2})^{k}(3+2\sqrt{2}) + (1-\sqrt{2})^{k}(3-2\sqrt{2})]$

$= \frac{1}{2}[(1+\sqrt{2})^{k+2} + (1-\sqrt{2})^{k+2}]$

由 1 , 2 及數學歸納法得證

現在既然 $F(2,n) = \frac{1}{2}[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}], \forall n \ge 0$

所以我們可得 $\lim_{n\to\infty}\frac{F(2,n)}{F(2,n-1)} = \lim_{n\to\infty}\frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}} = 1+\sqrt{2}$

也就是說 $\lim_{n\to\infty}\frac{F(2,n)}{F(2,n-1)}$ 的確存在而且就是 $1+\sqrt{2}$.

(丙) .三列性質之探討：

在探討三列性質前，我們要知道一些等式

$$F\begin{pmatrix} b \\ n,b \\ b \end{pmatrix}=F\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix}+F\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix}$$

$$F\begin{pmatrix} w \\ n,b \\ b \end{pmatrix}=F\begin{pmatrix} b \\ n,b \\ w \end{pmatrix}=F\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix}$$

$$F\begin{pmatrix} b \\ n,w \\ b \end{pmatrix}=F\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix}+F\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix}$$

$$F\begin{pmatrix} w \\ n,b \\ w \end{pmatrix}=F\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+F\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix}$$

原本我們打算模仿求二列遞迴公式的方法，實際演算後發現三列的遞迴公式並不如預期的那麼簡單，一開始我們推導如下：

$$\begin{aligned}\mathrm{F}(3,n) &= \mathrm{F}\begin{pmatrix} b \\ n,b \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} w \\ n,b \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n,w \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n,b \\ w \end{pmatrix}+\mathrm{F}\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} \\ &= \mathrm{F}(3,n-1) + \mathrm{F}\begin{pmatrix} w \\ n,b \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n,w \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n,b \\ w \end{pmatrix}+\mathrm{F}\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} \\ &= \mathrm{F}(3,n-1)+\mathrm{F}\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix}+\mathrm{F}\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix}+\end{aligned}$$

$$\mathrm{F}\begin{pmatrix} & w \\ n-1, & b \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & w \\ n-1, & b \\ & w \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n-1, & b \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & w \\ n-1, & b \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n-1, & w \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n-1, & b \\ & b \end{pmatrix} +$$

$$\mathrm{F}\begin{pmatrix} & b \\ n-1, & w \\ & b \end{pmatrix}$$

$$= \mathrm{F}(3, n-1) + 4\,\mathrm{F}(3, n-2) + 2\mathrm{F}\begin{pmatrix} & w \\ n-1, & b \\ & b \end{pmatrix} + 3\mathrm{F}\begin{pmatrix} & b \\ n-1, & w \\ & b \end{pmatrix} + 2\mathrm{F}\begin{pmatrix} & b \\ n-1, & b \\ & w \end{pmatrix} + \mathrm{F}\begin{pmatrix} & w \\ n-1, & b \\ & w \end{pmatrix}$$

$$= \mathrm{F}(3, n-1) + 4\mathrm{F}(3, n-2) + 2\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & b \end{pmatrix} + 2\mathrm{F}\begin{pmatrix} & b \\ n-2, & w \\ & b \end{pmatrix} + 2\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & w \end{pmatrix} + 3\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & b \end{pmatrix}$$

$$+ 3\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & w \end{pmatrix} + 3\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & b \end{pmatrix} + 3\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & w \end{pmatrix} + 2\,\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & b \end{pmatrix} + 2\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & b \end{pmatrix} +$$

$$2\,\mathrm{F}\begin{pmatrix} & b \\ n-2, & w \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n-2, & w \\ & b \end{pmatrix}$$

$$= \mathrm{F}(3, n-1) + 4\,\mathrm{F}(3, n-2) + 8\mathrm{F}(3, n-3) + 5\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-2, & w \\ & b \end{pmatrix}$$

$$+ 5\,\mathrm{F}\begin{pmatrix} & b \\ n-2, & b \\ & w \end{pmatrix} + 3\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & w \end{pmatrix}$$

$$= \mathrm{F}(3, n-1) + 4\,\mathrm{F}(3, n-2) + 8\mathrm{F}(3, n-3) + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & w \\ & b \end{pmatrix}$$

$$+ 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & w \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & w \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & w \\ n-2, & b \\ & w \end{pmatrix}$$

$$+ 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & b \end{pmatrix} + 5\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & w \\ & b \end{pmatrix} + 3\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & b \end{pmatrix} + 3\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & w \\ & b \end{pmatrix}$$

$$= \mathrm{F}(3, n-1) + 4\,\mathrm{F}(3, n-2) + 8\mathrm{F}(3, n-3) + 18\,\mathrm{F}(3, n-4) + 10\,\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & b \end{pmatrix}$$

$$+13\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & w \\ & b \end{pmatrix} +10\,\mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & w \end{pmatrix} +5\,\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & w \end{pmatrix}$$

$=\ldots\ldots\ldots\ldots\ldots\ldots$

因為有三種煩人的殘缺項（$\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & b \end{pmatrix} = \mathrm{F}\begin{pmatrix} & b \\ n-3, & b \\ & w \end{pmatrix}$、$\mathrm{F}\begin{pmatrix} & b \\ n-3, & w \\ & b \end{pmatrix}$、$\mathrm{F}\begin{pmatrix} & w \\ n-3, & b \\ & w \end{pmatrix}$）無法消去，因此在這裡卡了好一陣子，最後終於想到先利用 F(m,n)和這三種殘缺項的關係，列出三個方程式，再解方程式，而將殘缺項以 F(m,n)表示出來，然後再代回去他們的另一個關係式中，即可輕易的求出三列的遞迴公式，詳細運算過程如下:

$$\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n, & w \\ & b \end{pmatrix} + \mathrm{F}\begin{pmatrix} & b \\ n, & b \\ & w \end{pmatrix} + \mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & w \end{pmatrix} = \mathrm{F}(3,n) - \mathrm{F}(3,n-1)$$

$$2\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & b \end{pmatrix} + 3\mathrm{F}\begin{pmatrix} & b \\ n, & w \\ & b \end{pmatrix} + 2\mathrm{F}\begin{pmatrix} & b \\ n, & b \\ & w \end{pmatrix} + \mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & w \end{pmatrix} = \mathrm{F}(3,n+1) - \mathrm{F}(3,n) - 4\mathrm{F}(3,n-1)$$

$$5\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & b \end{pmatrix} + 5\mathrm{F}\begin{pmatrix} & b \\ n, & w \\ & b \end{pmatrix} + 5\mathrm{F}\begin{pmatrix} & b \\ n, & b \\ & w \end{pmatrix} + 3\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & w \end{pmatrix} = \mathrm{F}(3,n+2) - \mathrm{F}(3,n+1) - 4\mathrm{F}(3,n) - 8\mathrm{F}(3,n-1)$$

$$10\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & b \end{pmatrix} + 13\mathrm{F}\begin{pmatrix} & b \\ n, & w \\ & b \end{pmatrix} + 10\mathrm{F}\begin{pmatrix} & b \\ n, & b \\ & w \end{pmatrix} + 5\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & w \end{pmatrix} = \mathrm{F}(3,n+3) - \mathrm{F}(3,n+2) - 4\mathrm{F}(3,n+1) - 8\mathrm{F}(3,n) - 18\mathrm{F}(3,n-1)$$

令 $X=\mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & b \end{pmatrix} = \mathrm{F}\begin{pmatrix} & b \\ n, & b \\ & w \end{pmatrix}$ ， $Y=\mathrm{F}\begin{pmatrix} & b \\ n, & w \\ & b \end{pmatrix}$ ， $Z = \mathrm{F}\begin{pmatrix} & w \\ n, & b \\ & w \end{pmatrix}$

則我們由上式可得

2X+Y+Z = $F(3,n)$ - $F(3,n-1)$ ……………………………………………………..(1)

4X+3Y+Z = $F(3,n+1)$ - $F(3,n)$ - 4$F(3,n-1)$ ………………………………………(2)

10X+5Y+3Z = $F(3,n+2)$ - $F(3,n+1)$ - 4$F(3,n)$ - 8$F(3,n-1)$………………………(3)

20X+13Y+5Z = $F(3,n+3)$ - $F(3,n+2)$ - 4$F(3,n+1)$ - 8$F(3,n)$ - 18$F(3,n-1)$………...(4)

由 (1) (2) (3)

解 X = $\frac{1}{2}F(3,n+2)-F(3,n+1)-\frac{5}{2}F(3,n)-F(3,n-1)$

Y = $-\frac{1}{2}F(3,n+2)+\frac{3}{2}F(3,n+1)+\frac{3}{2}F(3,n)-\frac{1}{2}F(3,n-1)$

Z = $-\frac{1}{2}F(3,n+2)+\frac{1}{2}F(3,n+1)+\frac{9}{2}F(3,n)+\frac{3}{2}F(3,n-1)$

再代入 (4)

20X+13Y+5Z = $F(3,n+3)$ - $F(3,n+2)$ - 4$F(3,n+1)$ - 8$F(3,n)$ - 18$F(3,n-1)$

即得： $F(3,n) = 2F(3,n-1) + 6F(3,n-2) - F(3,n-4)$

所以我們得到：

<命題四> 三列的遞迴公式爲：

F(3 , 0) = 1，F (3 , 1) = 5，F (3 , 2) = 17

$F(3,n) = 2F(3,n-1) + 6F(3,n-2) - F(3,n-4)$ ∀n≥4,

接下來，我們一樣想求$\frac{F(3,n)}{F(3,n-1)}$的極限。

令 $b_n = \frac{F(3,n)}{F(3,n-1)}$

所以由 $\frac{F(3,n)}{F(3,n-1)} = 2\frac{F(3,n-1)}{F(3,n-1)} + 6\frac{F(3,n-2)}{F(3,n-1)} - \frac{F(3,n-4)}{F(3,n-1)}$

可得 $b_n = 2 + \frac{6}{b_n - 1} - \frac{1}{b_{n-3}b_{n-2}b_{n-1}}$.

設有極限值存在且為 X ,

得到 $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(2+\frac{6}{b_n-1}-\frac{1}{b_{n-3}b_{n-2}b_{n-1}})$,

所以 $X = 2 + \frac{6}{X} - \frac{1}{X^3}$,

因此 $X^4 = 2X^3 + 6X^2 - 1$,

即 $X^4 - 2X^3 - 6X^2 + 1 = 0$.

利用勘根定理得知：此方程式有四個根為 α_1，α_2，α_3，α_4 且

$-2 < \alpha_1 < -1$，$-1 < \alpha_2 < 0$，$0 < \alpha_3 < 1$，$3 < \alpha_4 < 4$

事實上，重複利用勘根定理，我們可得

$-1.568 < \alpha_1 < -1.567$, $-0.453 < \alpha_2 < -0.452$,

$0.388 < \alpha_3 < 0.389$, $3.631 < \alpha_4 < 3.632$,

所以我們猜測

<命題五> 存在常數 C_1，C_2，C_3，C_4 使得

$$F(3,n) = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + C_4\alpha_4^n,\ \forall n \geq 0 .$$

證明：

首先將 n = 0 , 1 , 2 , 3 代入命題五中,可得

$$C_1 + C_2 + C_3 + C_4 = 1,$$

$$C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 + C_4\alpha_4 = 5,$$

$$C_1\alpha_1^3 + C_2\alpha_2^3 + C_3\alpha_3^3 + C_4\alpha_4^3 = 63,$$

$$C_1\alpha_1^2 + C_2\alpha_2^2 + C_3\alpha_3^2 + C_4\alpha_4^2 = 17,$$

又因爲

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{vmatrix} = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4) \neq 0$$

所以必可解得出唯一解 C_1, C_2, C_3, C_4 之值

現在假設,對某個 $k \geq 4$ ，我們有

$$\text{F(3,n)} = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + C_4\alpha_4^n, \forall 0 \leq n \leq k-1.$$

則 F(3,k) = 2 F(3,k-1) + 6 F(3,k-2) - F (3,k-4)

$$= 2[C_1\alpha_1^{k-1} + C_2\alpha_2^{k-1} + C_3\alpha_3^{k-1} + C_4\alpha_4^{k-1}] + 6[C_1\alpha_1^{k-2} + C_2\alpha_2^{k-2} + C_3\alpha_3^{k-2} + C_4\alpha_4^{k-2}]$$

$$-[C_1\alpha_1^{k-4} + C_2\alpha_2^{k-4} + C_3\alpha_3^{k-4} + C_4\alpha_4^{k-4}]$$

$$= C_1\alpha_1^{k-4}(2\alpha_1^3 + 6\alpha_1^2 - 1) + C_2\alpha_2^{k-4}(2\alpha_2^3 + 6\alpha_2^2 - 1)$$

$$+ C_3\alpha_3^{k-4}(2\alpha_3^3 + 6\alpha_3^2 - 1) + C_4\alpha_4^{k-4}(2\alpha_4^3 + 6\alpha_4^2 - 1)$$

$$= C_1\alpha_1^{k-4}\alpha_1^4 + C_2\alpha_2^{k-4}\alpha_2^4 + C_3\alpha_3^{k-4}\alpha_3^4 + C_4\alpha_4^{k-4}\alpha_4^4$$

$$= C_1\alpha_1^k + C_2\alpha_2^k + C_3\alpha_3^k + C_4\alpha_4^k.$$

所以 $\forall n \geq 0, F(3,n) = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + C_4\alpha_4^n$ 而且

$$\lim_{n\to\infty}\frac{F(3,n)}{F(3,n-1)} = \lim_{n\to\infty}\frac{\alpha_4^n(C_4 + C_3(\frac{\alpha_3}{\alpha_4})^n + C_2(\frac{\alpha_2}{\alpha_4})^n + C_1(\frac{\alpha_1}{\alpha_4})^n)}{\alpha_4^{n-1}(C_4 + C_3(\frac{\alpha_3}{\alpha_4})^{n-1} + C_2(\frac{\alpha_2}{\alpha_4})^{n-1} + C_1(\frac{\alpha_1}{\alpha_4})^{n-1})} = \alpha_4$$,証畢

(丁) .四列以上性質之探討：

利用類似的方法我們可探討 m ≥ 4 的情形。

例如，m = 4 時，我們有

<命題六> $F(4,n)=4F(4,n-1)+9F(4,n-2)-5F(4,n-3)-4F(4,n-4)+F(4,n-5), \forall n \geq 5$,

而 F(4,0) =1, F (4,1) = 8, F(4,2) = 41, F(4,3) = 227, F(4,4) = 1234.

F(4,5) = 6743.

而用勘跟定理，我們知道，用來求極限方程式(即特徵方程式)

$$X^5 - 4X^4 - 9X^3 + 5X^2 + 4X - 1 = 0$$

有五相異實根 $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ ，其中

$-1.79087<\beta_1<-1.79086$, $-0.63176<\beta_2<0.63175$,

$0.21634<\beta_3<0.21635$, $0.74856<\beta_4<0.74857$,

$5.45770<\beta_5<5.45771$.

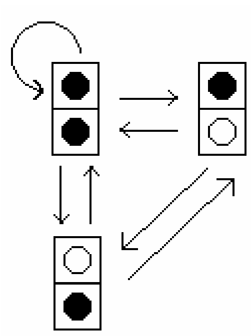
因此，我們可求得常數 $d_1, d_2, d_3, d_4 . d_5$ 而得到比內型公式如下：

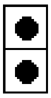
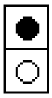
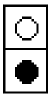
$$F(4,n) = d_1\beta_1^n + d_2\beta_2^n + d_3\beta_3^n + d_4\beta_4^n + d_5\beta_5^n .$$

(戊) .矩陣求法：

原先利用「導向排列」求排列個數的方法，其實也可以用「圖」表示，進而利用 Hamilton 定理，求出 m 列的遞迴公式，避開殘缺項問題，希望能藉由矩陣運算來簡化殘缺項反求法之複雜現象，並且希望藉由觀察矩陣的變化規則求得統一的遞迴公式。

例如：列數 m=2 時



若將  看成 1，看成 2，看成 3，則上圖可表示成矩陣 $\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$

第(1,2)元=1，代表 1 和 2 可以配在一起

第(2,2)元=0，代表 2 和 2 不能配在一起

因此，2 行 2 列的排法數可以矩陣 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$表示，所以 F(2,2)=7 即爲此矩陣中各元之總合，而 2 列 n 行的排法數可以矩陣 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n$表示，其中矩陣中各單位元素總合即爲 F(2,n)。

令 $A=\begin{pmatrix}1&1&1\\1&0&1\\1&1&0\end{pmatrix}$，由於 A 的特徵多項式爲 $-(X^3-X^2-3X-1)$，所以，由 Hamilton

定理得到 $A^3-A^2-3A-I=0$

所以，$A^{n+3}-A^{n+2}-3A^{n+1}-A^n=0$

因爲 F(2,n+2)=An 的各元之和

所以可得遞迴公式，F(2,n+2)=F(2,n+1)+3F(2,n)+F(2,n-1)

同理，我們可以利用矩陣求出三列以上的遞迴公式

但令我們困擾的是：利用 Hamilton 定理求出的公式並非最簡的遞迴公式，因爲

F(2,n+2)=2F(2,n+1)+F(2,n)

但另一方面，利用「矩陣求法」卻可幫助我們解釋一些現象。

1.我們發現利用「殘缺項反求法」所求出的遞迴公式中，領導係數爲 1，且其他各係數皆爲整數。試問：任何列數 m 所對應的遞迴公式，當其領導係數爲 1 時，其他各係數是否也皆爲整數?

當我們用「矩陣求法」時，發現這矩陣的各元皆爲整數，所以利用 Hamilton 定理可得到一遞迴公式，其領導項的係數爲 1，而所有係數皆爲整數，再由「高斯定理」即可看出：最簡遞迴公式中，若領導係數爲 1，其他各係數皆爲整數。我們相信用「殘缺項反求法」求得的遞迴公式極爲最簡公式（但我們迄今仍無法證明），所以我們相信對

任何列數 m 的遞迴公式，當其領導係數為 1 時，其他各係數也皆為整數。

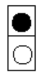
2.因為 A 是對稱矩陣，根據矩陣理論，矩陣所有的特徵值皆為實數，且此矩陣一定可被對角化，因此，我們就可看出不管列數 m 為何，其所對應之特徵方程式的跟皆為實數，且比內型公式恆存在。

(己) .殘缺項個數的探討：

在求 F(m,n)的過程中，我們發現殘缺項個數會隨著列數 m 改變，且對所求出的特徵方程式中的最高次有影響，因此，我們試著探討殘缺項與 m 的關係，希望在探討過程中，能啓發些許靈感。

在探討前，我們將原有的數據加以分析歸納，整理一表如下:

m 列	對稱型個數	非對稱型個數	殘缺項個數
二	0	1	1
三	2	1	3
四	1	3	4
五	4	4	8
六	2	9	11
七	7	13	20
八	4	25	29
九	12	38	50

註:
此表中，將上下顛倒視為同一種，例如:  和  為同一種。

對稱型排列的定義:由中央分隔線(偶數列)或中正中央的棋子(奇數列)向上及向下的棋子呈現對稱排列。

我們發現在奇數列與偶數列的特性不同，奇數列的中點為一棋子，而

偶數列中點爲一分隔線，因此我們將兩種情況分開討論:

(1) 奇數列:

a.對稱型

我們先探討奇數列，奇數列最中間那棋子(以下稱爲中位棋)將棋格分割成上下兩區，若是對稱圖形，則可視爲只有其中一區在變化，另一區爲其之相反排列，因此當對稱圖形且中位棋爲黑棋，則其個數爲[F($\frac{m-1}{2}$,1)-1]，若爲對稱圖形且中位棋爲白棋，則其個數爲[F($\frac{m-3}{2}$,1)]。

b.非對稱型

至於非對稱型，其必有與之上下顛倒之排法，所以非對稱型的個數(上下顛倒視爲一種)其個數爲 $\left[\frac{F(m,1)-F(1,\begin{smallmatrix}b\\ \vdots\\ b\end{smallmatrix})-\text{對稱型的個數}}{2}\right]$

因此奇數列殘缺項個數

$$S(m,1)=\left[F\left(\frac{m-1}{2},1\right)-1\right]+\left[F\left(\frac{m-3}{2},1\right)\right]+\left[\frac{F(m,1)-F(1,\begin{smallmatrix}b\\ \vdots\\ b\end{smallmatrix})-\text{對稱型的個數}}{2}\right]$$

令 $\frac{m-3}{2}=k$

經整理，

得奇數列殘缺項個數，

S(2k+3)= $\frac{F(1,2k+3)+F(1,k+1)+F(1,k)-2}{2}$

(2).偶數列

接下來，我們討論偶數列情況，和奇數列一樣地，分為對稱型與非對稱型來討論：

對稱型：

已知在中央分隔線兩旁必為 $\begin{matrix}b\\b\end{matrix}$ ，所以其個數為[F($\frac{m-2}{2}$,1)-1]

非對稱型：

而非對稱型方法數為 $\left[\frac{F(m,1)-F(1,\begin{matrix}b\\ \vdots\\ b\end{matrix})-\text{對稱型的個數}}{2}\right]$

所以偶數列殘缺項個數公式為

S(m,1)= [F($\frac{m-2}{2}$,1)-1]+ $\left[\frac{F(m,1)-F(1,\begin{matrix}b\\ \vdots\\ b\end{matrix})-\text{對稱型的個數}}{2}\right]$

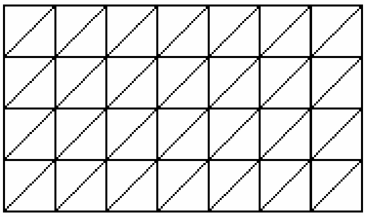
令 $\frac{m-2}{2}=k$

經整理，

得偶數列殘缺項個數，S(2k+2)= $\frac{F(1,2k+2)+F(1,k)-2}{2}$

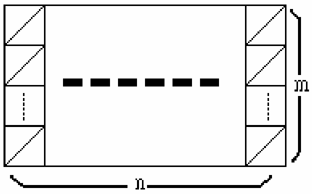
(庚) .對角棋盤：

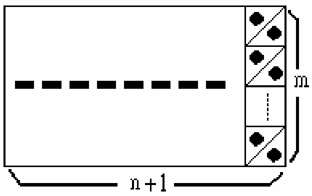
我們也嘗試將圖形推廣爲 m 列 n 行之特殊棋盤（ 如下圖 ），我們稱之爲「對角棋盤」



在探討其遞迴性質前，同樣地，我們定義：

定義一. m 列 n 行(共有 2mn 格)對角棋盤的放置方法數

= B(m,n)=B()

= B() = B(n+1, $\begin{matrix} b & b \\ b & b \\ \cdot & \cdot \\ \cdot & \cdot \\ b & b \end{matrix}$)

定義二. 任意兩相鄰格子(即兩者擁有共同邊)不可接放入白棋，而黑棋不受限

a.一列對角棋盤的遞迴性質之探討：

B(n,bw) = B(n,bb)
B(1,n) = B(n+1,bb)
= B(n,bb) + B(n,bw) + B(n,wb)
= 2B(n,bb) + B(n,wb)
= [2B(n-1,bb) + 2B(n-1,bw) + 2B(n-1,wb)]

$$
\begin{aligned}
&+ [B(n-1,bb) + B(n-1,wb)] \\
&= 5B(n-1,bb) + 3B(n-1,wb) \\
&= 6B(n-1,bb) + 3B(n-2,wb) – B(n-1,bb) \\
&= 3[B(n-1,bb) + B(n-2,bw) + B(n-2,wb)]– B(n-1,bb) \\
&= 3B(n,bb) – B(n-1,bb) \\
&= 3B(1,n-1) – B(1,n-2)
\end{aligned}
$$

<命題七> $B(1,n) = 3B(1,n-1) – B(1,n-2) \quad \forall n \in N, n \geq 2$

$B(1,0) = 1$，$B(1,1) = 3$

值得注意的一點，其實

$B(1,n) = F(1,2n)$

因此，

$$
\begin{aligned}
B(1,n) &= F(1,2n) \\
&= F(1,2n-1) + F(1,2n-2) \\
&= [F(1,2n-2) + F(1,2n-3)] + F(1,2n-2) \\
&= 2F(1,2n-2) + F(1,2n-3) \\
&= 2F(1,2n-2) + [F(1,2n-2)] - F(1,2n-4)] \\
&= 3F(1,2n-2) - F(1,2n-4) \\
&= 3B(1,n-1) - B(1,n-2)
\end{aligned}
$$

b.兩列對角棋盤的遞迴性質之探討：

同於探討 F(m,n)遞迴性質的方法，我們使用「殘缺項反求法」

$$
\begin{aligned}
B(n,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix}) &= B(n,\begin{smallmatrix} b & w \\ b & b \end{smallmatrix}) = B(n,\begin{smallmatrix} b & w \\ b & w \end{smallmatrix}) = B(n,\begin{smallmatrix} b & b \\ b & w \end{smallmatrix}) \\
&= B(n-1,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix}) + B(n-1,\begin{smallmatrix} w & b \\ b & b \end{smallmatrix}) + B(n-1,\begin{smallmatrix} w & b \\ w & b \end{smallmatrix}) + B(n-1,\begin{smallmatrix} w & b \\ b & w \end{smallmatrix}) \\
&\quad + B(n-1,\begin{smallmatrix} b & w \\ b & b \end{smallmatrix}) + B(n-1,\begin{smallmatrix} b & w \\ b & w \end{smallmatrix}) + B(n-1,\begin{smallmatrix} b & b \\ w & b \end{smallmatrix}) + B(n-1,\begin{smallmatrix} b & b \\ b & w \end{smallmatrix})
\end{aligned}
$$

$$B(n,\begin{matrix} w & b \\ b & b \end{matrix}) = B(n,\begin{matrix} w & b \\ b & w \end{matrix}) = B(n,\begin{matrix} b & b \\ w & b \end{matrix})$$

$$= B(n-1,\begin{matrix} b & b \\ b & b \end{matrix}) + B(n-1,\begin{matrix} w & b \\ b & b \end{matrix}) + B(n-1,\begin{matrix} w & b \\ w & b \end{matrix}) + B(n-1,\begin{matrix} w & b \\ b & w \end{matrix}) + B(n-1,\begin{matrix} b & b \\ w & b \end{matrix}) + B(n-1,\begin{matrix} b & b \\ b & w \end{matrix})$$

$$B(n,\begin{matrix} w & b \\ w & b \end{matrix}) = B(n-1,\begin{matrix} b & b \\ b & b \end{matrix}) + B(n-1,\begin{matrix} w & b \\ b & b \end{matrix}) + B(n-1,\begin{matrix} w & b \\ w & b \end{matrix}) + B(n-1,\begin{matrix} b & b \\ w & b \end{matrix})$$

由上列等式，經「殘缺項反求法」即可得到

<命題八> $B(2,n) = 8B(2,n-1) - 11B(2,n-2) + 4B(2,n-3) \quad \forall n \in N, n \geq 3$

$B(2,0) = 1$，$B(2,1) = 8$，$B(2,2) = 53$

(三). 棋盤可翻轉時，F(m , n)的遞迴性質：

前面所求的合法排列數 F(m , n)，是將矩形棋盤視爲固定不動。現在考慮在矩形棋盤上兩個合法排列，利用左右翻轉的方式，只要能因左右翻轉而形成一樣的排列者，我們就將它們看成同一種排法。那麼，在此觀點之下共有多少種排列方法呢？(因爲上下翻轉或左右翻轉的計數問題其實是相同的，所以只需討論左右翻轉，上下翻轉者同理可得。)

我們分情況討論如下：

(1) 當 n 爲奇數時，以第 $\frac{n+1}{2}$ 行爲正中央，切開左右兩邊各有 $\frac{n-1}{2}$ 行，考慮由左邊的 $\frac{n-1}{2}$ 行加上中間這行所形成的子棋盤。我們觀察到只要將這個子棋盤上的合法排列向右一翻就可得母棋盤上的左

右對稱合法排列。顯然母棋盤上所有的左右對稱合法排列皆可由此得到。因此左右對稱合法排列數爲 $F(m,\frac{n}{2}+1)$所以在左右翻轉之觀點下共有$\frac{F(m,n)+F(m,\frac{n+1}{2})}{2}$種不同的合法排列。

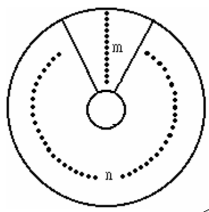
(2) 當 n 爲偶數時，若某合法排列爲左右對稱，則中間那兩行顯然必須放黑棋，故類似於(1)，我們得到在左右翻轉之觀點下共有

$\frac{F(m,n)+F(m,\frac{n}{2}-1)}{2}$種不同的合法排列。

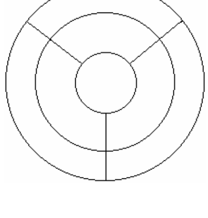
(四). C(m , n)的遞迴性質：

除了探討矩形棋盤及對角棋盤，另外地，我們還嘗試探討環狀棋盤的各種性質，在探討前我們先定義

定義： m 列 n 行(共有 mn 格)環狀棋盤的放置方法數

=C(m,n)=C(

).

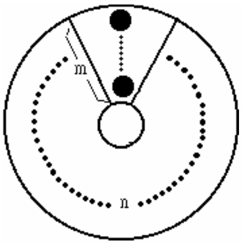
例：C(2,3) = C()

(甲)固定列數 m 的探討

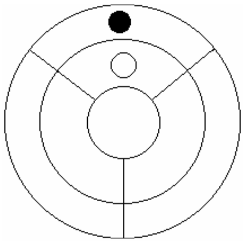
爲方便計算，我們使用諸如底下的符號

定義一. 在推導過程中，我們使用諸如底下之符號

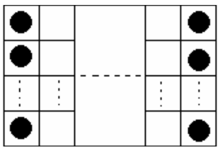
例如：　　　　　　　　　　　　↓有 m 個 b

C(

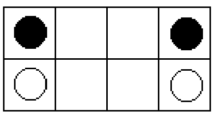
) $=C\left(n, \begin{matrix} b \\ b \\ \vdots \\ b \end{matrix}\right)$

C() $=C\left(3, \begin{matrix} b \\ w \end{matrix}\right)$

定義二. 在推導過程中，我們使用諸如底下之符號

F() $=F\left(\begin{matrix} b \\ b \\ \vdots \\ b \end{matrix}, n, \begin{matrix} b \\ b \\ \vdots \\ b \end{matrix}\right)$

又例如：

$C\left(3, \begin{matrix} b \\ w \end{matrix}\right) = F($  $) = F\left(\begin{matrix} b \\ w \end{matrix}, 4, \begin{matrix} b \\ w \end{matrix}\right)$

當 m=1 時，我們很容易可以得到：

$C(1,n) = C(n\ ,b) + C(n\ ,w) = F(1,n\text{-}1) + F(1,n\text{-}3)$

$\qquad = C(1,n\text{-}1) + C(1,n\text{-}3)\ ,\forall n \geq 3$

接下來，我們欲求 C(2,n)之遞迴公式。在底下之推導過程中，我們必須利用到底下之等式。

[1] $\forall n \geq 1, F(2,n)=F(2,n\text{-}1)+2F(n,\genfrac{}{}{0pt}{}{w}{b})$

[2] $\forall n \geq 1, C(2\ ,n) = F(2\ ,n\text{-}1) + 2F(\genfrac{}{}{0pt}{}{w}{b},n+1\ ,\genfrac{}{}{0pt}{}{w}{b})$

([2]其實是從公式底下 C(2,n)的推導過程中得到的)

現在我們有

$$\begin{aligned}
C(2,n) &= C\ (n\ ,\genfrac{}{}{0pt}{}{b}{b}) + C(n\ ,\genfrac{}{}{0pt}{}{w}{b}) + C(n\ ,\genfrac{}{}{0pt}{}{b}{w}) = F(2\ ,n\text{-}1) + 2C(n\ ,\genfrac{}{}{0pt}{}{w}{b}) \\
&= F(2\ ,n\text{-}1) + 2F(\genfrac{}{}{0pt}{}{w}{b},n+1\ ,\genfrac{}{}{0pt}{}{w}{b}) = F(2\ ,n\text{-}1) + 2[F(\genfrac{}{}{0pt}{}{w}{b}, n\ ,\genfrac{}{}{0pt}{}{b}{b}) + F(\genfrac{}{}{0pt}{}{w}{b}, n\ ,\genfrac{}{}{0pt}{}{b}{w})] \\
&= F(2\ ,n\text{-}1) + 2\ F(\genfrac{}{}{0pt}{}{w}{b}, n\ ,\genfrac{}{}{0pt}{}{b}{b}) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n\ ,\genfrac{}{}{0pt}{}{b}{w}) \\
&= F(2\ ,n\text{-}1) + 2\ F(\ n\ \text{-}1,\genfrac{}{}{0pt}{}{w}{b}) + 2[F(\genfrac{}{}{0pt}{}{w}{b}, n\text{-}1\ ,\genfrac{}{}{0pt}{}{b}{b}) + F(\genfrac{}{}{0pt}{}{w}{b}, n\text{-}1\ ,\genfrac{}{}{0pt}{}{w}{b})] \\
&= F(2\ ,n\text{-}1) + 2\ F(\ n\ \text{-}1,\genfrac{}{}{0pt}{}{w}{b}) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n\text{-}1\ ,\genfrac{}{}{0pt}{}{b}{b}) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n\text{-}1\ ,\genfrac{}{}{0pt}{}{w}{b}) \\
&= F(2\ ,n\text{-}1) + 2\ F(\ n\ \text{-}1,\genfrac{}{}{0pt}{}{w}{b}) + 2\ F(\ n\ \text{-}2,\genfrac{}{}{0pt}{}{w}{b}) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n\text{-}1\ ,\genfrac{}{}{0pt}{}{w}{b}) \\
&= F(2\ ,n\text{-}1) + [\ F(2\ ,n\text{-}1) - F(2\ ,n\text{-}2)] + [\ F(2\ ,n\text{-}2) - F(2\ ,n\text{-}3)] \\
&\quad + [\ C(2,n\text{-}2) - F(2\ ,n\text{-}3)] = C(2,n\text{-}2) + 2F(2,n\text{-}1) - 2F(2,n\text{-}3)
\end{aligned}$$

所以　　$C(2,n) - C(2,n-2) = 2F(2,n-1)-2F(2,n-3) \quad \forall n \ge 4$

(仔細檢討以上推導過程可知必須$\forall n \ge 4$)

又因爲　$\forall n \ge 2$, $F(2,n)=2F(2,n-1)+F(2,n-2)$

所以　$\forall n \ge 6$, $C(2,n) - C(2,n-2) =2[C(2,n-1) - C(2,n-3)]+ [C(2,n-2) - C(2,n-4)]$

因此，我們得到

$$C(2,n)= 2C(2,n-1) + 2C(2,n-2) - 2C(2 ,n-3) - C(2 ,n-4) \quad \forall n \ge 6$$

而 C(3,n)以上的遞迴公式，我們只要配合殘缺項反求法就能夠求出，如：

$$C(3,n)=11C(3,n-2)+12C(3,n-3)-11C(3,n-4)-14C(3,n-5)+C(3,n-6)+2C(3,n-7)$$

另外，我們也意外的觀察到，C(2,n)這個數列似乎滿足

$$\forall n \ge 2, C(2,n)=2 C(2,n-1)+ C(2,n-2)-2(-1)^n$$

這個較短的遞迴公式。我們重新檢討 C(2,n)之遞迴公式，利用

$$F(n,\genfrac{}{}{0pt}{}{w}{b})= F(\genfrac{}{}{0pt}{}{w}{b},n,\genfrac{}{}{0pt}{}{b}{b})+ F(\genfrac{}{}{0pt}{}{w}{b},n ,\genfrac{}{}{0pt}{}{w}{b}) +F(\genfrac{}{}{0pt}{}{w}{b},n ,\genfrac{}{}{0pt}{}{b}{w}) \quad \forall n \ge 3$$

我們推導如下：

$$C(2,n) = C (n ,\genfrac{}{}{0pt}{}{b}{b}) + C(n ,\genfrac{}{}{0pt}{}{w}{b}) + C(n ,\genfrac{}{}{0pt}{}{b}{w}) = F(2 ,n-1) + 2C(n ,\genfrac{}{}{0pt}{}{w}{b})$$

$$= F(2 ,n-1) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n+1,\genfrac{}{}{0pt}{}{w}{b}) = F(2 ,n-1) + 2[F(\genfrac{}{}{0pt}{}{w}{b}, n ,\genfrac{}{}{0pt}{}{b}{b}) + F(\genfrac{}{}{0pt}{}{w}{b}, n ,\genfrac{}{}{0pt}{}{b}{w})]$$

$$= F(2 ,n-1) +2 F(n-1,\genfrac{}{}{0pt}{}{w}{b}) + 2F(\genfrac{}{}{0pt}{}{w}{b}, n ,\genfrac{}{}{0pt}{}{b}{w})$$

$$= F(2 ,n-1) + 2 F(n-1,\genfrac{}{}{0pt}{}{w}{b}) + 2[F(n,\genfrac{}{}{0pt}{}{w}{b})- F(n-1 ,\genfrac{}{}{0pt}{}{w}{b}) - F(\genfrac{}{}{0pt}{}{w}{b}, n ,\genfrac{}{}{0pt}{}{w}{b})]$$

$= F(2,n-1) + 2F(n-1,\frac{w}{b}) - 2F(\frac{w}{b}, n, \frac{w}{b})$

$= F(2,n-1)+[F(2,n) - F(2,n-1)]-[C(2,n-1) - F(2,n-2)]$

$=-C(2,n-1)+F(2,n)+F(2,n-2)$

所以 $C(2,n)+C(2,n-1)=F(2,n)+F(2,n-2) \quad \forall n \ge 3$

得到 $C(2,n)+C(2,n-1)=[2F(2,n-1)+F(2,n-2)]+F(2,n-2)$

$= 2F(2,n-1)+2F(2,n-2), \forall n \ge 3$

得到 $C(2,n)-2F(2,n-1)= -[C(2,n-1)-2F(2,n-2)], \forall n \ge 3$

因此 $C(2,n)-2F(2,n-1)=(-1)^{n-1}, \forall n \ge 2$

$C(2,n)=2F(2,n-1)+(-1)^{n} \quad, \forall n \ge 2$

所以 $C(2,n)+(-1)^{n}= 2F(2,n-1), \forall n \ge 2$

故由 F(m , n)之遞迴性質立即可得

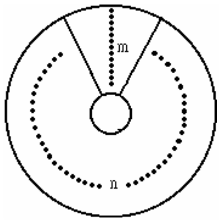
$\forall n \ge 4, C(2,n)=2C(2,n-1)+C(2,n-2)-2(-1)^{n}$

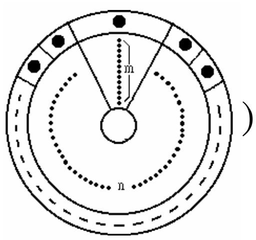
(乙).固定行數 n 的探討：

在環狀排列中，我們知道 F(m,n)並不一定等於 F(n,m)，固定ｍ列可求出一遞迴公式，同理固定ｎ行也可求出一遞迴公式

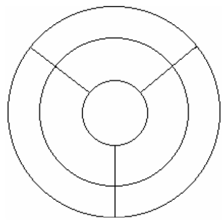
在討論前我們同樣定義。

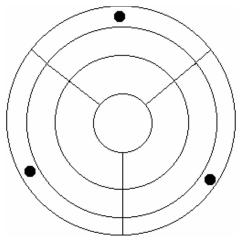
定義一：m 列 n 行(共有 mn 格)環狀棋盤的放置方法數

=C(m , n)=C()

=C() =C($\begin{matrix} b & b \\ b & b \\ . & . \\ . & . \\ b & b \end{matrix}$, m+1)

↑有 n 行黑棋　　↑有 n 個 b

例如：C(3,2)　=C()

= C() =　C($\begin{matrix} & b & \\ b & & b \end{matrix}$,3)

定義二. 任意兩相鄰格子(及兩者擁有共同邊)不可皆放入白棋，而黑棋不受限。

a. C(m,1)的遞迴性質之探討：

已知　C(m,1)=F(m,1)=F(1,m)

所以

C(m,1)=F(1,m)

　　=F(1,m-1)+F(1,m-2)

　　=C(m-1,1)+C(m-2,1)

得到

<命題九> $C(m,1)=C(m-1,1)+C(m-2,1) \quad \forall m\geq 2$

$C(0,1)=1$，$C(1,1)=2$

b. C(m,2)的遞迴性質之探討：

利用殘缺項反求法，

$$C(m,2)=C(\genfrac{}{}{0pt}{}{b}{b},m)+C(\genfrac{}{}{0pt}{}{b}{w},m)+C(\genfrac{}{}{0pt}{}{b}{w},m-1)$$

$$=C(m-1,2)+2[C(\genfrac{}{}{0pt}{}{b}{b},m-1)+c(\genfrac{}{}{0pt}{}{b}{w},m-1)]$$

$$=C(m-1,2)+2C(m-2,2)+2C(\genfrac{}{}{0pt}{}{w}{b},m-1)$$

得到

$$C(m,2)=C(m-1,2)+2C(\genfrac{}{}{0pt}{}{b}{w},m)$$

$$C(m,2)=C(m-1,2)+2C(m-2,2)+2C(\genfrac{}{}{0pt}{}{w}{b},m-1)$$

經整理得到

<命題十> $C(m,2)=2C(m-1,2)+C(m-2,2) \quad \forall m\geq 2$

$C(0,2)=1$，$C(1,2)=3$

C.三行以上遞迴性質之探討：

接經由「殘缺項反求法」而得

<命題十一> $C(m,3)=3C(m-1,3)+C(m-2,3) \quad \forall m\geq 2$

$C(0,3)=1$，$C(1,3)=4$

<命題十二> $C(m,4)=5C(m-1,4)+C(m-2,4)-C(m-3,4) \quad \forall m\geq 3$

C(0,4)=1，C(1,4)=7，C(2,4)=35

值得注意的一點，

四行 m 列的環狀棋盤，經由適當的變化，即可變成一個 $2\times 2\times m$ 的「長方體棋盤」

因此兩者性質相同，

C(m,4)=5C(m-1,4)+C(m-2,4)-C(m-3,4)

F(2,2,m)=5F(2,2,m-1)+F(2,2,m-2)-F(2,2,m-3)

<命題十三> C(m,5)=7C(m-1,5)+C(m-2,5)-C(m-3,5) $\forall m \geq 3$

C(0,5)=1，C(1,5)=11，C(2,5)=81

(五). 可翻轉或旋轉時，C(m , n)的遞迴性質：

(甲) 旋轉型環狀排列方法數之探討:

我們前面所探討的 C(2,n)是把環形棋盤看成固定不動而求出的方法數,現在給定一個 m 列 n 行的環狀棋盤，如果一個合法排列甲排列經由旋轉可以變成乙排列，則把甲排列跟乙排列看成相同，在此觀點之下，一共有多少種排列方法呢？我們稱此方法數爲旋轉型環狀排列方法數，記爲 T(m,n)。

定義:給定一個 m 列 n 行的環形棋盤,設甲排列爲此棋盤上的合法排列,若甲排列順時針旋轉 k 行($1 \le k \le n$)之後,所得排列跟甲排列完全一致的話,我們就稱甲排列爲 k 循環.

定義:若甲排列爲 k 循環,且對於任意的$1 \le l \le k-1$,甲排列不爲 l 循環,則我 1 稱甲排列爲真正 k 循環.

T(2,n)之求法**:**由前面的二列環狀排列公式可推出二列旋轉型公式爲

$$T(2,n) = 1 + \frac{2F(2,n-1) + (-1)^n - 1}{n} \quad n \ge 3, n\text{為質數}$$

當 n 爲合成數時,則必須用排容原理去求得.

例如:以 T(2,6)爲例

我們有

$$T(2,6)=\frac{\text{真正一循環的方法數}}{1}+\frac{\text{真正二循環的方法數}}{2}+\frac{\text{真正三循環的方法數}}{3}+\frac{\text{真正六循環的方法數}}{6},$$

因爲每兩個真正二循環的排列其實是同樣的旋轉排列,這是上式的第二項必須除以二的理由,同理,真正三循環的方法數必須除以三,真正六循環的方法數必須除以六.

另外，因爲二循環包括一循環在內,必須扣掉才是真正二循環，同理，三循環必須扣掉一循環才是真正三循環，而六循環必須扣掉一循環，二循環，三循環,才是真正六循環，但是在扣掉一循環，二循環，三循環

的過程當中，因爲有重複計算的問題，所以必須用到排容原理，所以真正六循環的方法數爲：

C(2,6)-1-C(2,2)-C(2,3)+1+1+1-1=C(2,6)-C(2,2)-C(2,3)+1

因此,我們就得到以下的等式

$$T(2,6)=1+\frac{C(2,2)-1}{2}+\frac{C(2,3)-1}{3}+\frac{C(2,6)-C(2,2)-C(2,3)+1}{6}$$
$$=1+3+4+30$$
$$=38$$

(乙) 環狀排列之內外翻轉問題：

定義:若我們將 m 列 n 行矩形棋盤上的某個合法排列之最左和最右二行並排在一起時，白棋與白棋仍未相鄰,則此排列，就相當於一個 m 列 n 行柱面棋盤上的合法排列，因此也就相當於一個 m 列 n 行環狀棋盤上的合法排列，因此，環狀排列的內外翻轉就相當於柱面排列的上下翻轉，也相當於在矩形棋盤上將對應的排列上下翻轉。

當列數 m 爲偶數時，因爲翻轉後必須對稱，而白棋不可以相鄰，所以中間二列顯然都是黑棋，因此，m 列的對稱排列的方法數爲 $C(\frac{m}{2}-1,n)$，而不對稱排列的方法數爲 $C(m,n)-C(\frac{m}{2}-1,n)$，所以若我們將上下翻轉後能重合的排列看成同一排列的話，就得到方法數爲

$$C(\frac{m}{2}-1,n)+\frac{C(m,n)-C(\frac{m}{2}-1,n)}{2}$$

另一方面，當列數 m 爲奇數時，利用$\frac{m+1}{2}$列環狀排列的翻轉，可得到 m 列的對稱環狀排列，顯然 m 列對稱環狀排列皆可由$\frac{m+1}{2}$列環狀排列的翻轉求得，因此，我們可利用 $C(\frac{m+1}{2},n)$的公式去求得對稱排列的方法數，而不對稱排列的方法數就等於 C(m,n)-$C(\frac{m+1}{2},n)$,所以若我們把上下翻轉後會重合的排列看成同一排列的話，就得到方法數爲

$$C(\frac{m+1}{2},n)+\frac{C(m,n)-C(\frac{m+1}{2},n)}{2}$$

(丙) **環狀旋轉翻轉問題:**

因爲環狀棋盤相當於柱面棋盤，可以上下翻轉或旋轉，所以我們考慮底下之計數問題:

在 m 列 n 行的環狀棋盤上，若兩個合法排列只經由旋轉或只經由內外翻轉或經由內外翻轉加旋轉之後能夠重合的話，就將此兩個排列視爲同一排列。請問在此觀點下，在 m 列 n 行的環狀棋盤上會有多少種排列呢?

首先以 m=2，n=6 爲例

爲了方便，我們以$(真 2)表示在固定環狀棋盤時，真正 2 循環排列之方法數，其餘依此類推.

顯然$(真 1)=1

其次，$(真 2)=$(真 2 對稱)+$(真 2 不對稱)=0+6.此時我們必須注意真正 2 循環而不對稱的排列可分兩大類.第一類是旋轉後會恰好變成翻轉的，第二類是再怎麼旋轉也不會變成翻轉的，而在旋轉翻轉的觀點下第一類每 2 個一組，第二類每 2×2=4 個一組，不難看出第一類只有 2 個，他們是由$\begin{pmatrix} b & w \\ w & b \end{pmatrix}$所引起的；而第二類有 4 個，乃是由$\begin{pmatrix} b & b \\ b & w \end{pmatrix}$引出來的.故對真正 2 循環排列而言，若考慮旋轉翻轉則共有$\frac{0}{2}+\frac{2}{2}+\frac{4}{2\times 2}$=2 種排列.

再者，$(真 3)=$(真 3 對稱)+$(真 3 不對稱)=0+12.因 3 爲奇數故可知真正 3 循環而不對稱的排列再怎麼旋轉也不會變成翻轉，故若考慮旋轉翻轉則共有$\frac{0}{3}+\frac{12}{2\times 3}$=2 種排列

最後，$(真 6)=$(真 6 對稱)+$(真 6 不對稱)=0+180.考慮第一類.觀察 bbbbbw，它經由一次次的旋轉可得 wbbbbb，bwbbbb，bbbwbb，bbbbwb，bbbbbw.

若我們考慮$\begin{pmatrix} bbbbbw \\ bbwbbb \end{pmatrix}$，則旋轉 3 次就變成翻轉.類似的$\begin{pmatrix} wbwbbb \\ bbbwbw \end{pmatrix}$亦然.檢查可知第一類就只包含這 2 個所引起的 6+6 種排列，也因此第二類共有 168 種排列.故若考慮旋轉翻轉則共有$\frac{0}{6}+\frac{12}{6}+\frac{168}{2\times 6}$=16 種排列。

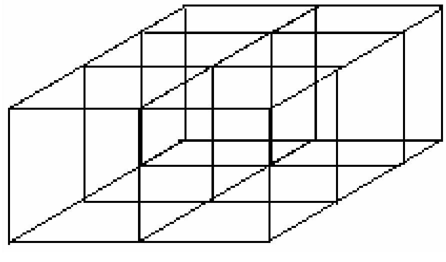
總結上述，當我們考慮旋轉翻轉時在 2 列 6 行的環狀棋盤上共有 1+2+2+16=21 種排列。

用相同的方法小心的分析可知在 3 列 6 行的環狀棋盤上，真正 2 循環而不對稱的排列只有 2 個屬於第一類，是由$\begin{pmatrix} bw \\ bb \\ wb \end{pmatrix}$所引起的；真正 6 循環而不對稱的排列有 5×6=30 個屬於第一類，分別由$\begin{pmatrix} bbbbbw \\ bbbbbb \\ bbwbbb \end{pmatrix}$、$\begin{pmatrix} bbbbbw \\ bbbbbb \\ bbwbbb \end{pmatrix}$、$\begin{pmatrix} bbbbbw \\ wbbwbb \\ bbwbbb \end{pmatrix}$、$\begin{pmatrix} bbbbbw \\ bwbbwb \\ bbwbbb \end{pmatrix}$、及$\begin{pmatrix} wbwbbb \\ bwbbwb \\ bbbwbw \end{pmatrix}$ 所引起的。從而可由計算得知考慮旋轉翻轉時在 3 列 6 行的環狀棋盤上共有 221 種排列.

(六). F (m , n ,k) 的性質：

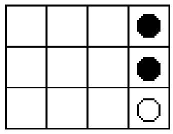
現在我們想將平面的棋盤推廣爲空間中的棋盤。首先，我們想像一個長方體棋盤，其長、寬、高分別爲 k、 m、 n 單位，因此總共有 m×n×k 個格子。那麼在我們限制的放法 (任何兩個放入白球的格子不可有任何一面相鄰) 之下，共有幾種放法呢？

例: 圖七爲 F(2,1,3) 的立體圖。

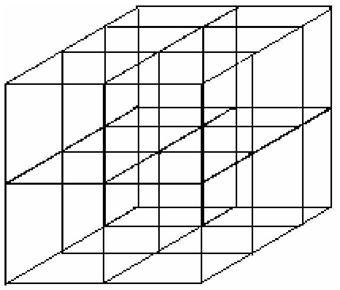


(圖七)

我們延續平面上三列遞迴公式的求法及符號，發展成空間中的求法與類似型式的符號。

在三列符號中， F() = $F(4,\begin{smallmatrix} b \\ b \\ w \end{smallmatrix})$,

類似地，在空間符號中，

F() = $F(2,2,3) = F(4,\begin{smallmatrix} bb \\ bb \end{smallmatrix})$.

我們延續平面上求三列遞迴公式的方法來討論空間中的遞迴公式。

例如：

$$F(2,2,k) = F(k+1,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix})$$

$$=F(k,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix})+F(k,\begin{smallmatrix} w & b \\ b & b \end{smallmatrix})+F(k,\begin{smallmatrix} b & w \\ b & b \end{smallmatrix})+F(k,\begin{smallmatrix} b & b \\ w & b \end{smallmatrix})+F(k,\begin{smallmatrix} b & b \\ b & w \end{smallmatrix})+F(k,\begin{smallmatrix} w & b \\ b & w \end{smallmatrix}) + F(k,\begin{smallmatrix} b & w \\ w & b \end{smallmatrix})$$

$$= F(2,2,k-1)+4F(k,\begin{smallmatrix} w & b \\ b & b \end{smallmatrix})+2F(k,\begin{smallmatrix} w & b \\ b & w \end{smallmatrix})$$

$$= 5F(2,2,k-1)+F(2,2,k-2)+F(2,2,k-3)$$

(七) F (m , n) 的大小順序：

我們將不同的棋盤所對應的方法數加以比較，顯然

F(1,0)<F(1,1)<F(1,2)……..，F(2,0)<F(2,1)<F(2,2)……..，

F(m,0)<F(m,1)<F(m,2)……..。而且 F(2,3)<F(3,4)<F(7,8)。也就是說，若棋盤 A 的行和列都比棋盤 B 的行和列少，則 F(A)<F(B)，我們能否馬上比

較出 F(A)和 F(B)之大小呢？首先，我們有

<命題十四>　若 $m\geq 2$，$n\geq 2$，則 $F(m,n)<F(1,m\times n)$

例如：$F(2,6)<F(1,12)$，$F(3,4)<F(1,12)$

證明：設棋盤 A 和如圖八所示有 m 列 n 行，其中每一格皆編上號碼

(1,1)，(1,2)，(1,3)…….，(m,n)。

若我們已在棋盤 A 中依限制條件擺滿黑、白棋，則如圖八所示，我們可

將此放法轉換成一列 mn 行之一種合法的排列方式：

(1 , 1)	(1 , 2)	(1 , 3)	…………..	(1 , n)
(2 , 1)	(2 , 2)	(2 , 3)	…………..	(2 , n)
(3 , 1)	(3 , 2)	(3 , 3)	…………..	(3 , n)
(4 , 1)	(4 , 2)	(4 , 3)	…………..	(4 , n)
⋮	⋮	⋮	⋮	⋮
(m , 1)	(m , 2)	(m , 3)	…………..	(m , n)

(1,1)	…	(1,n)	(2,n)	(2,n-1)	…	(2,1)	(3,1)	…	(3,n)	…	(m,k)

(當 m 是偶數時,k=1;當 n 是奇數時,k=n)

所以 A 上任一種合法的排列，皆可轉換成一個一列 mn 行棋盤上之合法排列，而且顯然 A 上任兩個不同的合法排列所轉換成的一列 mn 行上的排列是不同的，所以 F(m,n)≤F(1,mn)

另一方面，顯然有些一列 mn 行上的合法排列不能由行上的合法排列不能由 A 上的合法排列轉換而得。所以 F(m,n)<F(1,mn)，證畢。

那麼 F(2,6)和 F(3,4)之大小，或 F(2,8)和 F(4,4)之大小關係又如何？

一般而言，如果兩個棋盤 A 和 B 格數一樣多，我們能否馬上判斷出 F(A)和 F(B)之大小呢？由實際的數據，我們得發現一個現象：若棋盤越接近正方形，則其對應之方法數就越少。例如：

F(3 ,4)=227<239=F(2 ,6)，而 F(4 ,4)=1234<1393=F(2 ,8).

那麼這現象是否真的永遠成立呢？截至目前爲止，仍然在探討中。

更一般地，任給兩個棋盤 A 和 B，我們如何比較其大小呢？這是我們目前很感興趣，極力想解決問題。

伍・結論

(1)各列的遞迴公式爲

<a> $F(1,0)=1, F(1,1)=2, F(1,n) = F(1,n-1) + F(1,n-2), \forall n \geq 2$.

<b> $F(2,0)=1, F(2,1)=3, F(2,n) = 2F(2,n-1) + F(2,n-2), \forall n \geq 2$.

<c> $F(3,0)=1, F(3,1)=5, F(3,2)=17, F(3,3)=63, F(3,n) = 2F(3,n-1) + 6F(3,n-2) - F(3,n-4), \forall n \geq 4$.

<d> $F(4,0)=1, F(4,1)=8, F(4,2)=41, F(4,3)=227, F(4,4)=1234, F(4,5)=6743$

$$F(4,n) = 4F(4,n-1) + 9F(4,n-2) - 5F(4,n-3) - 4F(4,n-4) + F(4,n-5), \forall n \geq 5.$$

<e> $F(2,2,0)=1, F(2,2,1)=7, F(2,2,2)=35, F(2,2,3)=181$,

$$F(2,2,k) = 5F(2,2,k-1) + F(2,2,k-2) - F(2,2,k-3), \forall k \geq 3.$$

(2)我們已知道 $, \forall n \geq 0$,

$$F(1,n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{(n+2)}-\left(\frac{1-\sqrt{5}}{2}\right)^{(n+2)}\right],$$

$$F(2,n)=\frac{1}{2}\left[(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\right],$$

$$F(3,n)= C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + C_4\alpha_4^n,$$

$$F(4,n)= d_1\beta_1^n + d_2\beta_2^n + d_3\beta_3^n + d_4\beta_4^n + d_5\beta_5^n.$$

(3).殘缺項個數的公式爲

<a> 奇數列殘缺項個數，$S(2k+3) = \dfrac{F(1,2k+3)+F(1,k+1)+F(1,k)-2}{2}$

<b> 偶數列殘缺項個數，$S(2k+2) = \dfrac{F(1,2k+2)+F(1,k)-2}{2}$

(4).對角棋盤的遞迴公式爲

<a>B(1,0) = 1，B(1,1) = 3　　B(1,n) = 3B(1,n-1) - B(1,n-2)　　$\forall n \in N, n \ge 2$

<b>B(2,0) = 1，B(2,1) = 8，B(2,2) = 53

B(2,n) = 8B(2,n-1) − 11B(2,n-2) + 4B(2,n-3)　　$\forall n \in N, n \ge 3$

(5).可翻轉時，矩形棋盤的公式爲

<a> m 爲奇數時，共有 $\dfrac{F(m,n)+F(m,\frac{n+1}{2})}{2}$ 種不同的合法排列

<b> m 爲偶數時，共有 $\dfrac{F(m,n)+F(m,\frac{n}{2}-1)}{2}$ 種不同的合法排列

(6).環狀棋盤各行的遞迴公式爲

<a>C(1,n) = C(1,n-1) + C(1,n-3)，$\forall n \ge 3$

<b>C(2,n) = 2C(2,n-1) + C(2,n-2) + 2(-1)^{n}　，　$\forall n \ge 4$

(7).環狀棋盤各行的遞迴公式爲

<a>C(m,1) = C(m-1,1) + C(m-2,1)　$\forall m \ge 2$

<b>C(m,2) =2C(m-1,2) + C(m-2,2)　$\forall m \ge 2$

<c>C(m,3) =3C(m-1,3) + C(m-2,3)　$\forall m \ge 2$

<d>C(m,4) =5(m-1,4) + C(m-2,4) - C(m-3,4)　　$\forall m \ge 3$

<e>C(m,5) =7(m-1,5) + 5 C(m-2,5) - C(m-3,5)　　$\forall m \ge 3$

(8).可旋轉或翻轉時，環狀棋盤各列的公式爲

<a>.旋轉時

$$T(m,n) = \sum_{\substack{k \in N \\ k | n}} \frac{\text{真正}k\text{循環排列之方法數}}{k} ,$$

而真正 k 循環排列，必須使用排容原理

<b>翻轉時

m 爲奇數，$T'(m,n)=C(\frac{m+1}{2},n)+\frac{C(m,n)-C(\frac{m+1}{2},n)}{2}$

m 爲偶數，$T'(m,n)=C(\frac{m}{2}-1,n)+\frac{C(m,n)-C(\frac{m}{2}-1,n)}{2}$

(9).長方體棋盤各行的遞迴公式爲

F(2,2,k) = 5F(2,2,k-1) + F(2,2,k-2) - F(2,2,k-3)

(10).若 $m \geq 2, n \geq 2$ 則 $F(m,n) < F(1, m \times n)$

陸・討論

(1)我們用來求三列遞迴公式的方法，其實適用範圍很廣。例如，將規則改成 :「不但有一邊相鄰的格子不可放白棋，而且有共同頂點的格子也不可都放入白棋。」那麼，在新的規定下，怎麼求 F(m,n)呢？顯然 F(m,n)之求法，還是可以用我們發展出來的方法來求得。

(2)起源自兔子問題的費伯那契數列，神奇地在自然界中到處出現，那麼我們現在所探討的問題是否能在自然科學裡找到例子呢？效法費伯那契，我們暫且想像有某種生物系統(例如:類似基因的排列),或機械系統,是由甲、乙兩種(甚至更多種)單元,以類似於棋盤狀的方式排列而成。爲了讓這個系統能正常運作,乙單元不能連續出現,那麼有多

少種讓系統正常運作的排列方式呢?這時候我們的方法就可以運用上了。

(3)我們希望藉由 2 列、3 列……等公式的變化，而得到一個統一的公式。只要給定列數 m，由這統一的公式，我們馬上可以寫出 F (m , n) 的遞迴公式、求極限的方程式以及比內型公式等等。

(4)由數據顯示 $\frac{F(m,n)}{F(m,n-1)}$ 成波浪型上下起伏的變化，而終於趨近一個定值。爲什麼會上下起伏呢？當 m=2 時，令$\alpha=2a=1+\sqrt{2}$,$\beta=2b=1-\sqrt{2}$.　則 $F(2,n)=a\alpha^n+b\beta^n$　又因爲

$\frac{F(2,n)}{F(2,n-1)}-\frac{F(2,n+1)}{F(2,n)}=\frac{F(2,n)^2-F(2,n-1)\cdot F(2,n+1)}{F(2,n-1)\cdot F(2,n)}$所以經計算後，上式的分子$=-ab(\alpha-\beta)^2(\alpha\beta)^{n-1}=2(\alpha\beta)^{n-1}=2(-1)^{n-1}$

由此式可得當 n 爲偶數時，$\frac{F(2,n)}{F(2,n-1)}<\frac{F(2,n+1)}{F(2,n)}$，而當 n 爲奇數時，$\frac{F(2,n)}{F(2,n-1)}>\frac{F(2,n+1)}{F(2,n)}$，這就是關鍵所在。

(5)當 m≥2 時，我們也想探討 F(m,n)是否有其它許多類似於費伯那契數列的有趣性質。

(6)用「殘缺項反求法」，所求得的遞迴公式是否爲最短的公式呢？在矩形棋盤我們相信這是對的，雖然現在我們還不會證明。至於在環狀棋盤，則必須精巧地利用「殘缺項反求法」，才能得到最短的遞迴公式。如何才能得到最短的遞迴公式呢？我們希望找到一個系統化的方法。

柒・參考資料

[1]斐波那契數列(九章出版社)

[2] I.N.HERSTEIN, TOPICS IN ALGEBRA

附表一

以下爲 4 行 n 列之原始程式碼

```
10 CLS
20 DIM A#(8),B#(8)
30 FOR I = 1 TO8
    40  A#(I) =1
50      B#(I) =1
60 NEXT I
70 INPUT N%
80 FOR m =1 TO N%
90          A#(1) = B#(1) + B#(2) + B#(3) + B#(4) + B#(5) + B#(6) + B#97)
+ B#(8)
100     A#(2) = B#(1) + B#(5) + B#(6) + B#(7) + B#(8)
110     A#(3) = B#(1) + B#(5) + B#(6) + B#(8)
120     A#(4) = B#(1) + B#(5) + B#(7)
130     A#(5) = B#(1) + B#(2) + B#(3) + B#(4) + B#(7) +B#(8)
140     A#(6) = B#(1) + B#(2) + B#(3) + B#(7)
150     A#(7) = B#(1) + B#(2) + B#(4) + B#(5) + B#(6) + B#(8)
160        A#(8) = B#(1) + B#(2) + B#(3) = B#(5) + B#(7)
170     FOR J = 1 TO 8
180             SWAP A#(J), B#(J)
190     NEXT J
200 PRINT B#(1), m
210 NEXT m
```

以下爲 5 行 n 列之原始程式碼

```
10 CLS
20 DIM A#(13), B#(13)
30 FOR I = 1 TO 13
40       A#(I) = 1
50       B#(I) = 1
60 NEXT I
70 INPUT N%
80 FOR m = 1 TO N%
90         A#(1) = B#(1) + B#(2) + B#(3) + B#(4) + B#(5) + B#(6) + B#(7) +
                   B#(8) + B#(9) + B#(10) + B#(11) + B#(12) + B#(13)
100        A#(2) = B#(1) + B#(7) + B#(8) + B#(9) + B#(10) + B#(11) +
                   B#(12) + B#(13)
110        A#(3) = B#(1) + B#(7) + B#(8) + B#(9) + B#(12) + B#(13)
120        A#(4) = B#(1) + B#(7) + B#(8) + B#(12)
130        A#(5) = B#(1) + B#(7) + B#(9) + B#(10) + B#(11) + B#(13)
140      A#(6) = B#(1) + B#(7) + B#(8) + B#(10) + B#(12)
150        A#(7) = B#(1) + B#(2) + B#(3) + B#(4) + B#(5) + B#(6) + B#(10)
                 + B#(11) + B#(12) + B#(13)
160        A#(8) = B#(1) + B#(2 + B#(3) + B#(4) + B#(6) + B#(10) + B#(11)
                 + B#(13)
170      A#(9) = B#1) + B#(2) + B#(3) + B#(5) + B#(10) + B#(12)
180        A#(10) = B#(1) + B#(2) + B#(5) + B#(6) + B#(7) + B#(8) + B#(9)
                   + B#(12) + B#(13)
190      A#(11) = B#(1) + B#(2) + B#(5) + B#(7) + B#(8) + B#(12)
200        A#(12) = B#(1) + B#(2) + B#(3) + B#(4) + B#(6) + B#(7) + B#(9)
                   + B#(10) + B#(11) + B#(13)
210        A#(13) = B#(1) + B#(2) + B#(3) + B#(5) + B#(7) + B#(8) + B#(10)
                   + B#(12)
220      FOR J = 1 TO 13
230            SWAP A#(I), B#(J)
240      NEXT J
250 PRINT B#(1), m
260 NEXT m
```

以下爲 6 行 n 列之原始程式碼

```
10 CLS
20 DIM A#(21),B#(21)
30 FOR I = 1 TO 21
    40  A#(1) = 1
50      B#(1) = 1
60 NEXT I
70 INPUT N%
80 FOR m = 1 TO N%
90         A#(1) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(5)+ B#(6)+ B#(7)+
                   B#(8)+ B#(9)+ B#(10)+ B#(11)+ B#(12)+ B#(13)+
                   B#(14)+ B#(15)+   B#(16)+ B#(17)+ B#(18)+ B#(19)+
                   B#(20)+ B#(21)
100        A#(2) = B#(1)+ B#(10)+ B#(11)+ B#(12)+ B#(13)+ B#(14)+
                   B#(18)+ B#(19)+ B#(20)+ B#(21)
110        A#(3) = B#(1)+ B#(10)+ B#(11)+ B#(12)+ B#(13)+ B#(14)+
                   B#(18)+ B#(19)+ B#(21)
120        A#(4) = B#(1)+ B#(10)+ B#(11)+ B#(12)+ B#(14)+ B#(18)+
                   B#(19)+ B#(21)
130        A#(5) = B#(1)+ B#(10)+ B#(11)+ B#(13)+ B#(18)+ B#(20)
140        A#(6) = B#(1)+ B#(10)+ B#(13)+ B#(14)+ B#(15)+ B#(16)+
                   B#(17)+ B#(20)+ B#(20)
150        A#(7) = B#(1)+ B#(10)+ B#(13)+ B#(15)+ B#(16)+ B#(20)
160        A#(8) = B#(1)+ B#(10)+ B#(11)+ B#(12)+ B#(14)+ B#(15)+
                   B#(17)+ B#(18)+ B#(19)+ B#(21)
170        A#(9) = B#(1)+ B#(10)+ B#(11)+ B#(12)+ B#(14)+ B#(15)+
                   B#(16)+ B#(18)+ B#(19)+ B#(20)+B#(21)
180        A#(10) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(5)+ B#(6)+ B#(7)+
                   B#(8)+ B#(9)+ B#(15)+ B#(16)+ B#(17)+ B#(18)+
                   B#(19)+ B#(20)+ B#(21)
190      A#(11) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(5)+ B#(6)+ B#(7)+
               B#(8)+B#(9)+ B#(15)+ B#(16)+ B#(17)+ B#(20)+ B#(21)
200        A#(12) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(8)+ B#(15)+ B#(16)+
B#(20)
210        A#(13) = B#(1)+ B#(2)+ B#(3)+ B#(5)+ B#(6)+ B#(7)+ B#(8)+
                   B#(9)+ B#(15)+ B#(17)+B#(18)+B#(19)+B#(20)
```

```
220       A#(14) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(6)+ B#(8)+ B#(15)+
                   B#(16)+B#(18)+B#(20)
230       A#(15) = B#(1)+ B#(2)+ B#(6)+ B#(7)+ B#(8)+ B#(9)+ B#(10)+
                   B#(11)+ B#(12)+ B#(13)+ B#(14)+ B#(18)+ B#(19)+
                   B#(20)+ B#(21)
240       A#(16) = B#(1)+ B#(2)+ B#(6)+ B#(7)+ B#(9)+ B#(10)+ B#(11)+
                   B#(12)+ B#(14)+ B#(18)+B#(19)+B#(21)
250       A#(17) = B#(1)+ B#(2)+ B#(6)+ B#(8)+ B#(10)+ B#(11)+
                   B#(13)+ B#(18)+ B#(20)
260       A#(18) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(5)+ B#(8)+ B#(9)+
                   B#(10)+ B#(13)+ B#(14)+ B#(15)+ B#(16)+ B#(17)+
                   B#(20)+ B#(21)
270       A#(19) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(8)+ B#(10)+ B#(13)+
                   B#(15)+ B#(16)+ B#(20)+
280       A#(20) = B#(1)+ B#(2)+ B#(3)+ B#(5)+ B#(6)+ B#(7)+ B#(9)+
                   B#(10)+ B#(11)+ B#(12)+ B#(14)+ B#(15)+ B#(17)+
                   B#(18)+ B#(19)+ B#(21)
290       A#(21) = B#(1)+ B#(2)+ B#(3)+ B#(4)+ B#(6)+ B#(8)+
                   B#(10)+B#(11)+ B#(13)+ B#(15)+ B#(16)+ B#(18)+
                   B#(20)
300       FOR J=1 TO 21
310      SWAP A#(J),B#(J)
320      NEXT J
330 PRINT B#(1),m
340 NEXT m
```

Pattern Counting On Chessboards

I. Motivation

A problem in the lecture of our school's science camp states that "if we place black and white chess pieces in a row of cells with the restriction that no two adjacent cells be placed with white chess pieces, how many admissible patterns are there?" The answer is basically the famous Fibonacci sequence. We are, therefore, curious about the situation where the chessboards are rectangular or even of other shapes, for example, annular.

II. Purpose

Definition 1. *A pattern of black and white chess pieces on a chessboard is said to be admissible if no two lattices with a common edge are placed with white chess pieces.*

In this research, we wish to compute

1. the number R(m,n) of admissible patterns on an m×n rectangular chessboard;
2. the number D(m,n) of admissible patterns on an m×n rectangular chessboard with a diagonal joining the upper right-hand corner with the

lower left-hand corner added on each cell;

3. the number C(m,n) of admissible patterns on an $m\times n$ annular chessboard;
4. the number Q(m,n,k) of admissible patterns on an $m\times n\times k$ rectangular polyhedral chessboard;
5. the number U(m,n) of admissible patterns on an $m\times n$ rectangular chessboard up to the action flipping from left to right;
6. the number V(m,n) of admissible patterns on an $m\times n$ annular chessboard up to the action flipping inside out;
7. the number T(m,n) of admissible patterns on an $m\times n$ annular chessboard up to rotation.

All of the above notations are illustrated in Table 1.

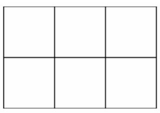
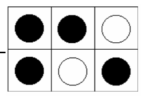
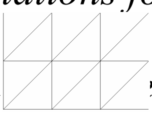
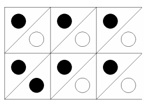
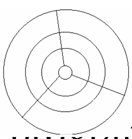
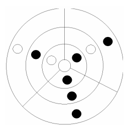
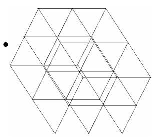
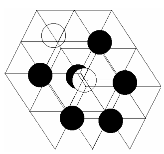
Notation	Shape of the chessboard	Example of admissible patterns
R(m,n)	 rectangular (m=2,n=3)	
D (m,n)	 diagonal (m=2,n=3)	
C(m,n)	 annular (m=3,n=3)	
Q(m,n,k)	 rectangular polyhedral (m=2,n=2,k=2)	
U(m,n)	rectangular (m=2,n=4)	Flipping from left to right, is transformed to , so they are identified.
V(m,n)	annular (m=2,n=3)	Flipping inside out, is transformed to , so they are identified.
T(m,n)	annular (m=2,n=3)	By rotation, is transformed to , so they are identified.

Table 1. *Notations for the numbers of admissible patterns on various kinds of chessboards*

III. Software

1. Spreadsheet program Excel
2. Maple VII.

IV. The process of research

Chapter 1. Method computing R(m,n)

Given an $m \times n$ rectangular chessboard A, how to count the number R(m,n) of admissible patterns ? We have the following methods.

Section 1.1. Brute force

The rawest idea to count R(m,n) is to list out all of the admissible patterns and count them one by one . It is suitable for m, n small. Once m, n grows, it is not easy (even horrible) to list out all admissible patterns. For example, what shown in Fig. 1 is all of the admissible patterns when A is 2×4 and 1×5, respectively.

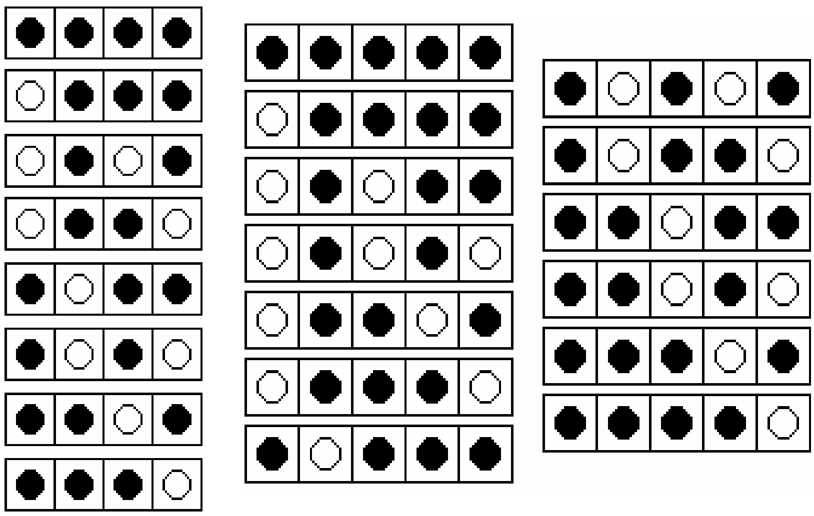


Fig.1

Section 1.2. Computing R(m,n) by tree diagrams .

It is quite a good method to compute R(m,n) by tree diagrams.

Definition 2. *A pattern on a column of cells is called a column pattern and* s*imilarly we define the so called* 「*row pattern*」.

Observing that any admissible pattern on an m×n rectangular chessboard is composed of column patterns, it is natural to compute R(m,n) by taking advantage of tree diagrams . For example, we show in Fig.2 the counting of R(A) where A is 2×5 . In this case, there are 3 column patterns, say $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$, $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$, and $\begin{bmatrix}\circ\\\bullet\end{bmatrix}$.

What can match with $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$ is $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$, $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$, and $\begin{bmatrix}\circ\\\bullet\end{bmatrix}$,what can match with $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$ is $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$ and $\begin{bmatrix}\circ\\\bullet\end{bmatrix}$ while $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$ and $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$ match with $\begin{bmatrix}\circ\\\bullet\end{bmatrix}$.

According to the above reasons, if we put a $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$ on the fourth column, then there are 3 patterns can be put on the fifth column, while there are only

two

patterns admissible on the fifth column if what put on the fourth column is $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$ or $\begin{bmatrix}\circ\\\bullet\end{bmatrix}$.

Now, if the third column position are placed $\begin{bmatrix}\bullet\\\bullet\end{bmatrix}$ then there are totally 3+2+2 patterns are admissible. Similarly, the corresponding number is 5 if there placed on the third column a $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$ or $\begin{bmatrix}\bullet\\\circ\end{bmatrix}$. Successively, we get R(2,5)=99.

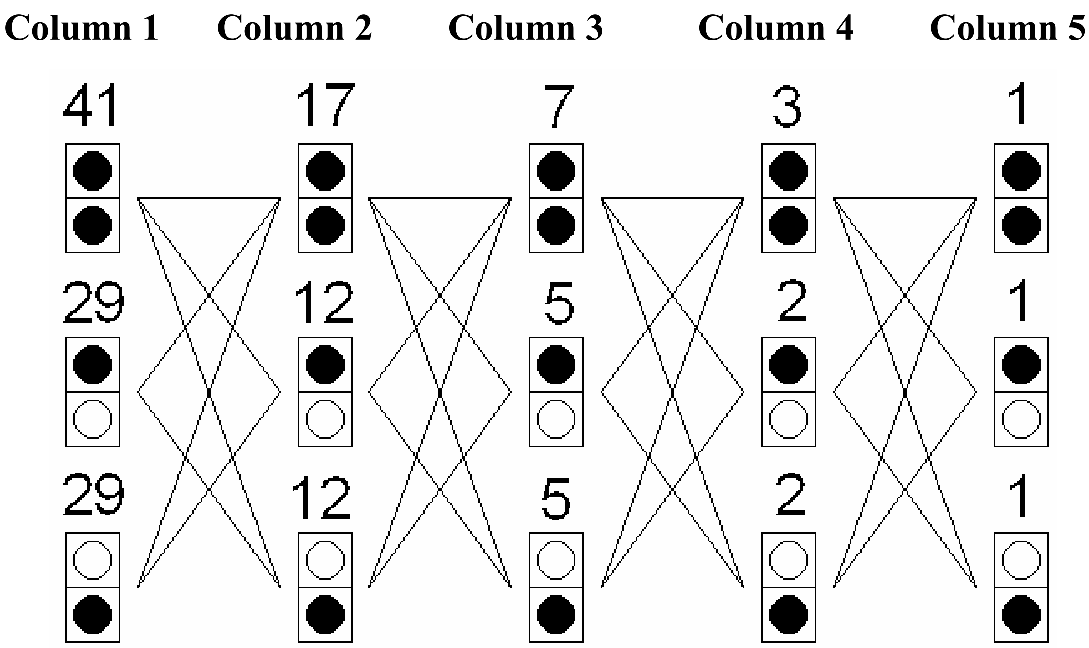
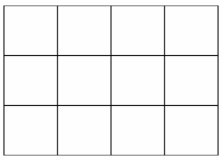
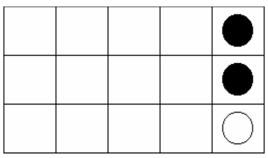


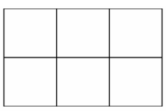
Fig. 2 *The tree diagram computing R(2,5)*

Section 1.3. Computing symbolically

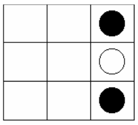
First we claim with examples the following identifications for the sake of convinience.

(1)  = [3,4]

(2)  = $\left[5, \begin{matrix} b \\ b \\ w \end{matrix}\right]$

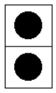
(3) R() = R([2,3]) = R(2,3)

(4) Extend R to chessboards having chesses placed in some lattices of

which, and R() = R([3, $\begin{smallmatrix}b\\w\\b\end{smallmatrix}$]) = R(3, $\begin{smallmatrix}b\\w\\b\end{smallmatrix}$)

In addition to the above identifications, we define here the notion of "complete term" and "residual terms". Roughly speaking, Terms of the form R(m,n) or R(n, $\begin{smallmatrix}b\\b\\\vdots\\b\end{smallmatrix}$) are complete, while R(n,P_0) with $P_0 \neq$($\begin{smallmatrix}b\\b\\\vdots\\b\end{smallmatrix}$) is residual . For example , R(2,3), R(3, $\begin{smallmatrix}b\\b\\b\end{smallmatrix}$) = R(3,2) are complete and R(3, $\begin{smallmatrix}b\\b\\w\end{smallmatrix}$) , R(4, $\begin{smallmatrix}b\\b\\w\end{smallmatrix}$) are residual.

Furthermore, we define

R(1,0) = R() = R(1,b) = 1, R(2,0) = R() = R(1, $\begin{smallmatrix}b\\b\end{smallmatrix}$) = 1.

And in general R(m,0)=1, for m$\in$n.

The following computation reveals the intimate relation between complete terms and residual terms.

Section 1.4. Counting by computers

Based on tree diagrams, one can use computers to get the values of R(m,n). In particular, we find it very convenient to perform such computations by the spreadsheet program Excel, as shown in Table 2.

m / n	1	2	3	4	5	6	7	8
1	2	3	5	8	13	21	34	55
2	3	7	17	41	99	239	577	1393
3	5	17	63	227	827	2999	10897	39561
4	8	41	227	1234	6743	36787	200798	1095851
5	13	99	827	6743	55447	454385	3729091	30584687
6	21	239	2999	36787	454385	5598861	69050253	851302029

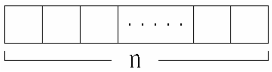
Table 2. *Some values of R(m,n) counted by the spreadsheet.*

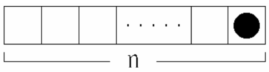
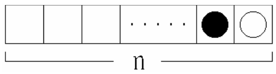
Chapter 2. The recursion property of R(m, n)

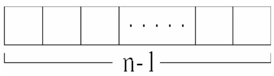
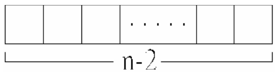
Since R (m,n) = R (n,m), we fix m below and study the properties of the sequence $<R(m,n)>_{n \in N}$.

Section 2.1. The recursion property of $<R(1,n)>_{n \in N}$

$<R(1,n)>_{n \in N}$ is basically the Fibonacci sequence, which is verified as follows：

R (1,n) = R()

= R() + R()

= R() + R()

= R(1,n-1) + R(1,n-2).

So we have

Proposition 1. *R(1,n) = R (1,n-1) + R(1,n-2).*

Moreover, R (1,0) = 1, R (1,1) = 2.

In a word, $<R(1,n)>_{n \in N}$ is the sequence 1,2,3,5,8,13,21……, with the first "1" of the Fibonacci sequence removed.

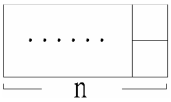
Section 2.2.The recursion property of $<R(2,n)>_{n \in N}$

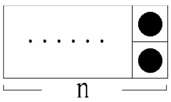
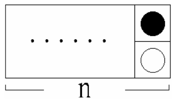
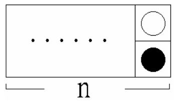
Observing Table 1, we conjecture that the recursion formula of

$\langle R(2,n) \rangle_{n \in N}$ is as follows

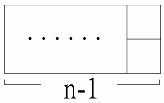
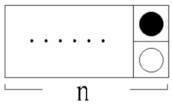
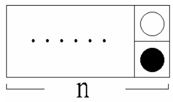
Proposition 2. *R (2,n) = 2R (2,n-1) + R (2,n-2), for* $n \geq 3$.

Proof

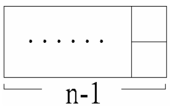
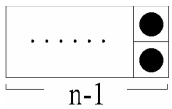
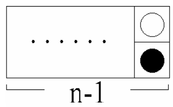
R (2, n) = R ()

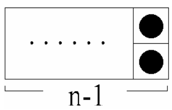
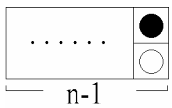
= R () + R () + R ()

$= R(n, \genfrac{}{}{0pt}{}{b}{b}) + R(n, \genfrac{}{}{0pt}{}{b}{w}) + R(n, \genfrac{}{}{0pt}{}{w}{b})$

= R() + R() + R()

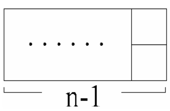
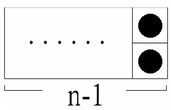
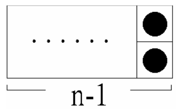
$= R(2, n\text{-}1) + R(n, \genfrac{}{}{0pt}{}{b}{w}) + R(n, \genfrac{}{}{0pt}{}{w}{b})$

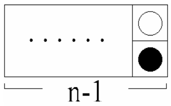
= R() + [R() + R()]

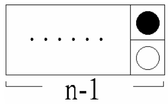
+ [R() + R()]

$= R(2, n\text{-}1) + [\, R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{b}) + R(n\text{-}1, \genfrac{}{}{0pt}{}{w}{b})\,] + [\, R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{b})$

$+ R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{w})\,]$

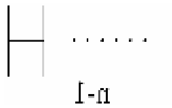
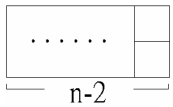
= R() + R() + [R()

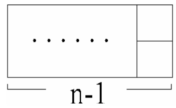
+R (

) + R (

)]

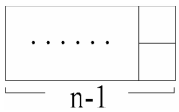
$$= R(2, n\text{-}1) + R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{b}) + [\, R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{b}) + R(n\text{-}1, \genfrac{}{}{0pt}{}{w}{b}) + R(n\text{-}1, \genfrac{}{}{0pt}{}{b}{w}) \,]$$

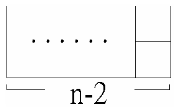
= R (
) + R (

) + R (

)

$$= R(2, n\text{-}1) + R(2, n\text{-}2) + R(2, n\text{-}1)$$

= 2R(

) + R(

)

$$= 2R(2, n\text{-}1) + R(2, n\text{-}2). \qquad \text{Q.E.D.}$$

Now we intend to compute the limit of $\frac{R(2,n)}{R(2,n-1)}$. Imitating the argument used to derive the Binet formula for Fibonacci sequence, we argue as follows.

Let $\frac{R(2,n)}{R(2,n-1)} = b_n$.

From $\frac{R(2,n)}{R(2,n-1)} = 2 + \frac{R(2,n-2)}{R(2,n-1)}$,

we get $b_n = 2 + \frac{1}{b_{n-1}}$.

Suppose that exist $\langle b_n \rangle$ converges to a real number x.

Taking limits,we get $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(2+\frac{1}{b_{n-1}})$,

hence $x = 2 + \frac{1}{x}$,

so that we obtain $x^2 - 2x - 1 = 0$,

which has two roots $1+\sqrt{2}$ and $1-\sqrt{2}$.

On the other hand, we know that for <R(2, n)>$_{n\in N}$, it holds that

$$R(1,n) = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{(n+1)} - (\frac{1-\sqrt{5}}{2})^{(n+1)}]$$

(Notice that the sequence <R(1, n)>$_{n\in N}$ is 1,2,3,5,8,13……).

Therefore, we predict that <R(2, n)>$_{n\in N}$ should have a similar formula .

Assume first that

$$R(2, n)=\alpha(1+\sqrt{2})^n+\beta(1-\sqrt{2})^n$$

Since R(2, 1)=3, R(2, 2)=7, we have

$$\begin{cases} 3 = \alpha(1+\sqrt{2}) + \beta(1-\sqrt{2}), \\ 7 = \alpha(1+\sqrt{2})^2 + \beta(1-\sqrt{2})^2, \end{cases}$$

from which we obtain $\alpha = \frac{1+\sqrt{2}}{2}$ and $\beta = \frac{1-\sqrt{2}}{2}$.

Hence we guess that

Proposition 3. $R(2,n) = \frac{1}{2}[\ (1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}, for\ \ n \geq 0.$

Proof we use iduction on n .

[1]. when n = 1, R(2, 1) $= \frac{1}{2}[(1+\sqrt{2})^{1+1} + (1-\sqrt{2})^{1+1}]$

$$= \frac{1}{2}[(3+2\sqrt{2}) + (3-2\sqrt{2})]$$

$$= 3.$$

when n= 2, R(2, 2) $= \frac{1}{2}[(1+\sqrt{2})^{2+1} + (1-\sqrt{2})^{2+1}]$

$= \frac{1}{2}[(7+5\sqrt{2}) + (7-5\sqrt{2})]$

=7.

[2]. Suppose that for $1 \le n \le k (k \ge 2)$,

$$R(2, n) = \frac{1}{2}[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}],$$

then R(2,k+1)=2R(2,k)+R(2,k–1)

$$= 2\cdot\frac{1}{2}[(1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1}] + \frac{1}{2}[(1+\sqrt{2})^{k} + (1-\sqrt{2})^{k}]$$

$$= \frac{1}{2}[2(1+\sqrt{2})^{k+1} + 2(1-\sqrt{2})^{k+1}] + \frac{1}{2}[(1+\sqrt{2})^{k} + (1-\sqrt{2})^{k}]$$

$$= \frac{1}{2}[(1+\sqrt{2})^{k}(3+2\sqrt{2}) + (1-\sqrt{2})^{k}(3-2\sqrt{2})]$$

$$= \frac{1}{2}[(1+\sqrt{2})^{k+2} + (1-\sqrt{2})^{k+2}].$$

Proposition 3 then is proved by mathematical induction . Q.E.D.

Now that for $n \ge 0$, R(2,n) $= \frac{1}{2}[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}]$,

we have

$$\lim_{n\to\infty} \frac{R(2,n)}{R(2,n-1)} = \lim_{n\to\infty} \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}} = 1+\sqrt{2},$$

which says that $\lim_{n\to\infty} \frac{R(2,n)}{R(2,n-1)}$ indeed exists and equals $1+\sqrt{2}$.

Section 2.3.The recursion formula for <R(3, n)>$_{n\in N}$

To study <R(3,n)>$_{n\in N}$, we need the following preliminary identities .

$$R\begin{pmatrix} b \\ n,b \\ b \end{pmatrix} = R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix},$$

$$R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} = R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} = R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix},$$

$$R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} = R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix},$$

$$R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} = R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix}.$$

Imitating the method deriving the recursion formula for $<R(2,n)>_{n\in N}$, we calculated as follows：

$$\begin{aligned}
R(3,n) &= R\begin{pmatrix} b \\ n,b \\ b \end{pmatrix} + R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} \\
&= R(3,n-1) + R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} \\
&= R(3,n-1) + R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix} \\
&\quad + R\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n-1,b \\ b \end{pmatrix} \\
&\quad + R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} \\
&= R(3,n-1) + 4\,R(3,n-2) + 2R\begin{pmatrix} w \\ n-1,b \\ b \end{pmatrix} + 3R\begin{pmatrix} b \\ n-1,w \\ b \end{pmatrix} + 2R\begin{pmatrix} b \\ n-1,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n-1,b \\ w \end{pmatrix}
\end{aligned}$$

$$= R(3,n-1)+4R(3,n-2)+2R\begin{pmatrix} b \\ n-2,b \\ b \end{pmatrix}+2R\begin{pmatrix} b \\ n-2,w \\ b \end{pmatrix}+2R\begin{pmatrix} b \\ n-2,b \\ w \end{pmatrix}$$

$$+3R\begin{pmatrix} b \\ n-2,b \\ b \end{pmatrix}+3R\begin{pmatrix} b \\ n-2,b \\ w \end{pmatrix}+3R\begin{pmatrix} w \\ n-2,b \\ b \end{pmatrix}+3R\begin{pmatrix} w \\ n-2,b \\ w \end{pmatrix}+2R\begin{pmatrix} b \\ n-2,b \\ b \end{pmatrix}$$

$$+2R\begin{pmatrix} w \\ n-2,b \\ b \end{pmatrix}+2R\begin{pmatrix} b \\ n-2,w \\ b \end{pmatrix}+R\begin{pmatrix} b \\ n-2,b \\ b \end{pmatrix}+R\begin{pmatrix} b \\ n-2,w \\ b \end{pmatrix}$$

$$= R(3,n-1)+4R(3,n-2)+8R(3,n-3)+5R\begin{pmatrix} w \\ n-2,b \\ b \end{pmatrix}+5R\begin{pmatrix} b \\ n-2,w \\ b \end{pmatrix}$$

$$+5R\begin{pmatrix} b \\ n-2,b \\ w \end{pmatrix} + 3\,R\begin{pmatrix} w \\ n-2,b \\ w \end{pmatrix}$$

$$= R(3,n-1)+4R(3,n-2)+8R(3,n-3)+5R\begin{pmatrix} b \\ n-3,b \\ b \end{pmatrix}+5R\begin{pmatrix} b \\ n-3,w \\ b \end{pmatrix}+5R\begin{pmatrix} b \\ n-3,b \\ w \end{pmatrix}$$

$$+5R\begin{pmatrix} b \\ n-3,b \\ b \end{pmatrix}+5R\begin{pmatrix} b \\ n-3,b \\ w \end{pmatrix}+5R\begin{pmatrix} w \\ n-3,b \\ b \end{pmatrix}+5R\begin{pmatrix} w \\ n-3,b \\ w \end{pmatrix}+5R\begin{pmatrix} b \\ n-3,b \\ b \end{pmatrix}$$

$$+5R\begin{pmatrix} w \\ n-3,b \\ b \end{pmatrix}+5R\begin{pmatrix} b \\ n-3,w \\ b \end{pmatrix}+3R\begin{pmatrix} b \\ n-3,b \\ b \end{pmatrix}+3R\begin{pmatrix} b \\ n-3,w \\ b \end{pmatrix}$$

$$= R(3,n-1)+4R(3,n-2)+8R(3,n-3)+18R(3,n-4)+10R\begin{pmatrix} w \\ n-3,b \\ b \end{pmatrix}$$

$$+13R\begin{pmatrix} b \\ n-3,w \\ b \end{pmatrix}+10\,R\begin{pmatrix} b \\ n-3,b \\ w \end{pmatrix}+5\,R\begin{pmatrix} w \\ n-3,b \\ w \end{pmatrix}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

It was annoying that we could not eliminated the 3 types of residual terms

$R\begin{pmatrix} w \\ n-k,b \\ b \end{pmatrix} = R\begin{pmatrix} b \\ n-k,b \\ w \end{pmatrix}$, $R\begin{pmatrix} b \\ n-k,w \\ b \end{pmatrix}$, and $R\begin{pmatrix} w \\ n-k,b \\ w \end{pmatrix}$, where k is an integer .

Fortunately, it came to us that we could solve this problem by the following algorithm, called "**The Method of Residual Terms**"(abbreviated as **MRT**).

Step 1 Taking advantage of the preliminary identities successively to get 4 equations relating complete terms and residual terms .

Step 2 Solving the first 3 equations to express residual terms in terms of complete terms .

Step 3 Substituting the result obtained in step 2 into the fourth equation gives us the desired recursion formula .

The details are as follows.

We have

$$R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} = R(3,n) - R(3,n-1),$$

$$2R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + 3R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + 2R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} = R(3,n+1) - R(3,n) - 4R(3,n-1),$$

$$5R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + 5R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + 5R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + 3R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} = R(3,n+2) - R(3,n+1) - 4R(3,n) - 8R(3,n-1),$$

$$10R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix} + 13R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix} + 10R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix} + 5R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix} = R(3,n+3) - R(3,n+2) - 4R(3,n+1) - 8R(3,n) - 18R(3,n-1).$$

Let X = $R\begin{pmatrix} w \\ n,b \\ b \end{pmatrix}$ = $R\begin{pmatrix} b \\ n,b \\ w \end{pmatrix}$, Y = $R\begin{pmatrix} b \\ n,w \\ b \end{pmatrix}$, Z = $R\begin{pmatrix} w \\ n,b \\ w \end{pmatrix}$,

we obtain that

2X+Y+Z = R(3,n)-R(3,n-1), ……………………………………………(1)

4X+3Y+Z = R(3,n+1)-R(3,n)-4R(3,n-1), ………..………………… (2)

10X+5Y+3Z = R(3,n+2)-R(3,n+1)-4R(3,n)-8R(3,n-1),………………...(3)

20X+13Y+5Z = R(3,n+3)-R(3,n+2)-4R(3,n+1)-8R(3,n)-18R(3,n-1).….(4)

By (1), (2), and (3), we get

$$X = \frac{1}{2}R(3,n+2) - R(3,n+1) - \frac{5}{2}R(3,n) - R(3,n-1),$$

$$Y = -\frac{1}{2}R(3,n+2) + \frac{3}{2}R(3,n+1) + \frac{3}{2}R(3,n) - \frac{1}{2}R(3,n-1),$$

$$Z = -\frac{1}{2}R(3,n+2) + \frac{1}{2}R(3,n+1) + \frac{9}{2}R(3,n) + \frac{3}{2}R(3,n-1).$$

Substitute them into (4)

we get at once that R(3,n) = 2R(3,n-1)+ 6R(3,n-2) – R(3,n-4).

Therefore, we have

Proposition 4. *The recursion formula of* $\langle R(3,n)\rangle_{n\in N}$ *is*

R(3, 0) = 1 ，R (3, 1) = 5 ，R (3, 2) = 17,

R(3,n)= 2R(3,n-1)+ 6 R(3,n-2) – R(3,n-4), for n≥4 .

Next, we want to calculate again the limit of $\frac{R(3,n)}{R(3,n-1)}$.

Let $b_n = \frac{R(3,n)}{R(3,n-1)}$.

From $\frac{R(3,n)}{R(3,n-1)} = 2\frac{R(3,n-1)}{R(3,n-1)} + 6\frac{R(3,n-2)}{R(3,n-1)} - \frac{R(3,n-4)}{R(3,n-1)}$, We get $b_n = 2 + \frac{6}{b_{n-1}} - \frac{1}{b_{n-3}b_{n-2}b_{n-1}}$.

Assume that <b_n> converges to a real number x.

We have $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(2+\frac{6}{b_{n-1}}-\frac{1}{b_{n-3}b_{n-2}b_{n-1}})$,

hence $x = 2 + \frac{6}{x} - \frac{1}{x^3}$,

So that, $x^4 - 2x^3 - 6x^2 +1 = 0$.

By intermediate value theorem, there are four roots α_1, α_2, α_3, and α_4 for this equation and

$$-2 < \alpha_1 < -1,\ -1 < \alpha_2 < 0,\ 0 < \alpha_3 < 1,\ 3 < \alpha_4 < 4.$$

In fact, using intermediate value theorem repeatedly, we have

$$-1.568 < \alpha_1 < -1.567 \quad , \quad -0.453 < \alpha_2 < -0.452,$$

$$0.388 < \alpha_3 < 0.389 \quad , \quad 3.631 < \alpha_4 < 3.632 .$$

As before we have

Proposition 5. There exist constants C_1, C_2, C_3, and C_4 such that

$$R(3,n)= C_1(\alpha_1)^n+ C_2(\alpha_2)^n+ C_3(\alpha_3)^n+ C_4(\alpha_4)^n, \text{ for } n\geq 0 .$$

Proof

Substituting n = 0, 1, 2, 3 into the identity asserted in proposition 5, we get

$C_1+C_2+C_3+C_4=1$,

$C_1\alpha_1+C_2\alpha_2+C_3\alpha_3+C_4\alpha_4=5$,

$C_1(\alpha_1)^3+C_2(\alpha_2)^3+C_3(\alpha_3)^3+C_4(\alpha_4)^3=63$,

$C1(\alpha_1)^2+C_2(\alpha_2)^2+C_3(\alpha_3)^2+C_4(\alpha_4)^2=17$.

Since

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{vmatrix} = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_4) \neq 0,$$

the above equation has a unique set of solutions C_1,C_2,C_3, and C_4.

Assume that, for some $k \geq 4$, we have

$$R(3,n) = C1(\alpha_1)^n + C_2(\alpha_2)^n + C_3(\alpha_3)^n + C_4(\alpha_4)^n, \text{ for } 0 \leq n \leq k-1.$$

Then

R(3,k) = 2 R(3,k-1) + 6 R(3,k-2) - R (3,k-4)

$$= 2[C_1\alpha_1^{k-1} + C_2\alpha_2^{k-1} + C_3\alpha_3^{k-1} + C_4\alpha_4^{k-1}] + 6[C_1\alpha_1^{k-2} + C_2\alpha_2^{k-2} + C_3\alpha_3^{k-2} + C_4\alpha_4^{k-2}]$$

$$-[C_1\alpha_1^{k-4} + C_2\alpha_2^{k-4} + C_3\alpha_3^{k-4} + C_4\alpha_4^{k-4}]$$

$$= C_1\alpha_1^{k-4}(2\alpha_1^3 + 6\alpha_1^2 - 1) + C_2\alpha_2^{k-4}(2\alpha_2^3 + 6\alpha_2^2 - 1)$$

$$+ C_3\alpha_3^{k-4}(2\alpha_3^3 + 6\alpha_3^2 - 1) + C_4\alpha_4^{k-4}(2\alpha_4^3 + 6\alpha_4^2 - 1)$$

$$= C\,\alpha^{k-4}\alpha^4 + C\,\alpha^{k-4}\alpha^4 + C\,\alpha^{k-4}\alpha^4 + C_4\alpha_4^{k-4}\alpha_4^4$$

$$= C\,\alpha^k + C\,\alpha^k + C\,\alpha^k + C_4\alpha_4^k.$$

Hence for $n \geq$ $n) = C\,\alpha^n + C\,\alpha^n + C\,\alpha^n + C_4\alpha_4^n$, and

$$\lim_{n\to\infty} \frac{(3,\ )}{(3,\ -1)} = \lim_{n\to\infty} \frac{\alpha_4^n (C_4 + C_3(\frac{\alpha_3}{\alpha_4})^n + C_2(\frac{\alpha_2}{\alpha_4})^n + C_1(\frac{\alpha_1}{\alpha_4})^n)}{\alpha_4^{n-1}(C_4 + C_3(\frac{\alpha_3}{\alpha_4})^{n-1} + C_2(\frac{\alpha_2}{\alpha_4})^{n-1} + C_1(\frac{\alpha_1}{\alpha_4})^{n-1})} = \alpha_4,$$ Q.E.D.

Section 2.4. The recursion formula of $<R(m,n)>_{n\in N}$, for m≥4

By means of MRT we can investigate the corresponding problems when

$m \geq 4$.

For example, for m = 4, we have

Proposition 6.

[1] For $n \geq 5$, *R(4,n)=4R(4,n-1)+9R(4,n-2)-5R(4,n-3)-4R(4,n-4)+R(4,n-5),*

and R(4,0) =1, R (4,1) = 8, R(4,2) = 41, R(4,3) = 227, R(4,4) =1234.

[2] For $n \geq 5$, *R(4,n)=* $d_1\beta_1 + d_2\beta_2 + d_3\beta_3 + d_4\beta_4 + d_5\beta_5$,

where the dj's are constants are constants and $\beta_1, \beta_2, \beta_3, \beta_4,$ *and* β_5

are distinct real numbers.

In fact, the characteristic equation

$$x^5 - 4x^4 - 9x^3 + 5x^2 + 4x - 1 = 0$$

gives us 5 distinct real roots β_1, β_2 , β_3, β_4, and β_5, where

$-1.79087<\beta_1<-1.79086$, $-0.63176<\beta_2< 0.63175$,

$0.21634<\beta_3<0.21635$, $0.74856<\beta_4<0.74857$,

$5.45770<\beta_5<5.45771$.

Section 2.5. The number of residual terms

Let $m \in N$ and $P_0 \neq \begin{pmatrix} b \\ b \\ . \\ . \\ b \end{pmatrix}$ be an admissible column pattern of length m .

Any residual terms of the form $R(n-k,P_0)$, k a positive integer, are viewed of the same type . Denote by S(m) the number of different types of residual terms present in deriving the recursion formula of $<R(m, n)>_{n \in N}$.

According to MRT, the order of the recursion formula equals S(m)+1 . With the expectation to get a unified recursion formula for R(m, n), we think it essential to derive a formula for S(m) .

Given an admissible column pattern P_0, denote by P_0^* the pattern obtained by flipping the upside of P_0 down . P_0, together with R(n-k, P_0), are said to be symmetric if $P_0 = P_0^*$. Obviously R(n-k, P_0)=R(n-k, P_0^*) .

It is apparent that, counting S(m) is equivalent to count the number of admissible column patterns up to symmetry with the pattern $\begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}$ excluded.

Denote by $S_1(m)$ the number of symmetric residual terms and $S_2(m)$ the number of nonsymmetric residual terms. Then we have

$$\begin{aligned} S(m) &= S_1(m)+S_2(m) \\ &= S_1(m)+\frac{1}{2}\left(R(m, 1)-R\left(1, \begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}\right)-S_1(m)\right) \\ &= \frac{1}{2}(R(m, 1)+S_1(m)-1) . \end{aligned}$$

Now we discuss below according to either m is even or odd .

Case 1 . m is odd .

If $P_0 \neq \begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}$ is a symmetric admissible column pattern of length m, then P_0 must equal to either

$A=\begin{pmatrix} q_1 \\ b \\ q^*_1 \end{pmatrix}$ or $B=\begin{pmatrix} q_2 \\ b \\ w \\ b \\ q^*_2 \end{pmatrix}$,

Where $q_1 \neq \begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}$ is admissible with length $\frac{m-1}{2}$ while q_2 is admissible with length $\frac{m-3}{2}$. As a result.

$S_1(m)=R(\frac{m-1}{2}, 1)-1+R(\frac{m-3}{2},1)$ and

$S(m)=\frac{1}{2}(R(m,1)+R(\frac{m-1}{2},1)+R(\frac{m-3}{2},1)-2)$.

Case 2. m is even.

If $P_0 \neq \begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}$ is a symmetric admissible column pattern with length m, then P_0 must be of the form $\begin{pmatrix} q \\ \\ q^* \end{pmatrix}$,

where $q \neq \begin{pmatrix} b \\ b \\ \cdot \\ \cdot \\ b \end{pmatrix}$ is admissible of length $\frac{m-2}{2}$. Thus

$S_1(m)=R(\frac{m-2}{2},1)-1$ and $S(m)=\frac{1}{2}(R(m,1)+R(\frac{m-2}{2},1)-2)$.

Combining the above discussions, we have

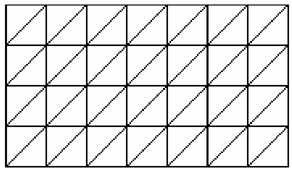
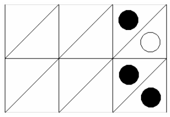
Proposition 7.

[1] If m is odd, then $S(m)=\frac{1}{2}(R(m,1)+R(\frac{m-1}{2},1)+R(\frac{m-3}{2},1)-2)$.

[2] If m is even, then $S(m)=\frac{1}{2}(R(m, 1)+R(\frac{m-2}{2}, 1)-2)$

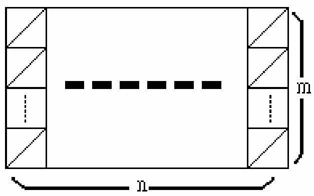
Chapter 3. Diagonal chessboards

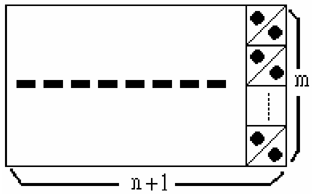
We consider here a kind of special chessboards, called "diagonal chessboards”. Note that a diagonal chessboard with m rows and n columns has 2mn triangular cells.

For convinience, We identify, for example,  with the notation <4,7> and  with $<3, \begin{matrix} b/w \\ b/b \end{matrix}>$.

We begin with the following definition.

Definition 3. *With the restriction that no two cells with a common edge are both placed with white chess pieces, the number of admissible patterns on an* $m \times n$ *diagonal chessboard, which has 2mn cells in total, is denoted by*

D(m,n)=D()

= D() $= D(n+1, \begin{matrix} b & b \\ b & b \\ \vdots & \vdots \\ b & b \end{matrix})$

It turns out that MRT works well here. For example, we discuss below the recursion property of D(1,n) and D(2,n), respectively.

Section 3.1. The recursion formula of D(1,n)

Since D(n,b/w) =D(n,b/b), we have

$$
\begin{aligned}
D(1,n) &= D(n+1,b/b) \\
&= D(n,b/b) + D(n,b/w) + D(n,w/b) \\
&= 2D(n,b/b) + D(n,w/b) \\
&= [2D(n-1,b/b) + 2D(n-1,b/w) + 2D(n-1,w/b)] \\
&\quad + [D(n-1,b/b) + D(n-1,w/b)] \\
&= 5D(n-1,b/b) + 3D(n-1,w/b) \\
&= 6D(n-1,b/b) + 3D(n-2,w/b) – D(n-1,b/b) \\
&= 3D(n-1,b/b) + D(n-2,b/w) + D(n-2,w/b)] – D(n-1,b/b) \\
&= 3D(n,b/b) – D(n-1,b/b) \\
&= 3D(1,n-1) – D(1,n-2).
\end{aligned}
$$

Therefore, we obtain

Proposition 8. *$D(1,n) = 3D(1,n-1) – D(1,n-2)$ for $n \geq 2$*

and $D(1,0) = 1$，$D(1,1) = 3$.

What worth noting is that

$$D(1,n) = R(1,2n)$$

Hence

$$
\begin{aligned}
D(1,n) &= R(1,2n) \\
&= R(1,2n-1) + R(1,2n-2) \\
&= [R(1,2n-2) + R(1,2n-3)] + R(1,2n-2) \\
&= 2R(1,2n-2) + R(1,2n-3) \\
&= 2R(1,2n-2) + [R(1,2n-2)] - R(1,2n-4)] \\
&= 3R(1,2n-2) - R(1,2n-4) \\
&= 3D(1,n-1) -D(1,n-2)
\end{aligned}
$$

Section 3.2. The recursion formula of D(2,n)

We start with the following basic identities.

$$D(n,\genfrac{}{}{0pt}{}{b/b}{b/b}) = D(n,\genfrac{}{}{0pt}{}{b/w}{b/b}) = D(n,\genfrac{}{}{0pt}{}{b/w}{b/w}) = D(n,\genfrac{}{}{0pt}{}{b/b}{b/w})$$

$$= D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{w/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{b/w})$$

$$+ D(n\text{-}1,\genfrac{}{}{0pt}{}{b/w}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{b/w}{b/w}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{w/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{b/w}),$$

$$D(n,\genfrac{}{}{0pt}{}{w/b}{b/b}) = D(n,\genfrac{}{}{0pt}{}{w/b}{b/w}) = D(n,\genfrac{}{}{0pt}{}{b/b}{w/b})$$

$$= D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{w/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{b/w})$$

$$+ D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{w/b}) + DB(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{b/w}),$$

$$D(n,\genfrac{}{}{0pt}{}{b/b}{w/b}) = D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{b/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{w/b}{w/b}) + D(n\text{-}1,\genfrac{}{}{0pt}{}{b/b}{w/b}).$$

Now MRT gives us that

Proposition 9. D D D D $n \geq 3$

D , D , D

Chapter 4. The number of admissible atterns u to s mmetr

In preceding sections the rectangular chessboards are viewed as fixed on a desk .We are curious about that, what is the number of admissible patterns if we identify patterns up to the action that flipping from left to right? (Since the problem of counting admissible patterns up to flipping upside down is the same as that of flipping from left to right, we consider only the latter.)

Definition 4. *Two admissible patterns P_1 and P_2 on a rectangular chessboard are said to be LR equivalent if P_1 can become identical with P_2 via flipping from left to right .If an admissible pattern P is invariant via flipping from left to right, it is called LR symmetric. The number of LR equivalence classes on an $m\times n$ rectangular chessboard is denoted by U(m,n).*

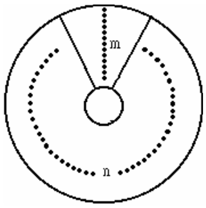
We compute U(m,n) as follows.

Case 1. If n is odd, consider the subchessdoard B formed by the $\frac{n+1}{2}$ columns of A counting from the left . Observe that by flipping an admissible pattern on B over the right half part of A, we obtain a LR symmetric admissible pattern on A . Obviously all of the LR symmetric admissible patterns on A can be obtained in this way . Hence the number of LR equivalence classes is $\dfrac{R(m,n)+R(m,\frac{n+1}{2})}{2}$.

Case 2. If n is odd, then evidently the midst two columns of A must be placed with black chess pieces . Argue as in case 1, we conclude that the number of LR equivalence classes is $\dfrac{R(m,n)+R(m,\frac{n}{2}-1)}{2}$.

Chapter 5. Admissible patterns on annular chessboards

In addition to rectangular and diagonal cases, we intend to study in this section the corresponding problem where the chessboards concerned are annular.

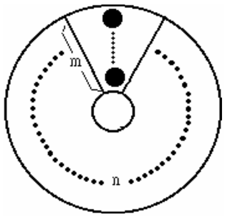
Let A = 

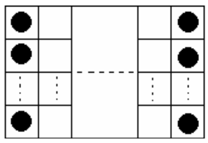
be an annular chessboard with m rows and n columns . We shall say abbreviately that A is an m×n annular chessboard . Choose any one of the n columns of A as the first column and then number the n columns clockwise .

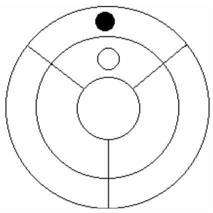
For the sake of convinience, we claim the following identifications.

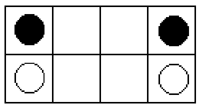
1. If A is an m×n annular chessboard, then A is identified with the pair (m,n) .
2. Let A be an m×n annular chessboard. The state that there placed on the first column of A an admissible column pattern P_0 is identified with the symbol (n,P_0).
3. Let A be an m×n annular chessboard. The state that there placed clockwise an admissible row pattern P_0 on the outmost row of A is identified with the symbol (m,P_0) .

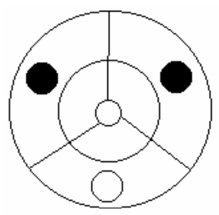
For example,

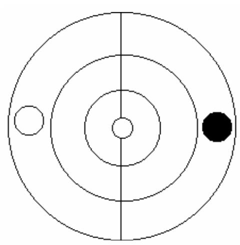
[1]  is identified with $\left(n, \begin{smallmatrix} b \\ \vdots \\ b \end{smallmatrix}\right)$ and also with the

rectangular chessboard  , thus with $\left(\begin{smallmatrix} b \\ b \\ \vdots \\ b \end{smallmatrix}, n+1, \begin{smallmatrix} b \\ b \\ \vdots \\ b \end{smallmatrix}\right)$

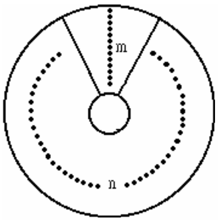
[2]  is identified with $\left(3, \begin{smallmatrix} b \\ w \end{smallmatrix}\right)$ and also with the rectangular

chessboard  , thus with $\left(\begin{smallmatrix} b \\ w \end{smallmatrix}, 4, \begin{smallmatrix} b \\ w \end{smallmatrix}\right)$.

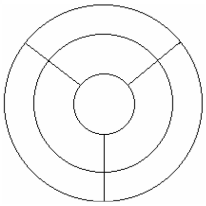
For example,  is identified with the (2, bwb)

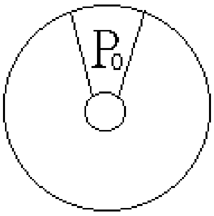
and  is identified with (3,bw) .

Now let A be an m×n annular chessboard belonging to one of the 3 types described above. Then we denote by C(A) the number of admissible patterns on A . With the above identifications, we have naturally the notations below.

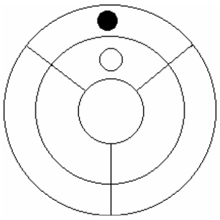
[1].C(m,n)=C(

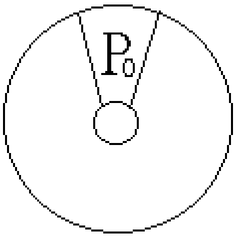
) .

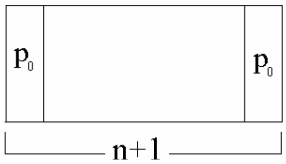
For example, C(2,3) = C() .

[2].C(

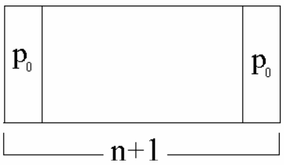
) = C(n ,P_0) .

For example, C() = C(3, $\begin{smallmatrix} b \\ w \end{smallmatrix}$) .

[3].Since 

is identified with 

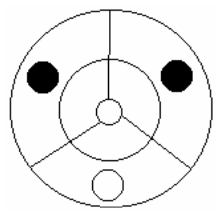
,

we see that C(

) = R(P_0, n+1,P_0) .

For example, C(3, $\begin{smallmatrix} b \\ w \end{smallmatrix}$) = R($\begin{smallmatrix} b \\ w \end{smallmatrix}$,4, $\begin{smallmatrix} b \\ w \end{smallmatrix}$) .

[4]. Let A be an m×n annular chessboard with an admissible row pattern P_0 on the outmost row. Then C(A) = C(m,P_0).

For example, C() = C(2, bwb).

Adopting the above notations, we want to study the recursion property of C(m,n) by fixing m and n respectively .

Section 5.1. The tree diagrams computing C(m,n)

For a pattern P on an m×n annular chessboard to be admissible, it is necessary that the first and m-th column patterns of P must match. It is this nature that makes things complicated and one might thought that there is no tree diagram computing C(m,n) at all. In fact, the tree diagram *does* exist and is composed of several subdiagrams, as shown in **Fig.3**. Notice that each of the subdiagrams is a variation of the diagram computing R(m,n) and the number of subdiagrams equals to that of residual terms.

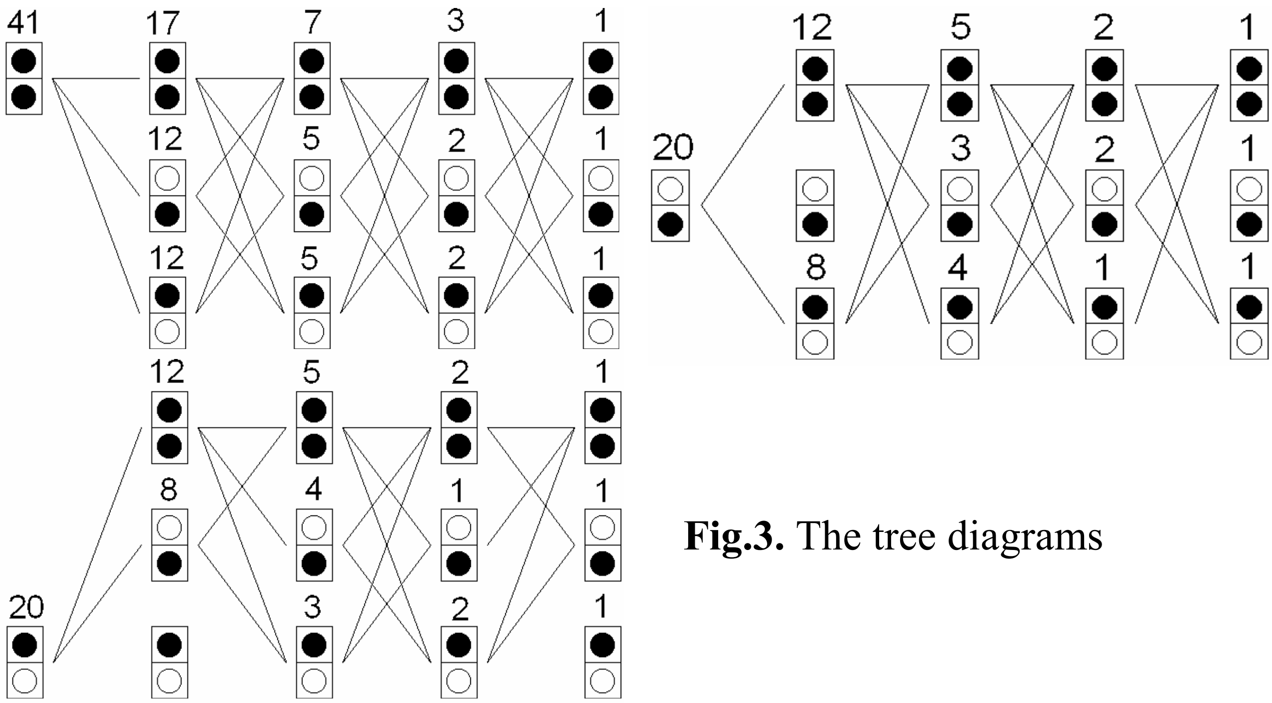


Fig.3. The tree diagrams

C(2,5)

Section 5.2. The recursion formula of $<C(m,n)>_{n\in N}$ for m fixed

When m = 1,it is very easy to get that, for $n \geq 3$

$C(1,n) = C(n,b) + C(n,w) = R(1,n\text{-}1) + R(1,n\text{-}3) = C(1,n\text{-}1) + C(1,n\text{-}3).$

Next, we desire to derive the recursion formula of $\langle C(2,n)\rangle_{n\in N}$. To deal with this problem the following identities is preliminary.

[1] For $n \ge 1$, $R(2,n)=R(2,n\text{-}1)+2R(n,{w \atop b})$.

[2] For $n \ge 1$, $C(2,n) = R(2,n\text{-}1) + 2R({w \atop b},n+1,{w \atop b})$.

(The identity [2] is, in fact, arised from the process we deriving the recursion formula of C(2,n) below.)

Now we have

$$C(2,n) = C(n,{b \atop b}) + C(n,{w \atop b}) + C(n,{b \atop w}) = R(2,n\text{-}1) + 2C(n,{w \atop b})$$

$$= R(2,n\text{-}1) + 2R({w \atop b},n+1,{w \atop b})$$

$$= R(2,n\text{-}1) + 2[R({w \atop b}, n,{b \atop b}) + R({w \atop b}, n,{b \atop w})]$$

$$= R(2,n\text{-}1) + 2R({w \atop b}, n,{b \atop b}) + 2R({w \atop b}, n,{b \atop w})$$

$$= R(2,n\text{-}1) + 2R(n\text{-}1,{w \atop b}) + 2[R({w \atop b}, n\text{-}1,{b \atop b}) + R({w \atop b}, n\text{-}1,{w \atop b})]$$

$$= R(2,n\text{-}1) + 2R(n\text{-}1,{w \atop b}) + 2R({w \atop b}, n\text{-}1,{b \atop b}) + 2R({w \atop b}, n\text{-}1,{w \atop b})$$

$$= R(2,n\text{-}1) + 2R(n\text{-}1,{w \atop b}) + 2R(n\text{-}2,{w \atop b}) + 2R({w \atop b}, n\text{-}1,{w \atop b})$$

$$= R(2,n\text{-}1)+[R(2,n\text{-}1) - R(2,n\text{-}2)]+[R(2,n\text{-}2) - R(2,n\text{-}3)] +[C(2,n\text{-}2) - R(2,n\text{-}3)]$$

$$=C(2,n\text{-}2)+2R(2,n\text{-}1)-2R(2,n\text{-}3),$$

so $\quad C(2,n) - C(2,n\text{-}2) = 2R(2,n\text{-}1)-2R(2,n\text{-}3)$, for $n \ge 4$.

(Checking in details tells us that the above identity holds only for $n \ge 4$.)

Furthermore, for $n \geq 2$, R(2,n)=2R(2,n-1)+R (2,n-2)

Thus, for $n \geq 6$,

$$C(2,n) - C(2,n-2) = 2[C(2,n-1) - C(2,n-3)] + [C(2,n-2) - C(2,n-4)]$$

Therefore, we obtain

Proposition 10.

C(2,n)= 2C(2,n-1) + 2C(2,n-2) – 2C(2,n-3) - C(2,n-4), for $n \geq 6$.

On the other hand, we found by chance that the sequence

$\langle C(2,n) \rangle_{n \in N}$ seems satisfies the following shorter formula that , for $n \geq 2$,

$$C(2,n) = 2\,C(2,n-1) + C(2,n-2) - 2(-1)^n.$$

Noticing that, for $n \geq 3$,

$R(n,{w \atop b}) = R({w \atop b},n,{b \atop b}) + R({w \atop b},n,{w \atop b}) + R({w \atop b},n,{b \atop w})$,

we verified the above shorter recursion formula as follows.

Since

$$\begin{aligned}
C(2,n) &= C(n,{b \atop b}) + C(n,{w \atop b}) + C(n,{b \atop w}) = R(2,n-1) + 2C(n,{w \atop b}) \\
&= R(2,n-1) + 2R({w \atop b}, n+1, {w \atop b}) = R(2,n-1) + 2[R({w \atop b}, n, {b \atop b}) + R({w \atop b}, n, {b \atop w})] \\
&= R(2,n-1) + 2\,R(n-1,{w \atop b}) + 2R({w \atop b}, n, {b \atop w}) \\
&= R(2,n-1) + 2\,R(n-1,{w \atop b}) + 2[R(n,{w \atop b}) - R(n-1,{w \atop b}) - R({w \atop b}, n, {w \atop b})] \\
&= R(2,n-1) + 2\,R(n-1,{w \atop b}) - 2R({w \atop b}, n, {w \atop b}) \\
&= R(2,n-1) + [R(2,n) - R(2,n-1)] - [C(2,n-1) - R(2,n-2)] \\
&= -C(2,n-1) + R(2,n) + R(2,n-2),
\end{aligned}$$

we have, for $n \geq 3$, C(2,n) +C(2,n-1) = R(2,n)+R(2,n-2) .

Therefore, for $n \geq 3$, C(2,n) +C(2,n-1) = [2R(2,n-1)+R(2,n-2)]+ R(2,n-2)

= 2R(2,n-1)+2R(2,n-2),

thus C(2,n) –2R(2,n-1) = -[C(2,n-1)-2R(2,n-2)] .

As a result, C(2,n) –2R(2,n-1) = $(-1)^n$.

Combining with the recursion property of R(2,n), we obtain that

Proposition 11. *C(2,n)=2 C(2,n-1)+ C(2,n-2)-2$(-1)^n$, for* $n \geq 4$.

Section 5.3. The recursion formula of $\langle C(m,n)\rangle_{m\in N}$ for n fixed

1.The recursion formula of $\langle C(m,1)\rangle_{m\in N}$

Since C(m,1) = R(m,1) = R(1,m),

We have C(m,1)=R(1,m)=R(1,m-1)+R(1,m-2)=C(m-1,1)+C(m-2,1)

thus we conclude that

Proposition 12. *For $m \geq 2$, C(m,1)=C(m-1,1)+C(m-2,1),*

and C(0,1)=1 ，C(1,1)=2 .

2. The recursion formula of $\langle C(m, 2)\rangle_{m\in N}$

We have

$$C(m,2)= C(\genfrac{}{}{0pt}{}{b}{b},m)+C(\genfrac{}{}{0pt}{}{b}{w},m)+C(\genfrac{}{}{0pt}{}{b}{w},m-1)$$

$$= C(m-1,2)+2[C(\genfrac{}{}{0pt}{}{b}{b},m-1)+c(\genfrac{}{}{0pt}{}{b}{w},m-1)]$$

$$= C(m-1,2)+2C(m-2,2)+2C(\genfrac{}{}{0pt}{}{w}{b},m-1).$$

Hence

$$C(m,2) = C(m-1,2)+2C(\begin{smallmatrix} b \\ w \end{smallmatrix},m) = C(m-1,2)+2C(m-2,2)+2C(\begin{smallmatrix} w \\ b \end{smallmatrix},m-1).$$

Using MRT, we get

Proposition 13. *For $m \geq 2$, $C(m,2)=2C(m-1,2)+C(m-2,2)$,*

and $C(0,2)=1$，$C(1,2)=3$.

3. The recursion formula of $\langle C(m, n)\rangle_{m\in N}$ where n≥3 is fixed .

By means of MRT, we obtain

Proposition 14. *For $m \geq 2$, $C(m,3)=3C(m-1,3)+C(m-2,3)$*

and $C(0,3)=1$，$C(1,3) = 4$.

Proposition 15. *For $m \geq 3$, $C(m,4)=5C(m-1,4)+C(m-2,4)-C(m-3,4)$*

and $C(0,4)=1$，$C(1,4)=7$，$C(2,4)=35$.

Proposition 16. *For $m \geq 3$, $C(m,5)=7C(m-1,5)+C(m-2,5)-C(m-3,5)$,*

and $C(0,5)=1$，$C(1,5)=11$，$C(2,5)=81$

What worth noting is that, deforming suitably, an m×4 annular chessboard can become a 2×2×m rectangular polyhedral chessboard . Therefore, the numbers of admissible patterns on the two chessboards are equal. That is, C(m,4)= Q(2,2,m) and so we get from **Proposition 15** that,

Proposition 17. *$Q(2,2,m)=5Q(2,2,m-1)+Q(2,2,m-2)-Q(2,2,m-3)$, for $m \geq 3$.*

Chapter 6. The number of admissible patterns on annular chessboards up to symmetries

The number C(m,n) is the number of admissible patterns on an m×n annular chessboard. As before, we are interested with the problem of counting admissible patterns up to some symmetry.

Definition 5. *Given an m×n annular chessboard A.*

[1]Two patterns P_1 and P_2 on A are said to be rotational equivalent if P_1 can become identical with P_2 via rotation.

[2]A pattern P on A is called k-cyclic if P is invariant under the rotation that rotates the first column of P to the (k+1)th column of P.

[3]A pattern on A is said to be of order k if P is k-cyclic while P is not l-cyclic for $1 \le l \le k-1$.

[4]The number of rotation equivalence chess pieces on A is denoted by T(m,n).

[6] The number of admissible patterns of order k on A is denoted by $C^*(m,n,k)$.

Definition 6. *Given an m×n annular chessboard A.*

[1] Two patterns P_1 and P_2 on A are said to be reflexively equivalent if P_1 can become identical with P_2 via the action flipping inside out.

[2] A pattern P on A is called reflexively symmetric if P is invariant under the action flipping inside out.

[3] The number of reflexive equivalence classes of admissible patterns on A is denoted by V(m,n).

Section 6.1. Rotational symmetry

To start, notice that, if p is prime, then we get at once that

$$T(m,p)=1+\frac{C(m,p)-1}{P}.$$

In particular, according to the equality that

$$C(2,n)=2R(2,n-1)+(-1)^n, \text{ for } n\geq 2,$$

we obtain for p prime that

$$T(2,p)=1+\frac{2R(2,p-1)+(-1)^p-1}{p}.$$

In general, for a composite number n, we have $T(m,n)=\sum_{k|n}\frac{C^*(m,n,k)}{k}$.

However, the key point is, how to compute C(m,n,k) ?

Since the only admissible pattern that is 1-cyclic is the one with all cells placed with black chess pieces, we have $C^*(m,n,1)=1$.

Moreover, if k>1 is a factor of n, a k-cyclic pattern on an m×n annular chessboard is in fact produced by coping an admissible pattern on an m×k annular chessboard $\frac{n}{k}$ times, so that the number of k-cyclic patterns on an m×n annular chessboard is the same as C(m,k).

On the other hand, a k_1-cyclic pattern on an m×n chessboard must be

k_2-cyclic whenever k_1 is a factor of k_2. Thus to compute $C^*(m,n,k)$, the number of admissible patterns of order k on an $m\times n$ annular chessboard, one must take advantage of set theoretical techniques .

We take T(2,6) as an example .

First, $T(2,6)=\frac{C^*(2,6,1)}{1}+\frac{C^*(2,6,2)}{2}+\frac{C^*(2,6,3)}{3}+\frac{C^*(2,6,6)}{6}$.

Due to the above argument, we have

$$T(2,6)=1+\frac{C(2,2)-1}{2}+\frac{C(2,3)-1}{3}+\frac{C(2,6)-C(2,2)-C(2,3)+1}{6}=1+3+4+30=38$$

Section 6.2. The symmetry with respect to flipping inside out

A moment of reflection tells us that we can completely follow the argument counting LR equivalence classes on rectangular chessboards, and we have the following result.

Proposition 18.

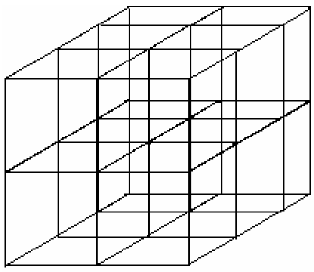
[1] If m is even, then V(m,n)= $\frac{C(m,n)+C(\frac{m}{2}-1,n)}{2}$.

[2] If m is odd, then V(m,n)= $\frac{C(m,n)+C(\frac{m+1}{2},n)}{2}$.

Chapter 7. Recursion formulae for rectangular polyhedral chessboards.

Now we consider the corresponding problem for rectangular polyhedral chessboards “ Under the restriction that no two cells with a common face are not placed with white chess pieces, how many patterns are admissible on an m×n×k rectangular polyhedral chessboard ?"

Denote by Q(m,n,k) the corresponding number. Imitating preceding notations, we have, for instance, the following expressions

Q() = Q (2,2, 3) = Q (4, $\begin{smallmatrix} bb \\ bb \end{smallmatrix}$).

As before, MRT tells us the recursion formulae for this case. For example, we have

$$Q(2,2,k) = Q(k+1,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix})$$

$$= Q(k,\begin{smallmatrix} b & b \\ b & b \end{smallmatrix})+Q(k,\begin{smallmatrix} w & b \\ b & b \end{smallmatrix})+Q(k,\begin{smallmatrix} b & w \\ b & b \end{smallmatrix})+Q(k,\begin{smallmatrix} b & b \\ w & b \end{smallmatrix})+Q(k,\begin{smallmatrix} b & b \\ b & w \end{smallmatrix})$$

$$+Q(k,\begin{smallmatrix} w & b \\ b & w \end{smallmatrix}) + Q(k,\begin{smallmatrix} b & w \\ w & b \end{smallmatrix})$$

$$= Q(2,2,k-1)+4Q(k,\begin{smallmatrix} w & b \\ b & b \end{smallmatrix})+2Q(k,\begin{smallmatrix} w & b \\ b & w \end{smallmatrix}),$$

computing in this way successively, we get by MRT that, for $k \geq 3$,

$$Q(2,2,k)= 5Q(2,2,k-1)+Q(2,2,k-2)+Q(2,2,k-3),$$

which is identical with **Proposition 17** .

Chapter 8. Inequalities concerning R(m,n)

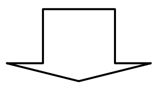
It is trivial that R(m,n) increases with m and n. In addition to it, we have the following inequality.

Proposition 19. *If $m \geq 2$ and $n \geq 2$, then $R(m,n) < R(1,mn)$*

For example, $R(2,6) < R(1,12)$ and $R(3,4) < R(1,12)$.

Proof As shown in **Fig.4**, any admissible pattern on the m×n chessboard A can be transformed to an admissible pattern on the $1 \times (mn)$ chessboard B. Since this transformation is evidently injective, we have $R(m,n) \leq R(1,mn)$.

(1, 1)	(1, 2)	(1, 3)	…………..	(1, n)
(2, 1)	(2, 2)	(2, 3)	…………..	(2, n)
(3, 1)	(3, 2)	(3, 3)	…………..	(3, n)
(4, 1)	(4, 2)	(4, 3)	…………..	(4, n)
⋮	⋮	⋮	⋮	⋮
(m, 1)	(m, 2)	(m, 3)	…………..	(m, n)



(1,1)	…	(1,n)	(2,n)	(2,n-1)	…	(2,1)	(3,1)	…	(3,n)	…	(m,k)

Fig.4 *(when m is even, k=1;when n is odd, k=n)*

On the other hand, since $m \geq 2$ and $n \geq 2$,it is obvious that there exists an admissible pattern on B that is not transformed from any admissible pattern on A . As a result, R(m,n)<R(1,mn). Q.E.D.

In general, if $m_1 \times n_1 = m_2 \times n_2$, can we compare $R(m_1,n_1)$ with $R(m_2,n_2)$ without knowing the values ? Numerical data reveals that, the more the shape of the rectangular chessboard is similar to a square, the less the number of admissible patterns is. For examples,

R(3,4)=227<239=R(2,6) and R(4,4)=1234<1393=R(2,8) .

However, so far we do not know how to verify this phenomenon.

V. Results

1. Some values of R(m,n) computed with the spreadsheet

m / n	1	2	3	4	5	6	7	8
1	2	3	5	8	13	21	34	55
2	3	7	17	41	99	239	577	1393
3	5	17	63	227	827	2999	10897	39561
4	8	41	227	1234	6743	36787	200798	1095851
5	13	99	827	6743	55447	454385	3729091	30584687
6	21	239	2999	36787	454385	5598861	69050253	851302029

Some values of C(m,n) computed with the spreadsheet

m / n	4	5	6	7	8	9	10	11
1	7	11	18	29	47	76	123	199
2	35	81	199	477	1155	2785	6727	16237
3	181	621	2309	8303	30277	109753	398857	1447931

2. Recursion formulae for rectangular chessboards

For m = 1,2, and 3 we have

(A) R(1,0) = 1,R(1,1) = 2, R(1,n) = R(1,n-1)+R(1,n-2), for $n \geq 2$.

(B) R(2,0) = 1,R(2,1) = 3, R(2,n) = 2R(2,n-1)+R(2,n-2),for $n \geq 2$.

(C) R(3,0) = 1,R(3,1) = 5, R(3,2) = 17, R(3,3) = 63,

R(3,n) = 2R(3,n-1)+6R(3,n-2) – R(3,n-4), for $n \geq 4$.

3. Recursion formulae for diagonal chessboards

For m = 1, and 2 we have

(A) D(1,0) = 1, D(1,1) = 3, and D(1,n) = 3D(1,n-1)-D(1,n-2), for $n \geq 2$

(B) D(2,0) = 1, D(2,1) = 8, D(2,2) = 53,

D(2,n)=8D(2,n-1)–11D(2,n-2)+4D(2,n-3), for $n \in N, n \geq 3$

4. Recursion formulae for rectangular polyhedral chessboards

For m = 2, n = 2 we have

Q(2,2,0) = 1, Q(2,2,1) = 7, Q(2,2,2) = 35, Q(2,2,3) = 181,

Q(2,2,k) = 5Q(2,2,k-1)+Q(2,2,k-2)–Q(2,2,k-3), for $k \geq 3$.

5. Recursion formulae for annular chessboards with the number n of columns fixed

For n = 1,2, and 3 we have

(A) C(0,1) = 1, C(1,1) = 2, C(m,1) = C(m-1,1)+C(m-2,1), for $m \geq 2$

(B) C(0,2) = 1, C(1,2) = 3, C(m,2) = 2C(m-1,2)+C(m-2,2), for $m \geq 2$

(C) C(0,3) = 1, C(1,3) = 4, C(m,3) = 3C(m-1,3)+C(m-2,3), for $m \geq 2$

6. Annular chessboards with the number m of rows fixed

(A) C(1,n) = C(1,n-1)+C(1,n-3), for $n \geq 3$

(B) C(2,n) = 2C(2,n-1)+C(2,n-2)+2$(-1)^n$, for $n \geq 4$

7. The number U(m,n) of equivalence classes in which two patterns on a rectangular chessboard are identified as the same upon flipping from left to right

(A) If m is odd, then U(m,n) = $\dfrac{R(m,n)+R(m,\frac{n+1}{2})}{2}$.

(B) If m is even, then U(m,n) = $\dfrac{R(m,n)+R(m,\frac{n}{2}-1)}{2}$.

8. The numbers of equivalence classes up to symmetries on an annular chessboard

(A) Symmetry with respect to rotation

T(m,n) = $\sum_{k|n}\frac{1}{k}$(The number of admissible patterns of order k).

(B) Symmetry with respect to flipping inside out

if m is odd, then V(m,n) = $C(\frac{m+1}{2},n)+\dfrac{C(m,n)-C(\frac{m+1}{2},n)}{2}$.

if m is even, then V(m,n) = $C(\frac{m}{2}-1,n)+\dfrac{C(m,n)-C(\frac{m}{2}-1,n)}{2}$.

VI. Discussion

1. It is highly desirable to obtain a formula for R(m,n) in terms of m and n. Thus far, we are able to solve for R(m,n) once an individual m is given.
2. In deriving the recursion formula of $\langle C(m,n)\rangle_{n\in N}$, for each fixed m, the result obtained does not appear to be the shortest. On the other hand, we believe that the recursion formula of $\langle R(m,n)\rangle_{n\in N}$ derived by MRT is indeed the case. (So far we are unable to supply the proof of this claim.)
3. The number of admissible patterns on a square chessboard in which two patterns are regarded as the same if rotation is taken into consideration,

remains open.

4. It is evident that the case of annular chessboards is isomorphic to that of cylindrical chessboards.
5. If m≥2, n≥2 then the inequality R(m,n)<R(1,mn) holds.
6. Using tables and figures generated by the spreadsheet program Excel, we find that for each fixed m, $\frac{R(m,n+1)}{R(m,n)}$ appears to approach a finite value, as shown in Fig.5 and Fig.6

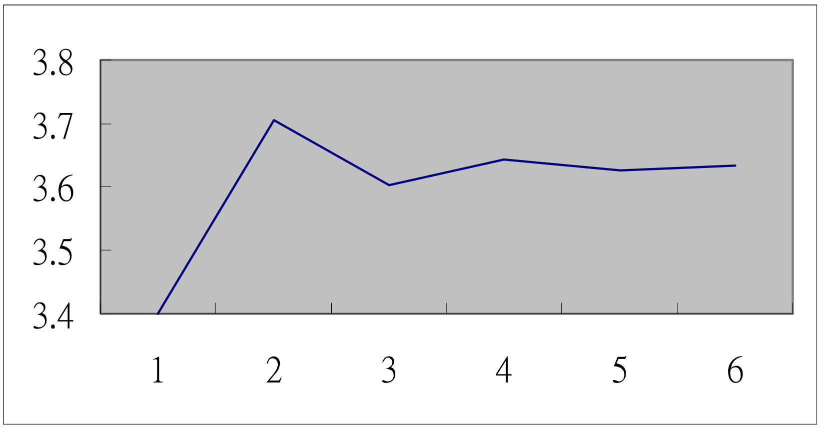


Fig.5 *Graph of R(3,n+1)/R(3,n)*

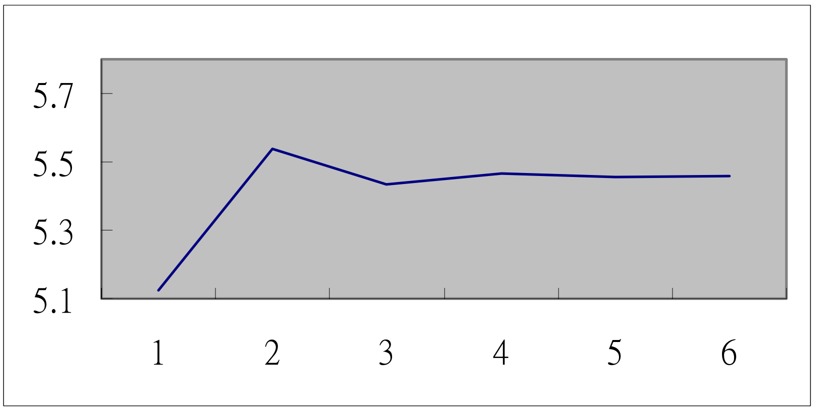


Fig.6 *Graph of R(4,n+1)/R(4,n)*

For the cases m = 3, 4, the limiting values of $\frac{R(m,n+1)}{R(m,n)}$ can be obtained by means of the Binet formula after solving the characteristic equations with the software Maple. For example, from the Binet formula

$$R(2,n) = \frac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right], \text{ for } n \geq 0,$$

we obtain $1+\sqrt{2}$ as the limiting value for m = 2.

VII. Conclusion

We went through the process of (1) brute force, (2) tree diagram, (3) spreadsheet computation, and (4) MRT recursion formulation in seeking the solution to our combinatorial problem. We believe that a large number of similar problems can be tackled this way.

VIII. References

1. Conway, J. H. and Guy, R. K., Fibonacci Numbers, In *The Book of Numbers*, New York, Springer-Verlag, pp. 111-113, 1996.

2. Hoggatt, V. E. Jr., *The Fibonacci and Lucas Numbers*, Boston, MA, Houghton Mifflin, 1969.

3. Honsberger, R., A Second Look at the Fibonacci and Lucas Numbers, Ch. 8 in *Mathematical Gems III*, Washington, DC, Math. Assoc. Amer., 1985.

評　　　語

（1）本文考量棋盤上不得兩白棋相鄰之所有排列數，文中以較嚴謹方法證明，並可以迴歸方式，得到 nxm 棋盤情形之計算公式。

（2）有關函數，文字敘述等表示方法，均有待改進。