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Recursive sequences and chaos phenomena

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我從一歲半就開始玩數字，數學可以說是我的第二生命，小學時曾就近到東吳大學數學系旁聽，六年級時在許文化老師的帶領下，開始研究科展，很幸運的以「撲克圈尋極限」，得到全國高小組數學科第一名。升上國中，繼續研究數學，國一時以「田字方塊深探祕」得到了全國國中組第三名，國二時以「巧解乘法多面體」得到了全國第一。目前就讀於建中數理資優班，在傅承德教授和導師游森棚指導下，開始寫論文並繼續研究數學，我相信數學是一切科學的基礎，也唯有腳踏實地，努力耕耘，才能達成目標。

遞迴數列及渾沌現象

研究摘要

給定一個 $p\in(0, 1)$，令 $k_0 = 0$, $p_0 = p$,定義 k_1 爲能使 $\frac{p^k}{1-p} \leqq 1$ 的最小正整數 k，而 $p_1 = \frac{p^{k_1}}{1-p}$ ；相同的，對於給定的 k_{n-1}，k_n 爲能使 $\frac{p^k}{1-p_{n-1}} \leqq 1$ 的最小正整數 k，$p_n = \frac{p^{k_n}}{1-p_{n-1}}$ 。若存在 k_n 使得 $\frac{p^{k_n}}{1-p_{n-1}} = 1$，則稱 $p \in I_n$ ； 若對於所有的 n 與 k_n，$\frac{p^{k_n}}{1-p_{n-1}} < 1$，則稱 $p \in I_\infty$。如此區間(0, 1)可分解成集合 $I_1, I_2, \ldots, I_\infty$。

進行電腦模擬時，常須要使用亂數。本主題研究對於不同的 p 值($0 < p < 1$)，觀察遞迴數列 p_n ($n\geqq 0$)的性質，並探討 p 值與其所屬的集合 I_n 的關係。討論對任意 $n = 1, 2, 3,\ldots,\infty$，$I_n$ 是否爲空集合，以及 p 值在哪些範圍時，數列 p_n 會產生渾沌現象，及哪些 p 可產生亂數。主要的結果如下：

(1) 當 $0 < p \leqq \frac{1}{4}$ 時，則 $p \in I_\infty$，且數列 p_n 遞增，並收斂到極限值 $p_\infty = \frac{1-\sqrt{1-4p}}{2}$ 。

(2) 當 $\frac{1}{4} < p < 1$ 且 p 爲 $\frac{1}{3}$ 和 $\frac{1}{2}$ 以外的有理數時，則 $p \in I_\infty$，且不存在相異的兩數 a 與 b，使得 $p_a = p_b$。

(3) 對於任意正整數 n，I_n 爲非空集合。首先針對較小的 n 值，找到了一些屬於 I_n 的元素，例如 $\frac{1}{2} \in I_1$, $\frac{1}{3} \in I_3$, $\frac{2-\sqrt{2}}{2} \in I_5$, $\frac{5-\sqrt{5}}{10} \in I_7$, $2-\sqrt{3} \in I_9$。

(4) 在討論 I_n 是否爲空集合時，我們考慮當 $k_n = 1$，把表示 p_n 的分數展開，則得到的結果可用以下遞迴式表示：$a_0(p) = 1$,

$a_1(p) = 1 - p,\quad a_{n+1}(p) = a_n - p\, a_{n-1}\ (n \geqq 1),\quad p_n = \dfrac{pa_{n-1}(p)}{a_n(p)}$ 。

要使 $p_n = 1$，就要使分母與分子相等，亦即 $a_n(p) - pa_{n-1}(p) = 0$，也就是 $a_{n+1}(p) = 0$。此多項式 $a_n(p)$與 Fibonacci 多項式

$f_0(x) = 1,\quad f_1(x) = x,\quad f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x)\ (n \geqq 1)$相關。經過轉換後，求得 $p_n = 1$ 的解爲$\dfrac{1}{4\cos^2 \dfrac{\pi}{n+3}} \in I_n$。

(5) 根據第(2)部分得到的結果，發現選定適當的 p 值代入，遞迴數列 p_n 會產生渾沌現象：決定一 p 值($0.25 < p < 1$)，並代入多次後，各 p_n 值在(p, 1)間的分布出現渾沌情況。而當 p 接近 1 時，p_n 接近均勻分布，這時就可利用每次產生的 p_n 作爲亂數值了。

關鍵字：渾沌，Fibonacci 多項式，遞迴數列

Recursive sequences and chaos phenomena

Abstract

For a given fixed number $p \in (0,1)$, let $n_0=0$, $p_0=p$,

$$k_1 = \inf\{k \geq 1: \frac{p^k}{1-p} \leq 1\},$$

and

$$p_1 := \frac{p^{k_1}}{1-p}.$$

By induction, for given p_{n-1}, we define

$$k_n = \inf\{k \geq 1: \frac{p^k}{1-p_{n-1}} \leq 1\},$$

and

$$p_n := \frac{p^{k_n}}{1-p_{n-1}}.$$

Then, $\{p_n, n \geq 0\}$ is a sequence of numbers belongs to (p,∞). For each $n=1,2,\ldots$, let $I_n := \{p \in (0,1)$; there exists k_n such that $p^{k_n}/(1-p_{n-1}) = 1\}$, and denote $I_\infty = \{p \in (0,1);\ p^{k_n}/(1-p_{n-1}) < 1$ for all k and $k_n\}$. It is noted that the open interval (0,1) can be decomposed as a countably union of $I_1, I_2, \ldots$, and I_∞.

Motivated by the problem of random number generation in computer simulation, in this article, we study the behavior of the recursive sequence $\{p_n, n \geq 0\}$ for various $p \in (0,1)$. In particular, we explore the relationship between p and I_n, for $n=1,2,\ldots,\infty$. Our main results shows that

1. When $0 < p \leq 1/4$, then $p \in I_\infty$ and the sequence $\{p_n, n \geq 0\}$ is monotone increasing with limit $(1-\sqrt{1-4p})/2$.

2. When $1/4 < p < 1$ and is a rational number except 1/3 and 1/2, then $p \in I_\infty$ and p_n are distinct from each other for different n. That is, $p_i \neq p_j$ for $i \neq j$.

3. For each n=1,2,…,I_n is not empty. We first find explicit element in each I_n for small n, for instance, $\frac{1}{2} \in I_1$, $\frac{1}{3} \in I_3$, $\frac{2-\sqrt{2}}{2} \in I_5$, $\frac{5-\sqrt{5}}{10} \in I_7$, $2-\sqrt{3} \in I_9$.

4. In general, we let k_n=1, and expand the ratio of p_n to get recursive $a_0(p) = 1$, $a_1(p) = 1-p$, $a_{n+1}(p) = a_n(p) - p\, a_{n-1}(p)$ $(n \geq 1)$,

$$p_n = \frac{p\, a_{n-1}(p)}{a_n(p)} .$$

In order to make $p_n = 1$, we need $a_n(p) - p\, a_{n-1}(p) = 0$, that is $a_{n+1}(p) = 0$. Then the polynomial $a_n(p)$ has relation with the Fibonacci polynomial $f_0(x) = 1$, $f_1(x) = x$, $f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x)$ $(n \geqq 1)$. A simple transformation leads that

$$\frac{1}{4\cos^2 \frac{\pi}{n+3}} \in I_n, \text{ for } n=1,2,\ldots.$$

5. By using the result form part 2, we find that for a suitable initial point $p \in (0,1)$, the recursive sequence $\{p_n, n \geq 0\}$ can provide an example of chaotic systems. To be more precise, when $.25 < p < 1$, a chaotic phenomena happens and it approaches to uniform distribution as p approaches to 1. Then the sequence $\{p_n, n \geq 0\}$ can be used to produce random numbers.

Keywords. chaos, Fibonacci polynomial, recursive sequence.

一、研究動機：

生活上大部分使用到亂數是一序列的亂數值，研究時常需要使用電腦模擬一些自然的現象，亂數能反映出實際的情況。亂數產生器(Random Number Generator)可利用遞迴函數反覆代入來產生亂數值。因此我設定 p 值使 $0<p<1$，並且 $p_0 = p$, $p_n = \dfrac{p^{k_n}}{1-p_{n-1}}$ ，其中 k_n 是能使 $p_n \leqq 1$ 的最小正整數，以觀察每次產生的數值，在區間(p,1)中的分布情形？

二、研究過程：

作電腦模擬時，使用遞迴函數 $p_n = \dfrac{p^{k_n}}{1-p_{n-1}}$ ，k_n 是能使 $p_n \leqq 1$ 的最小正整數，觀察此函數的值域分布情形。

（一）電腦模擬觀察到一些現象：p 小於等於 0.25 時，p_n 會逐漸增加，但 p_∞ 等於某個小於等於 0.5 的值。
證明：

1. 用數學歸納法證明任何 p_n 都不會大於 0.5

   當 $n = 1$ 時，$p_1 = \dfrac{p}{1-p_0} \leqq \dfrac{0.25}{1-0.25} = \dfrac{1}{3} < 0.5$，成立，

   設當 $n = k$ 時成立，亦即 $p_k \leqq 0.5$，

   則當 $n = k+1$ 時，$p_{k+1} = \dfrac{p}{1-p_k} \leqq \dfrac{0.25}{1-0.5} = 0.5$，故成立。

2. 再證明 $p_n > p_{n-1}$

   當 $n = 1$ 時，$\dfrac{p}{1-p_0} > p = p_0$，成立，

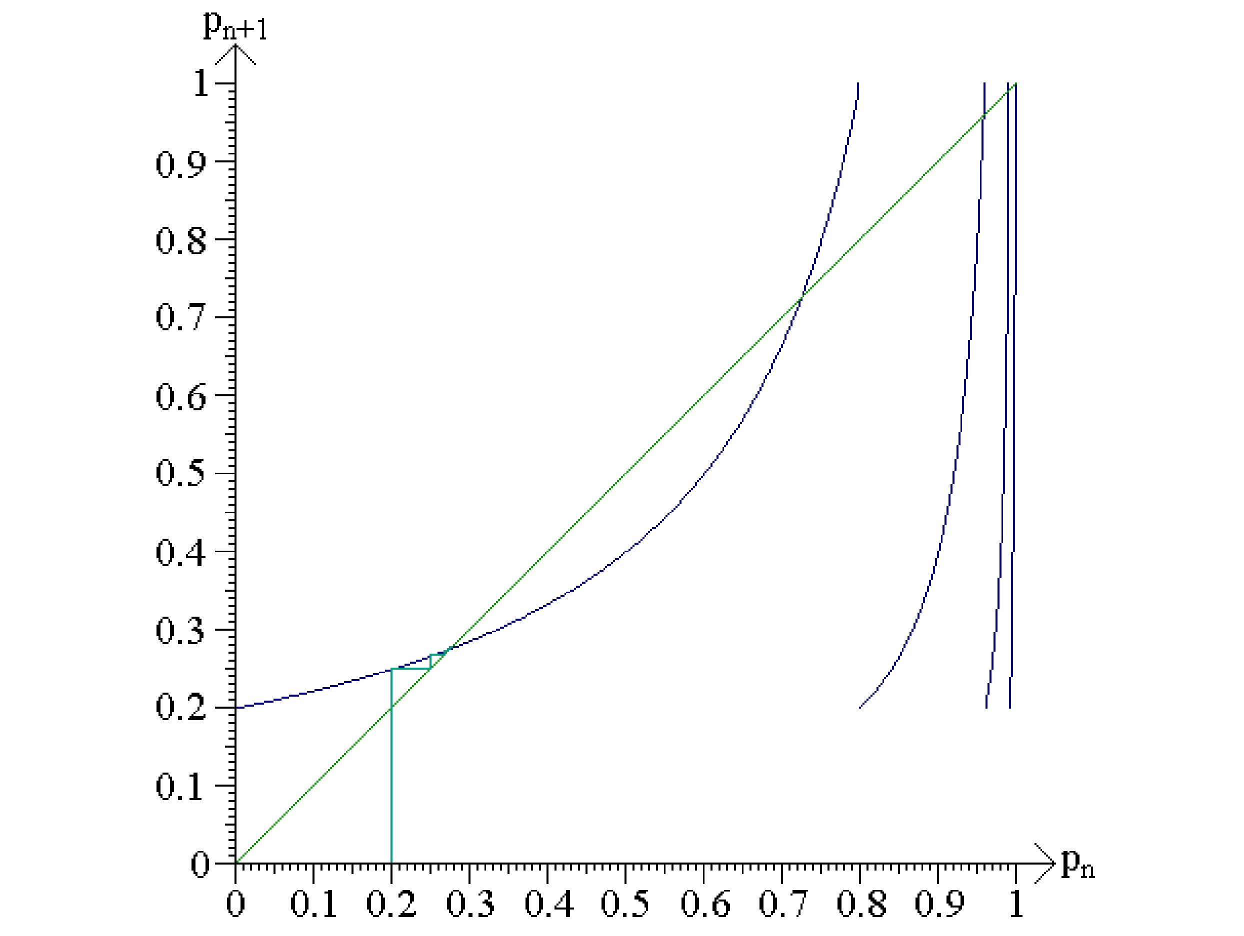
   設當 $n = k$ 時成立，亦即 $p_k > p_{k-1}$，

   則當 $n = k + 1$ 時，$p_{k+1} = \dfrac{p}{1-p_k} > \dfrac{p}{1-p_{k-1}} = p_k$，成立。

3. 由此，p_n將收斂到一個≦0.5 的數值 p_∞。

例如 p=0.2：

圖 1



上圖 1 中橫軸爲 p_n，縱軸爲 p_{n+1}，藍色線爲 $p_{n+1} = \dfrac{p^{k_{n+1}}}{1-p_n}$ （k_{n+1} 爲可使 p_{n+1}≦1 的最小正整數），綠色線爲 $p_{n+1}=p_n$。兩線相交處

$\dfrac{p^{k_{n+1}}}{1-p_n} = p_n$ 即爲 p_∞的值。藍色線是由不連續的曲線構成，且 p_n 靠近 1 時，每一段圖形接近直線，原因是 p_n 增加到接近 1 時，分母會變得很小，所以 $\dfrac{p^{k_n}}{1-p_n}$ 中的 k_n 值增大的速度也越來越快。

4. 計算 p_∞的值：

∵$p, p_n < 0.5$，故 $p < 1 - p_n$，k_n 值只要取 1，就可使 $\frac{p^{k_n}}{1-p_n} < 1$。

令 $x = p_\infty$，則 $\frac{p}{1-x} = x$，

$p = x(1-x)$,
$p = x-x^2$,
$x^2-x+p = 0$,
$x = \frac{1 \pm \sqrt{1-4p}}{2}$,但 $\frac{1+\sqrt{1-4p}}{2} \geqq \frac{1}{2}$ 故不合，

只有 $\frac{1-\sqrt{1-4p}}{2}$ 符合，

∴$p_\infty = \frac{1-\sqrt{1-4p}}{2}$ 。

（三）當 0.25<p<1 時，證明除了 $\frac{1}{3}$ 及 $\frac{1}{2}$ 以外的所有有理數都屬於 I_∞，且不會有任何兩個 I_a 和 I_b 相同：

1. 當 p 為有理數，並且分子不為 1 的情形：

(1)先以 p= $\frac{4}{11}$ 和 p= $\frac{3}{7}$ 為例，分別代入數次後，觀察規律：

① p= $\frac{4}{11}$

表 1

p_0	4/11
p_1	4/7
p_2	28/33
p_3	48/55
p_4	320/847
p_5	308/527
p_6	2108/2409
p_7	14016/36421
p_8	13244/22405
p_9	89620/100771
p_{10}	586304/1349271

$p=\frac{4}{11}$ 的分子爲偶數，且每次代入後所得到的值之分子也是偶數。

②但 p 的分子爲奇數時，p_n 的分子不一定爲奇數，

例如 $p=\frac{5}{19}$ ：

表 2

p_0	5/19
p_1	5/14
p_2	70/171
p_3	45/101
p_4	505/1064
p_5	280/559
p_6	2795/5301
p_7	1395/2506
p_8	12530/21109
p_9	5555/8579
p_{10}	42895/57456

雖然分子不全爲奇數，但都是 5 的倍數，推測這可能是由於 p 的分子爲 5 的緣故。

③再把分子改爲 3，觀察 p_n 的分子是否爲 3 的倍數：$p=\frac{3}{7}$

表 3

p_0	3/7
p_1	3/4
p_2	36/49
p_3	9/13
p_4	117/196
p_5	36/79
p_6	237/301
p_7	387/448
p_8	1728/2989
p_9	549/1261
p_{10}	3783/4984

代入 10 次後，發現每次得到的分子均爲 3 的倍數。
以下設 p 的分子爲質數 m 的倍數，證明所有 p_n 的分子均含有質因數 m：

(2)利用歸納法證明當 p 爲有理數，並且化爲最簡分數後，分子含有質因數 m（設爲 ma_0），分母不爲 m 的倍數（設爲 mb_0+c_0）時，即 $p=\frac{ma_0}{mb_0+c_0}$，則任何 p_n 的分子均爲 m 的倍數，分母則不是 m 的倍數：

①當 n=0 時，$p_0=p=\frac{ma_0}{mb_0+c_0}$(成立)

②若 $n=\ell$成立，亦即 $p_\ell=\frac{ma_\ell}{mb_\ell+c_\ell}$，

則當 $n=\ell+1$ 時，

$$p_{\ell+1}=\frac{p^{k_{\ell+1}}}{1-p_\ell}=\frac{\left(\frac{ma_0}{mb_0+c_0}\right)^{k_{\ell+1}}}{1-\frac{ma_\ell}{mb_\ell+c_\ell}}=\frac{mb_\ell+c_\ell}{m(b_\ell-a_\ell)+c_\ell}\times\frac{(ma_0)^{k_{\ell+1}}}{(mb_0+c_0)^{k_{\ell+1}}},$$

其中左邊的 $\frac{mb_\ell+c_\ell}{m(b_\ell-a_\ell)+c_\ell}$ 分子與分母都不是 m 的倍數，

右邊的 $\frac{(ma_0)^{n_{\ell+1}}}{(mb_0+c_0)^{n_{\ell+1}}}$ 的分子是 ma_0 的乘冪，仍爲 m 的倍數；分母是 mb_0+c_0 的乘冪，所以不爲 m 的倍數。兩分數相乘後，可得知 $p_{\ell+1}$ 的分子爲 m 的倍數，分母則不是。如果此分數可約分，則因分母無 m 的因數，故無法把分子中 m 的因數約去，約分後的分數分子仍爲 m 的倍數，分母仍不是。由此 $n=\ell+1$ 也成立。

③由①,②，n 爲任何數時均成立。

(3) 上面的試驗過程中，發現由同一 p 值得到的各 p_n 值均不相同：分母和分子逐漸增大，雖然有時會經由約分而減小，但增大的速度仍然較快，沒有發生同一 p_n 值重複出現的情況。

如果某一 p_a 與先前出現的 p_b 相同，但 p_{a-1} 與 p_{b-1} 相異，則表示不同的兩數代入後，會得到相同的數值。

現在以 $p = \frac{2}{5}$ 爲例，計算除了 $p_1 = \frac{2}{3}$ 以外，是否還有別的 p_n 能使 $p_{n+1} = \frac{2}{3}$ ：

$$\frac{(\frac{2}{5})^{k_n}}{1-p_n} = \frac{2}{3}$$， 其中 k_n 爲正整數，

$$\frac{1-p_n}{(\frac{2}{5})^{k_n}} = \frac{3}{2}$$，

$$1-p_n = \frac{3}{2} \times (\frac{2}{5})^{k_n}$$，

當 $k_n = 1, 2, 3, \ldots,$

$$p_n = \frac{2}{5}，\frac{19}{25}，\frac{113}{125}，\frac{601}{625}，\ldots,$$

但因 p 的分子爲偶數，故只有 $\frac{2}{5}$ 符合。

由此可見，須取適當的 k_n 值，才能使 p_n 的分子含有與 p 的分子相同的質因數。

以下爲證明：

(4) 當 $p = \frac{ma_0}{mb_0+c_0}$ 時，由已知的 p_{n+1} (設爲 $\frac{ma_{n+1}}{mb_{n+1}+c_{n+1}}$)逆推，求得 p_n 可能的數值：

由於 $p_{n+1} = \frac{p^{k_{n+1}}}{1-p_n}$，

$$1 - p_n = \frac{p^{k_{n+1}}}{p_{n+1}}$$，

可得 $p_n = 1 - \frac{p^{k_{n+1}}}{p_{n+1}}$，

即 $p_n = 1 - \dfrac{\left(\dfrac{ma_0}{mb_0+c_0}\right)^{k_{n+1}}}{\dfrac{ma_{n+1}}{mb_{n+1}+c_{n+1}}}$

$= 1 - \dfrac{mb_{n+1}+c_{n+1}}{ma_{n+1}} \times \dfrac{(ma_0)^{k_{n+1}}}{(mb_0+c_0)^{k_{n+1}}}$

因 p_n 的分子必爲 m 的倍數，且分母不是 m 的倍數，故

$1 - p_n = 1 - \dfrac{ma_{n+1}}{mb_{n+1}+c_{n+1}} = \dfrac{m(b_{n+1}-a_{n+1})+c_{n+1}}{mb_{n+1}+c_{n+1}}$

$= \dfrac{mb_{n+1}+c_{n+1}}{ma_{n+1}} \times \dfrac{(ma_0)^{k_{n+1}}}{(mb_0+c_0)^{k_{n+1}}}$

約簡後的分子、分母均不爲 m 的倍數。$mb_{n+1}+c_{n+1}$ 與 $(mb_0+c_0)^{n_{k+1}}$都不是 m 的倍數，所以$(ma_0)^{k_{n+1}}$與 ma_{n+1} 所含因數 m 個數必須相等，約簡後才會抵銷。設 a_{n+1} 含有 d 個 2 的因數，而 a_0 含有 e 個 2 的因數，則 k_n 必等於$\dfrac{d+1}{e+1}$。p, p_{n+1}, k_{n+1} 均已確定，由此可得到唯一的 p_n，亦即只有唯一合理的 p_n 值可產生特定的 p_{n+1}。

(5) 假設 p_0, p_1, p_2,…中有兩數重複，而最先重複的兩數分別爲 p_a 以及 p_b(b>a)，則由前面(2)可知 p_{a-1} 與 p_{b-1} 也相同，這與原先的假設 “p_a 與 p_b 爲最先重複的兩數”相矛盾，因此不可能有任何兩數重複出現。

2. 若 p 爲有理數，且分子爲 1 的情形：

大於 0.25 的真分數中，分子爲 1 的只有 $\dfrac{1}{3}$ 和 $\dfrac{1}{2}$ 。

(1) $p=\frac{1}{2}$ ，$p_1=\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=1$，故 $\frac{1}{2}\in I_1$ 。

(2) $p=\frac{1}{3}$ ，$p_1=\dfrac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}$ ，$p_2=\dfrac{\frac{1}{3}}{1-\frac{1}{2}}=\frac{2}{3}$ ， $p_3=\dfrac{\frac{1}{3}}{1-\frac{2}{3}}=1$，

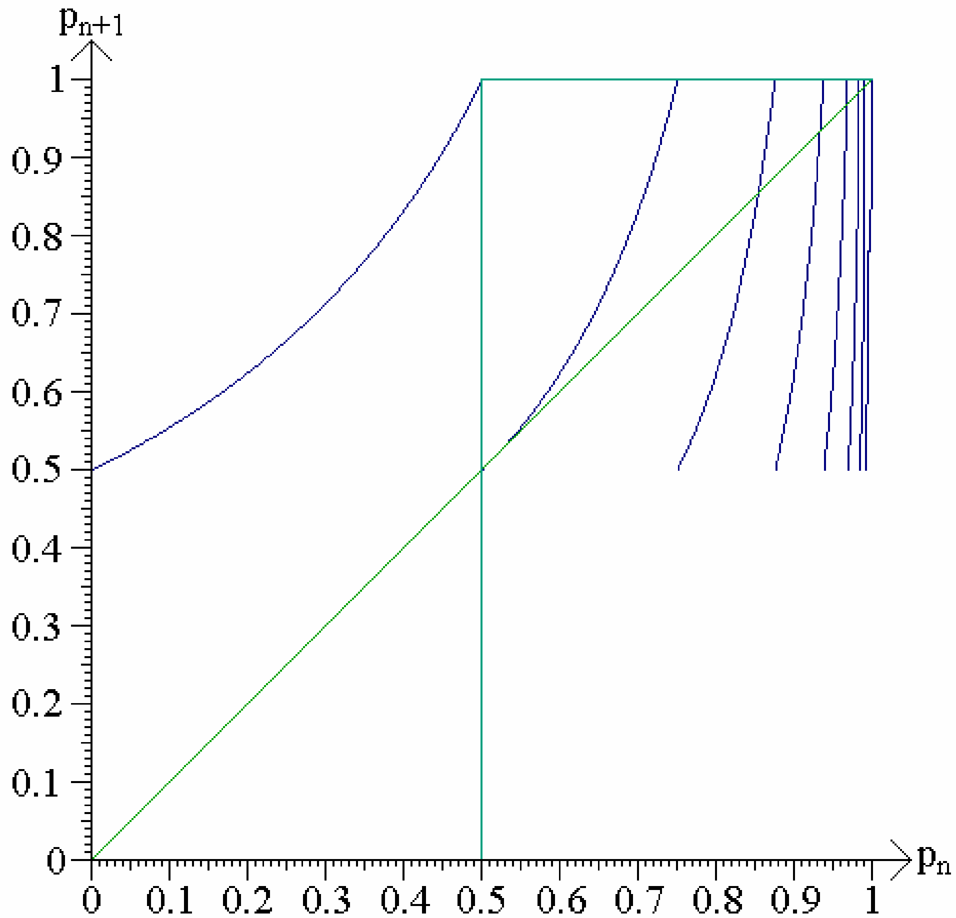
故 $\frac{1}{3}\in I_3$。

（三）$0.25<p \leqq 0.5$ 時，有一些 p 值經過有限步的運算後，可以得到 1：

1. 例如前面的 $\frac{1}{2}$ 和 $\frac{1}{3}$ ，圖形分別如下：

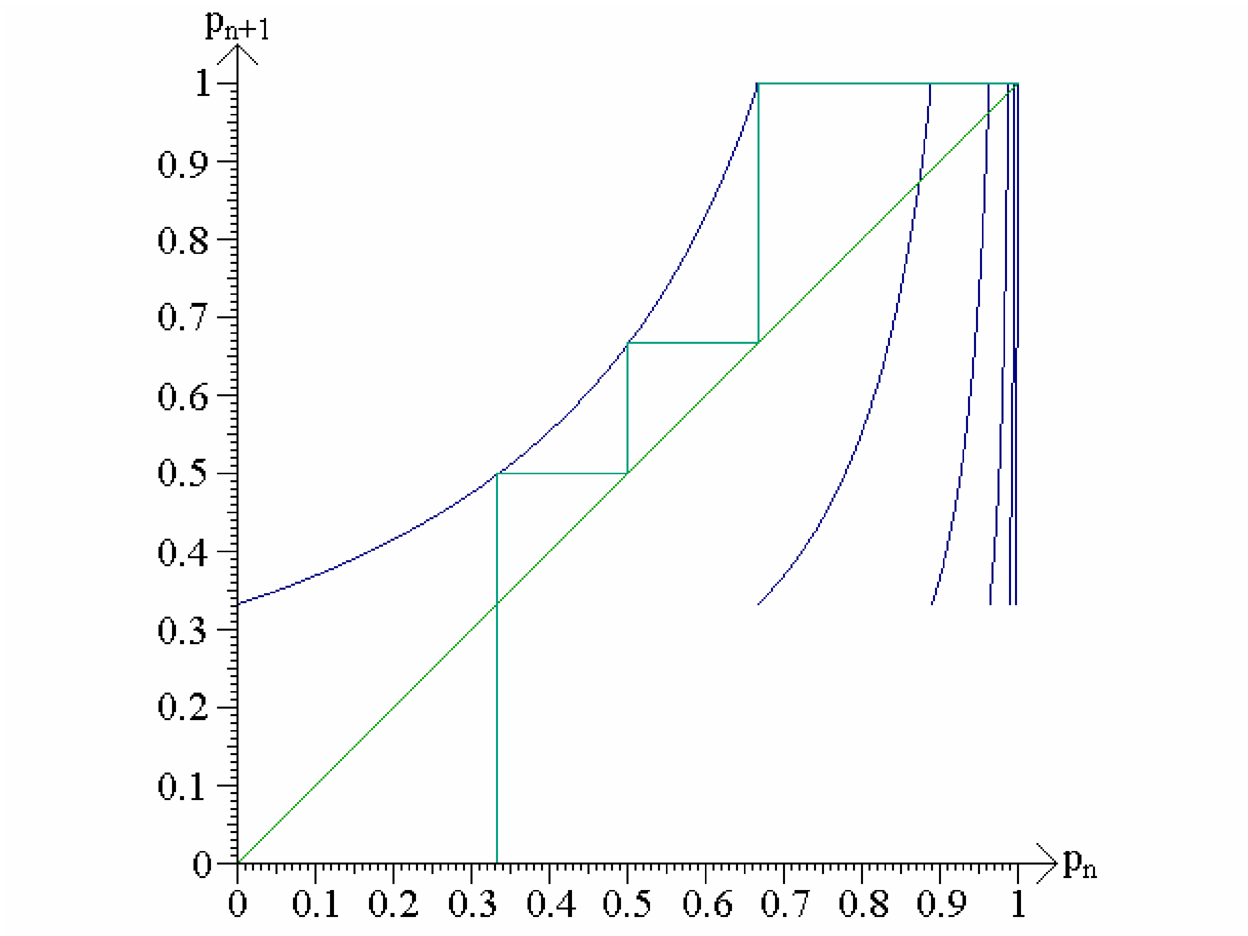
$p=\frac{1}{2}$

圖 2



$p = \frac{1}{3}$

圖 3



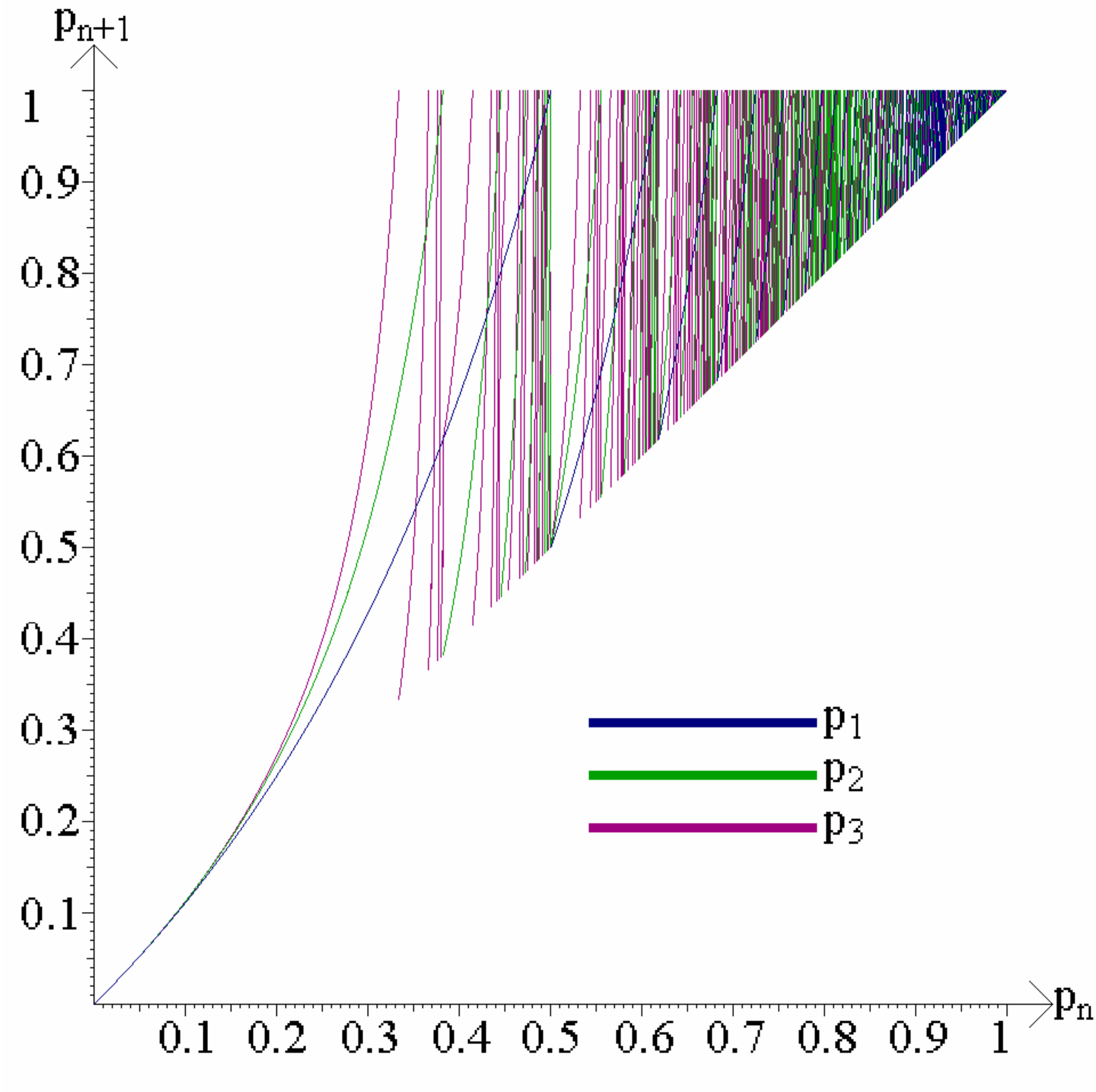
2. 檢驗還有哪些數能夠經過代入後得到 1：

(1) 試驗方法：

先決定一 n 值(n=1,2,3,…)，再設法使 $p_n=1$。p 從 0.25 開始逐漸增加，此時 p_n 值也相對增加。調整 p 值到 $p_n=1$ 爲止，即可找到 $p\in I_n$。

下圖 4 爲各 p 值代入 3 次所得到 p_1, p_2, p_3 的值。由圖中可看出 p 從 0 開始增加時，p_n 會隨之增加，但 p_n 達到 1 後，如果 p 值繼續增加，p_n 就會減小到接近 p 值。用此方法可以判斷一個 p 值大於或小於能使 $p_n=1$ 的值。

圖 4



(2) p_7=1 的試驗情況：

① p 從 0.25 開始增加，到 $p_7 = 1$ 爲止，p_7 未達到 1 以前，p_7 隨 p 增加，但超過此範圍時，p_7 又回到與 p 相近的值，這時就要把 p 減小了。

表 4

p	p_7	p 値過大或過小
0.25	0.44444 44444 44444	過小
0.26	0.52087 17671 73351	過小
0.27	0.68191 59224 35899	過小
0.275	0.89008 27805 49313	過小
0.28	0.48331 38401 55949	過大
0.276	0.96475 80974 21005	過小
0.277	0.29445 79839 85269	過大
0.2765	0.27934 10174 99825	過大
0.2764	0.27657 81470 72783	過大
0.2763	0.99128 91275 48333	過小
0.27635	0.99593 32506 04733	過小
0.27637	0.99780 96069 24777	過小
0.27638	0.99875 18499 75209	過小
0.27639	0.99969 68190 50126	過小

②由此可以看出 $0.27639 < p < 0.27640$，如果繼續試驗，能夠確定 p = 0.27639 32022 50021…。

此 p 値代入 7 次的情形爲：

表 5

p_0	0.27639 32022 50021
p_1	0.38196 60112 50105
p_2	0.44721 35954 99958
p_3	0.50000 00000 00000
p_4	0.55278 64045 00042
p_5	0.61803 39887 49895
p_6	0.72360 67977 49978
p_7	0.99999 99999 99998

③從上表 5 觀察到的現象：

最後的 p_7 並不剛好爲 1，是因爲有效位數不足。

另外 p_3 等於 $\frac{1}{2}$ ，而 p_5 應爲 $\frac{\sqrt{5}-1}{2}$ ，p_1 是 $1 - p_6$ 的值，也就是 $\frac{3-\sqrt{5}}{2}$ 。

④利用 $p_1 = \frac{3-\sqrt{5}}{2}$ 來計算 p 值：

設 $\frac{p}{1-p} = p_1 = \frac{3-\sqrt{5}}{2}$ ，

則 $p=\frac{3-\sqrt{5}}{2}(1-p)$, $p + \frac{3-\sqrt{5}}{2} p = \frac{3-\sqrt{5}}{2}$,

$\frac{5-\sqrt{5}}{2} p = \frac{3-\sqrt{5}}{2}$, $p = \frac{3-\sqrt{5}}{5-\sqrt{5}} = \frac{5-\sqrt{5}}{10}$ 。

⑤現在把 p 值實際代入 7 次檢驗：

$p_1 = \frac{3-\sqrt{5}}{2}$, $p_2 = \frac{\sqrt{5}}{5}$, $p_3 = \frac{1}{2}$, $p_4 = \frac{5-\sqrt{5}}{5}$,

$p_5 = \frac{\sqrt{5}-1}{2}$, $p_6 = \frac{5+\sqrt{5}}{10}$, $p_7 = 1$ 。

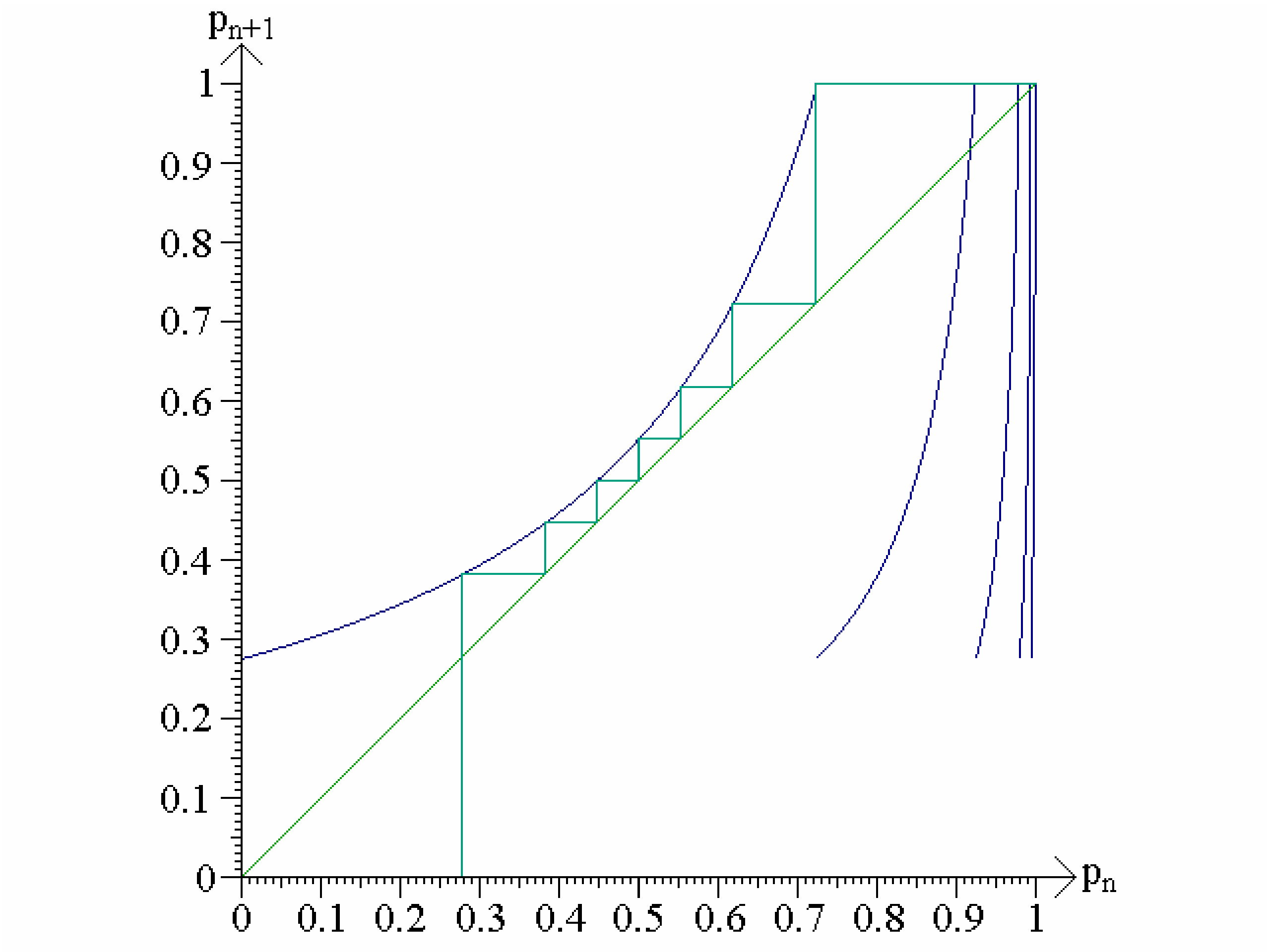
p_7 確實爲 1，故 $\frac{5-\sqrt{5}}{10} \in I_7$ 。

另外觀察到 $p = 0.2$ 時，p_∞ 的值爲

$\frac{1-\sqrt{0.2}}{2} = \frac{5-\sqrt{5}}{10}$ ，正好是此數值。

下圖 5 是 $\frac{5-\sqrt{5}}{10}$ 代入 7 次達到 1 的情形：

圖 5



3. $p_5 = 1$ 及 $p_9 = 1$ 的情形：

(1) 經過試驗後，找到以下兩個 p 值，分別代入 5 次及 9 次後會相當接近 1：

表 6

p_0	0.29289 32188 13452	0.26794 91924 31122
p_1	0.41421 35623 73094	0.36602 54037 84437
p_2	0.49999 99999 99998	0.42264 97308 10372
p_3	0.58578 64376 26902	0.46410 16151 37752
p_4	0.70710 67811 86451	0.49999 99999 99996
p_5	0.99999 99999 99977	0.53589 83848 62240
p_6		0.57735 02691 89617
p_7		0.63397 45962 15546
p_8		0.73205 08075 68845
p_9		0.99999 99999 99877

(2) 左欄中的 p 值屬於 I_5，代入的過程中，可觀察到

$p_1 \doteqdot \sqrt{2}-1,\ p_2 \doteqdot \dfrac{1}{2}\ ,\ p_4 \doteqdot \dfrac{\sqrt{2}}{2}$ ，

$\because \dfrac{p}{1-p_1} = p_2,\quad \therefore \dfrac{p}{2-\sqrt{2}} = \dfrac{1}{2}$ ，

$p = \dfrac{2-\sqrt{2}}{2}$ 。

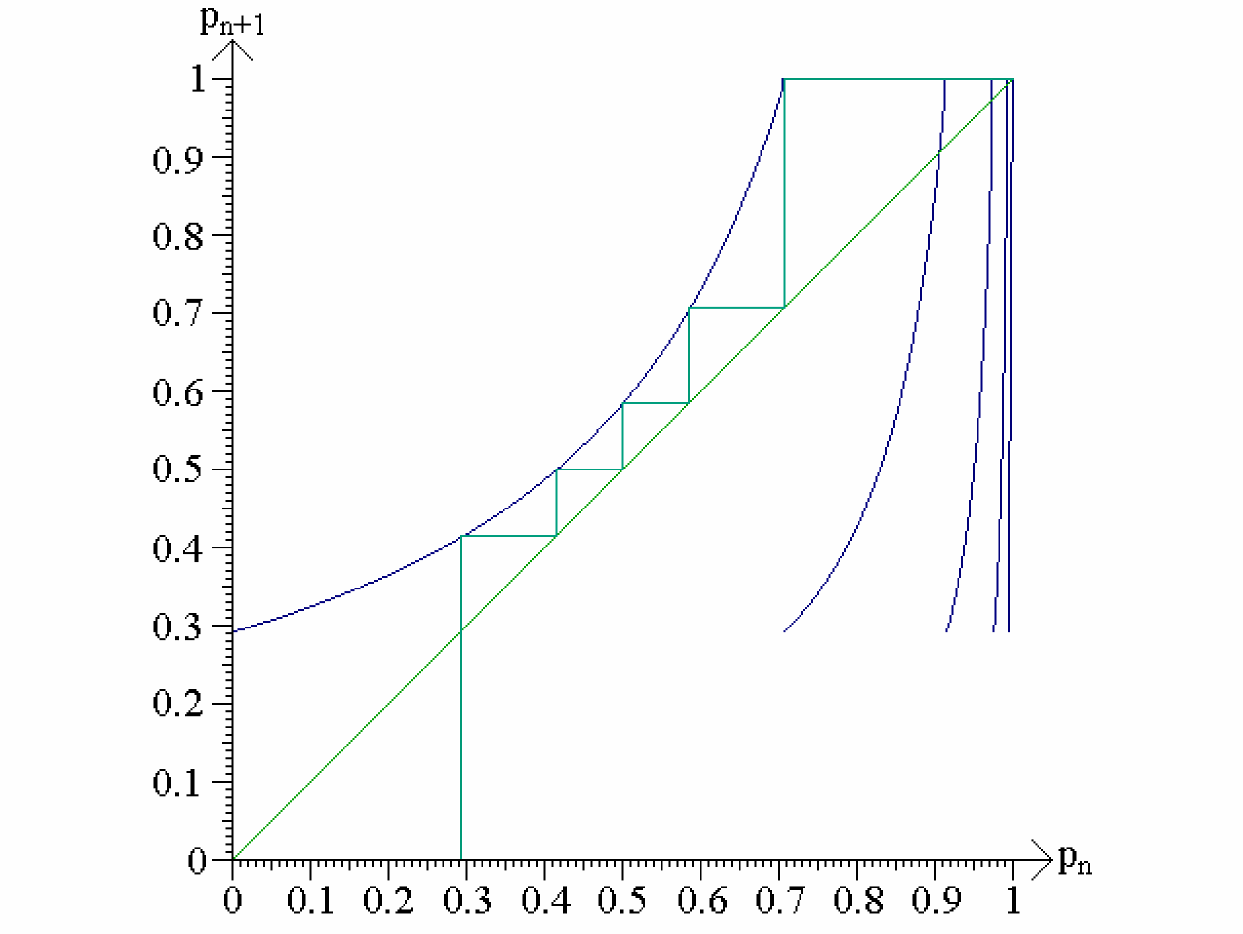
以 $p = \dfrac{2-\sqrt{2}}{2}$ 實際代入的情形：

$p_1 = \sqrt{2}-1$ ， $p_2 = \dfrac{1}{2}$ ， $p_3 = 2-\sqrt{2}$ ， $p_4 = \dfrac{\sqrt{2}}{2}$ ， $p_5=1$

證實 $\dfrac{2-\sqrt{2}}{2} \in I_5$ 。

下圖 6 爲 $p = \dfrac{2-\sqrt{2}}{2}$ 的代入情形：

圖 6



(3) p.13 表 6 右欄中的 p 值則屬於 I_9，代入的過程中，可觀察到：

$p_3 \doteqdot 2\sqrt{3}-3$ ， $p_4 \doteqdot \frac{1}{2}$ ， $p_8 \doteqdot \sqrt{3}-1$ ，

$\because \frac{p}{1-p_3} = p_4$ ，

$\therefore \frac{p}{4-2\sqrt{3}} = \frac{1}{2}$ ，

$p = 2-\sqrt{3}$

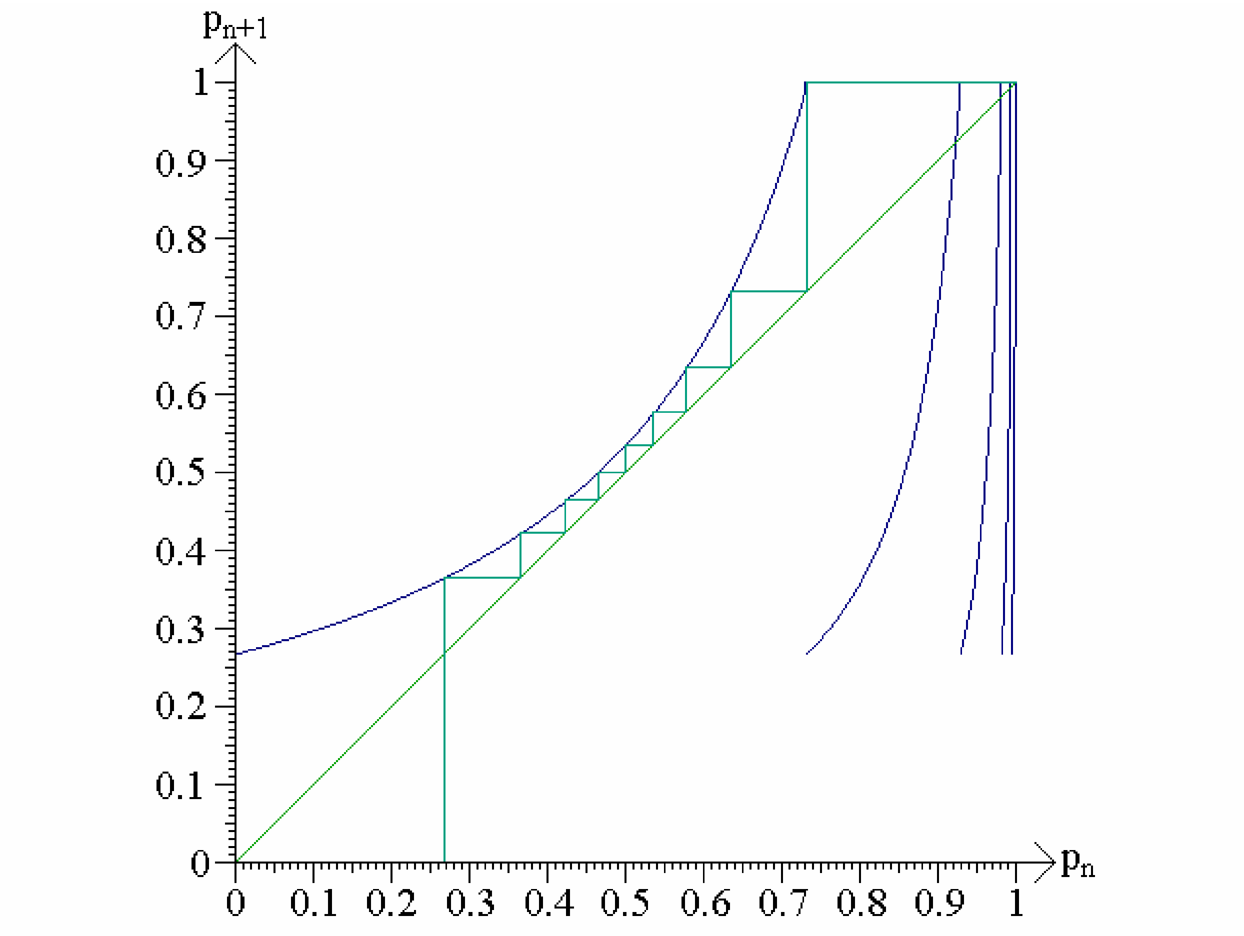
以$2-\sqrt{3}$ 實際代入的情形：

$p_1 = \frac{\sqrt{3}-1}{2}$ ， $p_2 = \frac{3-\sqrt{3}}{3}$ ， $p_3 = 2\sqrt{3}-3$ ， $p_4 = \frac{1}{2}$ ，

$p_5 = 4-2\sqrt{3}$ ， $p_6 = \frac{\sqrt{3}}{3}$ ， $p_7 = \frac{3-\sqrt{3}}{2}$ ， $p_8 = \sqrt{3}-1$ ， $p_9 = 1$ 。

證實 $2-\sqrt{3} \in I_9$ 。

圖 7



（四） 集合 I_n 有無限多個，所以無法用試驗的方式確定所有的 I_n 是否都有元素存在。考慮當 $k_n = 1$，探討是否存在能使 $p_n = 1$ 的 p 值：

1. 在 $k_n = 1$ 的情況下，將 $p_0, p_1, p_2, \ldots$ 用 p 表示：

$p_0 = p$，

$$p_1 = \frac{p}{1-p_0} = \frac{p}{1-p} \quad ,$$

$$p_2 = \frac{p}{1-p_1} = \frac{p-p^2}{1-2p} \quad ,$$

$$p_3 = \frac{p}{1-p_2} = \frac{p-2p^2}{1-3p+p^2} \quad ,$$

$p_4 = \frac{p}{1-p_3} = \frac{p-3p^2+p^3}{1-4p+3p^2}$ ，

$p_5 = \frac{p}{1-p_4} = \frac{p-4p^2+3p^3}{1-5p+6p^2-p^3}$ ，

$p_6 = \frac{p}{1-p_5} = \frac{p-5p^2+6p^3-p^4}{1-6p+10p^2-4p^3}$ 。

2. 設 $p_n = \frac{a(p)}{b(p)}$ ，則 $p_{n+1} = \frac{p}{1-p_n} = \frac{p\,b(p)}{b(p)-a(p)}$ ，其中分子爲 p_n 的分

母乘以 p，分母中的 a(p)則相當於 p_{n-1} 的分母乘以 p，所以 p_{n+1} 的分母等於 p_n 的分母減去 p_{n-1} 的分母乘上 p。

故對於任意的自然數 n，p_n 可表示爲 $\frac{p\,a_{n-1}(p)}{a_n(p)}$ ，其中 $a_0(p) = 1$, $a_1(p) = 1 - p$, $a_{n+1}(p) = a_n(p) – p\,a_{n-1}(p)$，如下表 7：

表 7

	常數	p	p^2	p^3	p^4	p^5
$a_0(p)$	1					
$a_1(p)$	1	-1				
$a_2(p)$	1	-2				
$a_3(p)$	1	-3	+1			
$a_4(p)$	1	-4	+3			
$a_5(p)$	1	-5	+6	-1		
$a_6(p)$	1	-6	+10	-4		
$a_7(p)$	1	-7	+15	-10	+1	
$a_8(p)$	1	-8	+21	-20	+5	
$a_9(p)$	1	-9	+28	-35	+15	-1
$a_{10}(p)$	1	-10	+36	-56	+35	-6

3. 以上各多項式是由 Fibonacci 多項式轉變而成：

(1) Fibonacci 多項式的遞迴公式爲 $f_{n+1}(x) = x\,f_n(x) + f_{n-1}(x)$，而

此處是 $a_{n+1}(p) = a_n(p) - p\ a_{n-1}(p)$，乘以 p 倍的是 a_{n-1} 而非 a_n，因此係數將與 Fibonacci 多項式順序相反，亦即高次項與低次項對調。又因 $a_n(p)$和 $a_{n-1}(p)$之間的運算符號是 "-"，所以 $a_n(p)$帶有符號，其中常數項爲正，正負相間。

(2) Fibonacci 多項式中 $f_1 = 1, f_2 = x$，次數不同，所以接下來所有的多項式，相鄰的兩項次數都差 2，亦即每兩項之間都有缺項。而此處的 $a_n(p)$則不會發生缺項。

4. 求出 $p_n = 1$ 的解：

(1) 欲使 $p_n = \dfrac{p\ a_{n-1}(p)}{a_n(p)} = 1$，則必須使 $p\ a_{n-1}(p) = a_n(p)$，移項得 $a_n(p) - p\ a_{n-1}(p) = 0$，可代換爲 $a_{n+1}(p) = 0$。

(2) Fibonacci 多項式 $f_n(x) = 0$ 的解爲
$x \in \{2i \cos \dfrac{j\pi}{n}, j = 1, 2, \ldots, n-1\}$。

(3) 若把 Fibonacci 多項式中的 x^2 以 p 代換（如果各次項爲奇數，則先約去 x），得到多項式 $g_n(p)$，則 $g_n(p) = 0$ 的每個解均爲 $f_n(x) = 0$ 之解的平方：
$p \in \{-4 \cos^2 \dfrac{j\pi}{n}, j = 1, 2, \ldots, n-1\}$。
又因 $\cos \dfrac{j\pi}{n} = \cos \dfrac{(n-j)\pi}{n}$，所以有一半的解是重複的，相異的解有：
$p \in \{-4 \cos^2 \dfrac{j\pi}{n}, j = 1, 2, \ldots, [\dfrac{n}{2}]\}$
例：$f_6(x) = x^5 + 4x^3 + 3x$, 則 $g_6(p) = p^2 + 4p + 3$,
$g_6(p) = 0$ 的解有-3, -1, 0。

(4) 再把 $g_n(p)$中每項改爲正負相間，其中領導係數爲正，得 $h_n(p)$，則 $h_n(p)$的解爲 $g_n(p)$解之相反數：

$p \in \{4\cos^2 \frac{j\pi}{n}, j = 1, 2, \dots, [\frac{n}{2}]\}$。

例：$h_6(p) = p^2 - 4p + 3$,　$h_6(p) = 0$ 的解有 3, 1, 0。

(5) 最後把 $h_n(p)$每項的係數順序倒轉，即可得到最初的 $a_{n-2}(p)$（因爲 $h_2(p)$對應的是 $a_0(p)$），$a_{n-2}(p) = 0$ 的解爲 $h_n(p)$ 解之倒數：$p \in \{\frac{1}{4\cos^2 \frac{j\pi}{n}}, j = 1, 2, \dots, [\frac{n-1}{2}]\}$。$j = \frac{n}{2}$並不合，因爲 $\cos \frac{\pi}{2} = 0$，故 $p = \frac{1}{0}$ 無意義。

例：$a_4(p) = 3p^2 - 4p + 1$，$a_4(p) = 0$ 的解有 $\frac{1}{3}$, 1。

(6) $p_n = 1$ 的解相當於 $a_{n+1}(p) = 0$ 的解，也就是 $p \in \{\frac{1}{4\cos^2 \frac{j\pi}{n+3}}, j = 1, 2, \dots, [\frac{n+2}{2}]\}$。

5. 證明 $a_{n+1} = 0$ 的所有解當中，只有$\frac{1}{4\cos^2 \frac{\pi}{n+3}}$，亦即 $j = 1$ 符合：

(1) $p_{n+1} = 0$ 的解必定小於 $p_n = 0$ 的解，因爲當 p_0 增大時，p_n 在未達到 1 之前，也會跟著增大，所以欲使 $p_n = 1$ 延後出現，就必須降低 p_n 的值。

(2) 用數學歸納法：

當 $n = 1$ 時，$a_2(p) = 0$ 只能得到一個解：$\frac{1}{2}$，此時 $j = 1$，成立。

設當 $n = k$ 時成立，亦即合理的解爲$\frac{1}{4\cos^2 \frac{\pi}{k+3}}$，

則當 $n = k + 1$ 時，若 j 取 2 以上，由於$\frac{2\pi}{k+4} > \frac{\pi}{k+3}$，

$\cos\frac{2\pi}{k+4} < \cos\frac{\pi}{k+3}$，

$\frac{1}{4\cos^2\frac{2\pi}{k+4}} > \frac{1}{4\cos^2\frac{\pi}{k+3}}$ ，

故不合，所以 j 只能取 1。

(3) 由此，$\frac{1}{4\cos^2\frac{\pi}{n+3}} \in I_n$。所以對於任意的自然數 n，至少可以找到一個屬於 I_n 的 p 值。

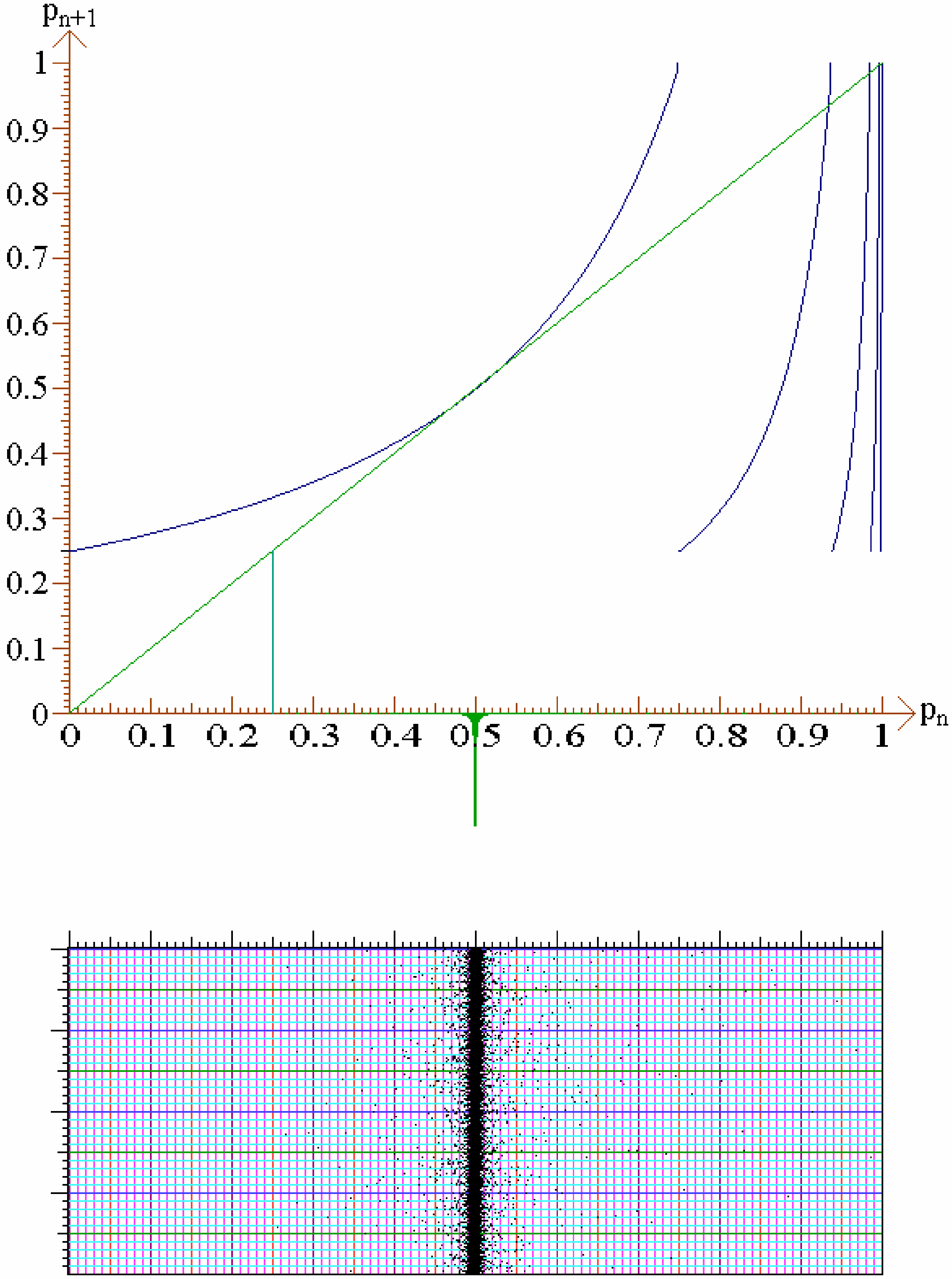
（五） 觀察特定 p 值的代入結果：

1. 接近 0.25 的 p 值經過反覆代入多次，每次出現的 p_n 值分布情況並不平均，而是集中在 0.5 附近出現：

(1) p=0.250001 時 p_n 的分布情況：

下圖 8 上方深藍曲線是以 $p_{n+1} = \frac{p^{n_k}}{1-p_n}$ ，綠色斜線是 $p_{n+1} = p_n$，淺藍色線段是 p 在橫軸上的位置，中間的直方圖則是把 0~1 分成 500 個區間，顯示 p 代入 25,000 次時，每次出現的 p_n 在各區間分布的情形。下方的方格則是把每個區間再分成 200 等分，當 p_n 值落在某一區間時，相對應的位置就點上一黑點。

圖 8



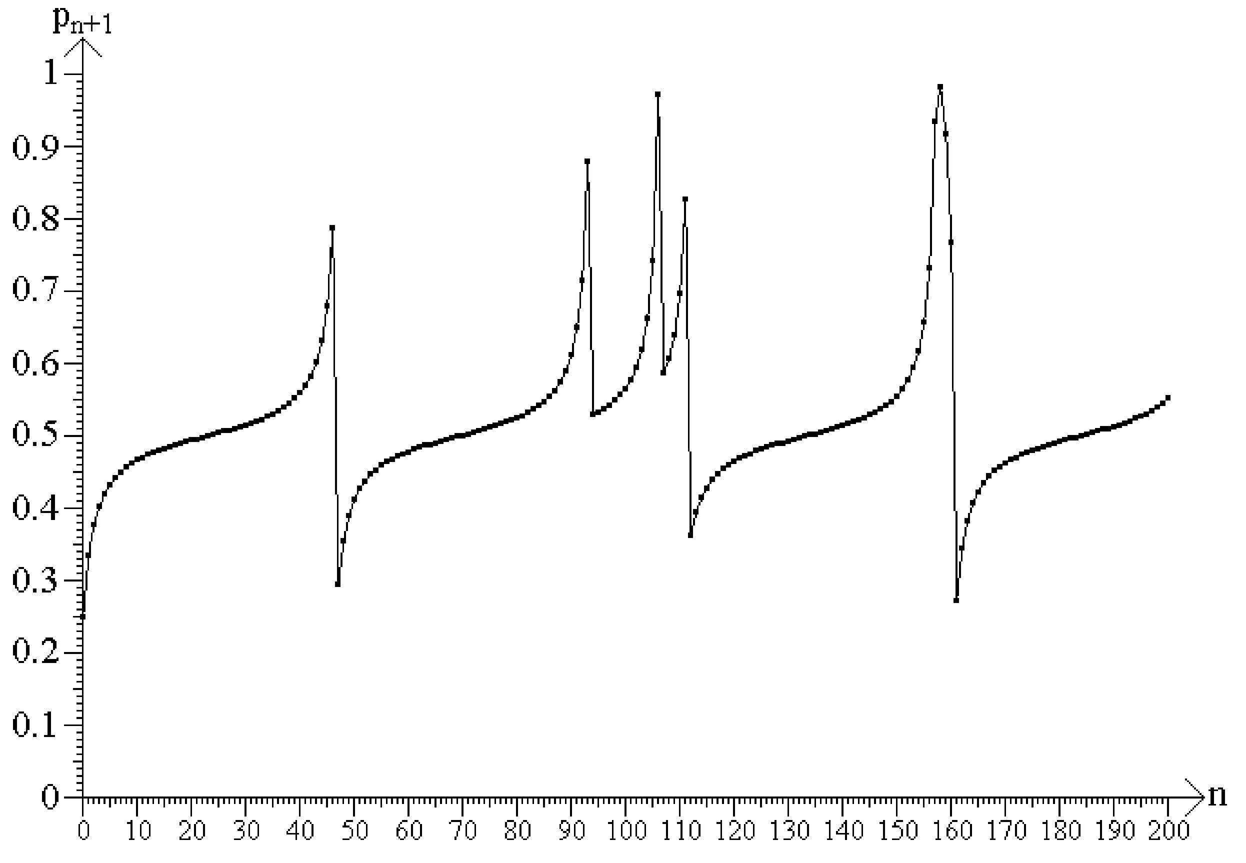
下表 8 爲圖 8 中央部分直方圖 500 條直線方別代表的數字，共 25 列 20 欄，由上而下，由左而右。如果此區間的數小於 p，而無 p_n 值在此範圍中，則用 “-”符號表示。

表 8

P0= .250001

-	-	-	-	-	0	0	0	2	78856		8	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	0	2	61752		6	2	0	0	0	1	0	0	0
-	-	-	-	-	0	0	1	3	8	637	4	3	0	0	0	1	0	0	0
-	-	-	-	-	1	1	0	0	6	327	7	2	1	0	1	0	0	0	0
-	-	-	-	-	0	0	0	1	9	196	5	0	0	0	0	0	0	0	0
-	-	-	-	-	0	0	1	1	10	133	3	3	0	1	0	0	0	0	0
-	-	-	-	-	0	0	0	3	11	94	4	0	1	1	0	0	0	0	0
-	-	-	-	-	0	0	1	2	11	72	5	2	1	0	0	0	1	1	0
-	-	-	-	-	1	1	2	2	15	55	5	1	0	1	2	0	0	0	0
-	-	-	-	-	1	1	0	2	15	44	2	1	1	0	0	0	1	0	1
-	-	-	-	-	0	0	1	3	18	37	2	0	0	0	0	0	0	0	0
-	-	-	-	-	0	0	1	1	21	29	5	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	1	3	26	27	3	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	1	3	29	21	3	1	0	1	0	0	0	0	0
-	-	-	-	-	0	0	2	2	37	19	3	2	2	0	2	1	0	0	0
-	-	-	-	-	2	0	0	5	45	18	1	2	0	0	0	1	0	0	0
-	-	-	-	-	1	1	3	1	55	14	2	1	1	0	0	1	0	0	0
-	-	-	-	-	0	0	0	4	71	12	3	0	0	1	0	0	1	0	0
-	-	-	-	-	0	1	1	4	95	13	3	2	0	0	0	0	0	0	1
-	-	-	-	-	0	0	0	5	132	12	2	0	2	0	1	0	0	0	0
-	-	-	-	-	0	2	1	2	198	7	3	0	0	1	0	0	0	0	0
-	-	-	-	-	0	0	0	5	326	10	2	3	0	0	0	0	0	0	0
-	-	-	-	-	1	0	3	4	639	8	1	0	0	0	0	0	0	1	0
-	-	-	-	-	0	2	1	51682		8	1	0	0	1	0	0	0	0	0
-	-	-	-	-	0	1	1	58858		6	2	2	3	0	0	0	0	0	0

(2) 以 p = 0.251 代入 200 次，並把 p_n 值畫成折線圖：
圖 9



從上圖 9 中可看出 p_n 從 0.25 開始遞增，到接近 0.5 時，增加速度減慢，然後又繼續增加到接近 1，下一個值則較無規律，可能爲(p,1)間的任何位置，必須視前一個 p_n 而定。接下來重複此過程，其中 p_n 增加最慢處，仍在 0.5 附近，這是因爲當 p 接近 0.25 且 p_n 接近 0.5 時，$\frac{p}{1-p_n} \fallingdotseq \frac{0.25}{1-0.5} = 0.5$，所以得出的 p_{n+1}，與 p_n 的差異並不大。

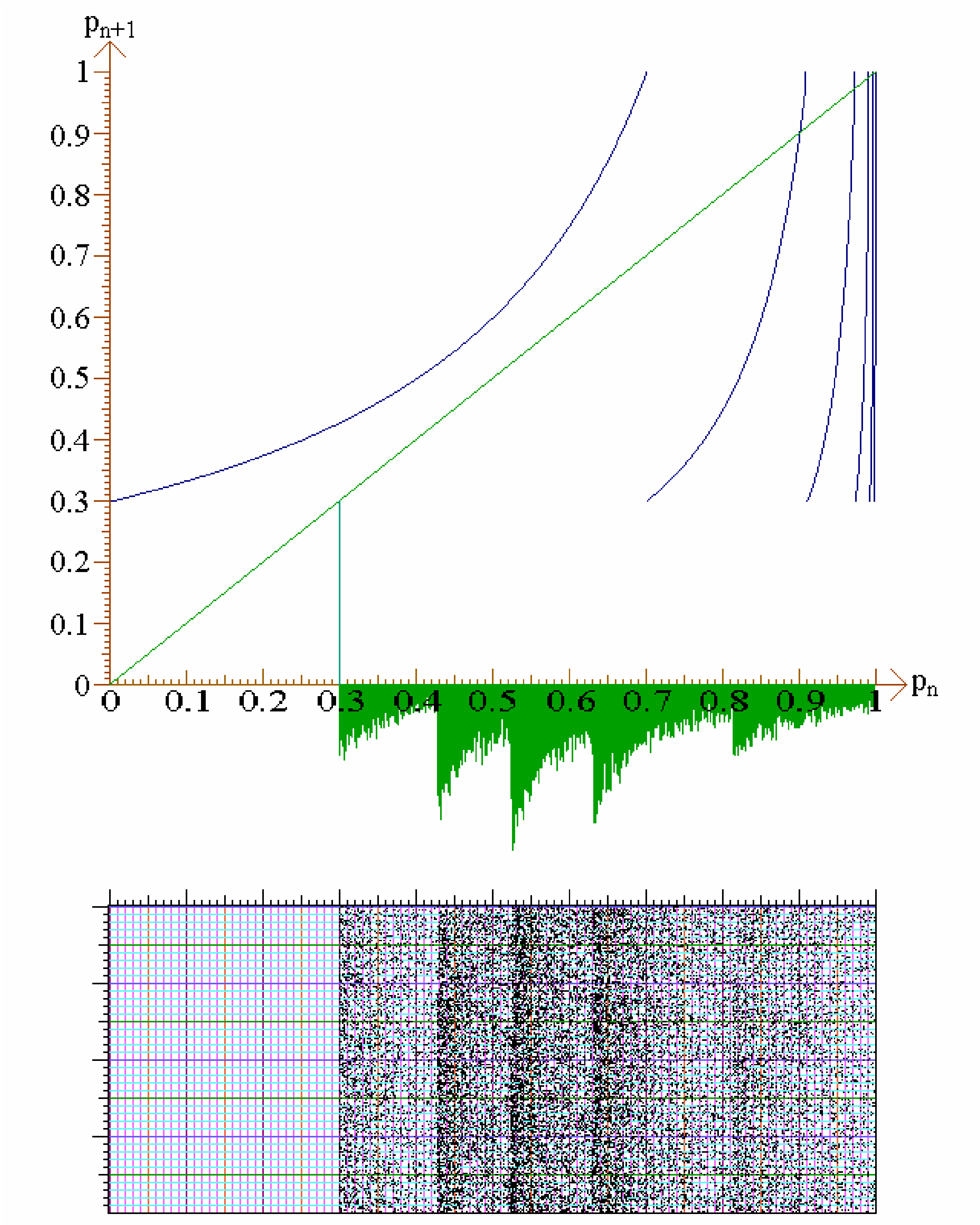
由此可見 p 接近 0.25 時，產生的 p_n 值大部分集中在 0.5 附近，並不是一個均勻的分布。

2. 0.25<p<0.5 的情形：

以 p = 0.3 為例

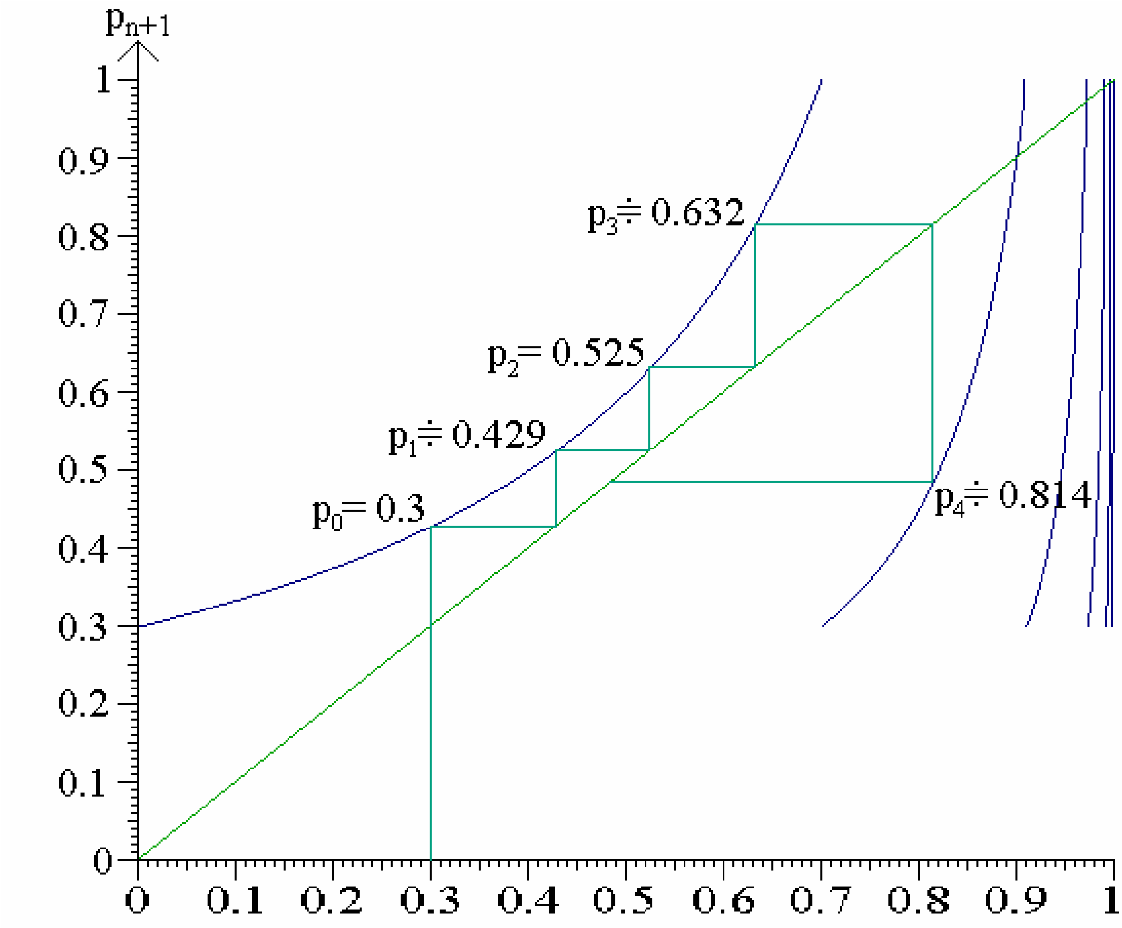
依照 p.21 圖 8 的方式，以 p = 0.3 代入的情況如下：

圖 10



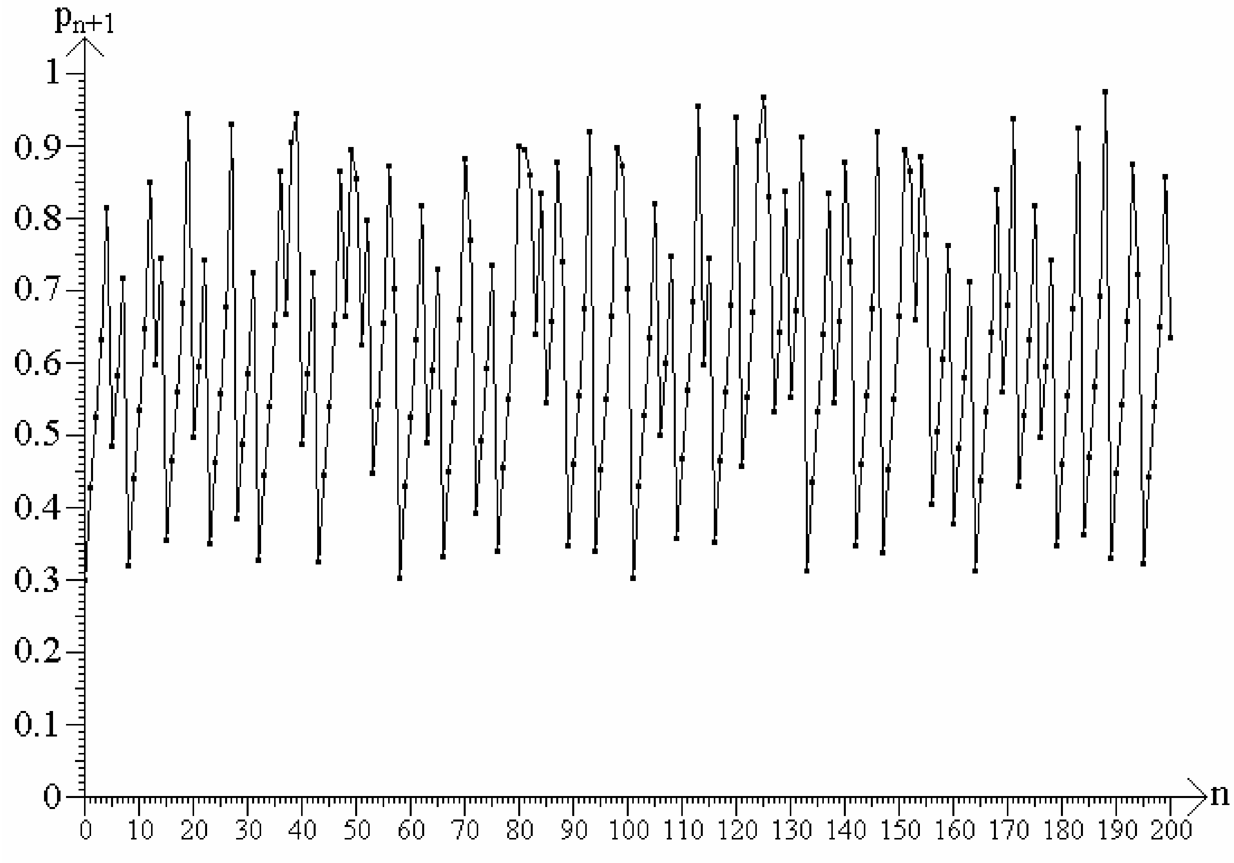
下圖 11 是 p = 0.3 代入 5 次，p_1~p_5 的位置圖：

圖 11



前頁圖 10 中可看出 p_n 在大約 0.300, 0.428, 0.524, 0.632, 0.814 等 5 個值附近出現的頻率特別高，且與上圖 11 中 p = 0.3 代入得到的 p_0~p_4 位置相當。現在把 p 代入 200 次，將每次的 p_n 值畫成折線圖，觀察各 p_n 值的分布情形：

圖 12



從上圖 12 可觀察到：

(1) 開始時 $p = 0.3$，p_n 連續遞增，經過 4 次代入，就已達到 0.8 以上，與前頁圖 11 對照，可看出前 4 次代入時，k_n 均為 1，而第 5 次代入時，因 $p_4 > 0.7$，$\dfrac{0.3}{1-p_4} > \dfrac{0.3}{1-0.7} = 1$，故 $k_5 > 1$。由此，第 5 次代入時，p_5 的值將視 p_4 與 k_5 而定，而非繼續增加。

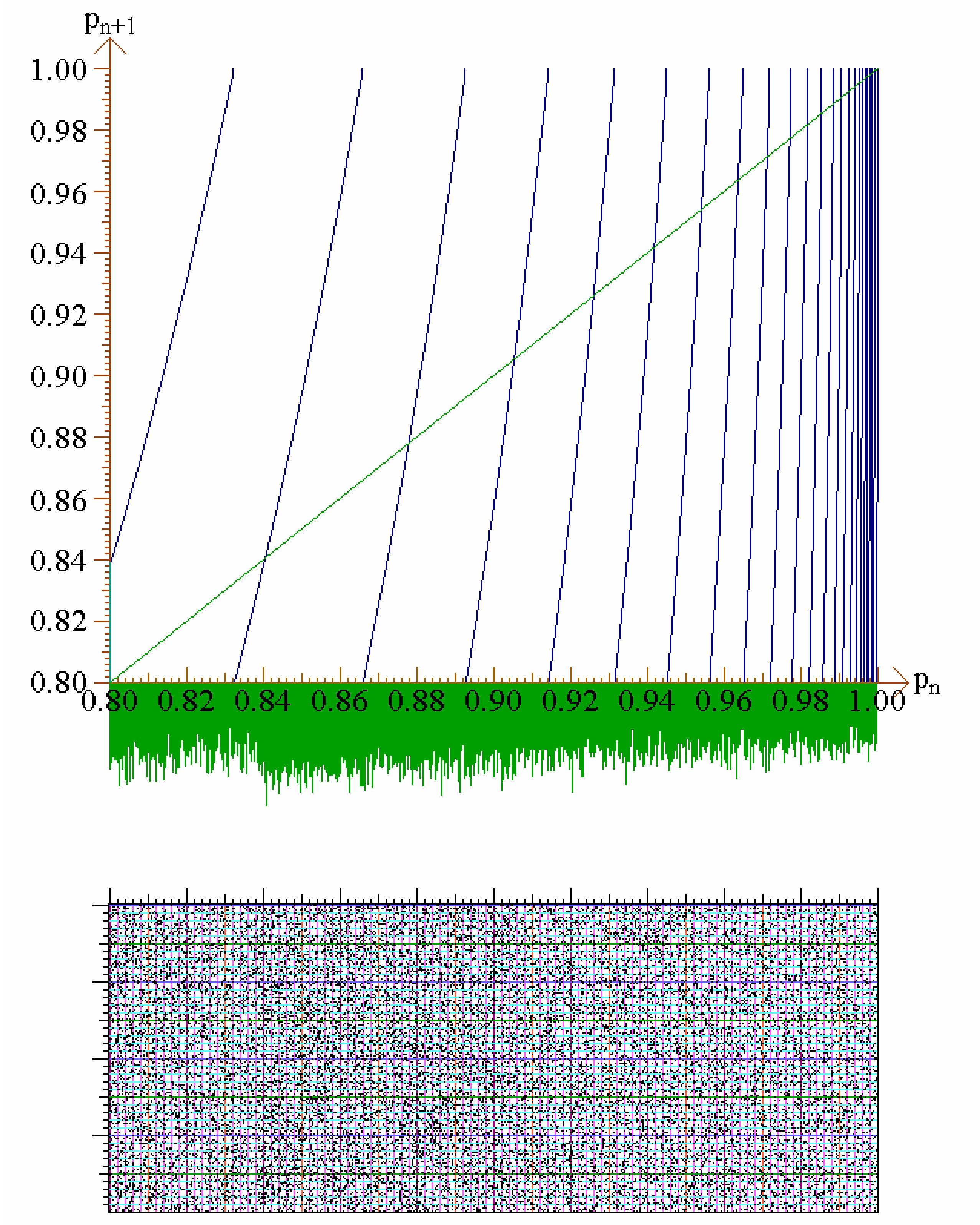
(2) 接下來重複此過程：當 $p_n < 0.7$ 時，k_{n+1} 取 1，且 $p_{n+1} > p_n$，直到某一數值 $p_{n+m} > 0.7$。由圖 10 可得知，$p_{n+m} > 0.7$ 時，p_{n+m+1} 將可能為$(p, 1)$間的任何數值，但較小的數值出現的頻率較高，原因是圖 10 中 p_{n+1} 較小時，曲線較平緩，故較大範圍的 p_n 會對應到較小的 p_{n+1}。

(3) 如此，重複代入多次後，若某一次達到 0.7 以上，則下一次的數值，接近 p 值(0.3)的情況較多，其他的數值則較少。這些數值繼續代入 4 次，分別會接近 p_1, p_2, p_3, p_4，但第 5 次代入後，不一定接近 p_5，因為 $p_4 > 0.7$，從圖 10 中可看出，p_n 值越大時，只要有微小的差異，將可能使 p_{n+1} 相差很大。因此圖 10 中間部分的直方圖中，接近 p_0, p_1, p_2, p_3, p_4 的區域，p_n 特別集中。

(4) 因每次達到 0.7 以後，下一次代入得到的值較無規律，而是分布在(p, 1)之間，而且越小的數，出現的頻率越高，又因 0.3 為有理數，**每次所得到的 p_n 將不重複，故發生渾沌(Chaos)現象，把 p 代入多次後，p_n 值將難以預測。**
渾沌現象是運用了某些圖形的碎形特性，這種性質從主要圖形的一連串局部重複累積而成。確定一組規則，使之隨機反覆運作，就能掌握某種形狀特質，重複這些規則，尺度一階階縮小，由此看來，形狀越碎形化，規則也會越簡單。

2. p > 0.5 的情形：
p 越大時，各 p_n 的分布越平均，以 p = 0.8 為例
下圖顯示 p = 0.8 的代入情形，為方便觀察，把圖形比例放大，只顯示 0.8~1 的範圍，因為 $p_n \geqq 0.8$。
圖形上半部的座標圖，表示 $0.8 \leqq p_n < 1$ 時，把 p_n 代入，所得到 p_{n+1} 的值。中間則是把 0.8 ~ 1 分為 500 個區間，將 p 值代入 25,000 次後，統計每區間的 p_n 值數量，所畫成的直方圖。下面的方格則是把每一區間再分為 200 等分，當某一 p_n 值落在一區間當中時，相對應的位置就畫上一黑點。

圖 13


pn+1
1.00
0.98
0.96
0.94
0.92
0.90
0.88
0.86
0.84
0.82
0.80
0.80 0.82 0.84 0.86 0.88 0.90 0.92 0.94 0.96 0.98 1.00
pn

下表 9 爲圖 13 中央部分直方圖 500 條直線方別代表的數字，共 25 列 20 欄，由上而下，由左而右。

表 9

P0= .8

56	45	47	45	51	56	59	42	57	49	61	61	49	47	47	52	50	54	53	41
52	64	48	54	65	52	48	54	54	50	43	43	71	47	34	36	52	41	38	45
48	50	44	61	80	65	52	55	62	64	58	53	44	44	53	43	56	58	40	39
50	48	53	29	57	68	50	54	52	71	45	52	46	56	50	38	35	37	35	52
50	50	41	47	60	51	55	46	67	44	49	45	49	44	47	55	56	57	42	38
52	64	41	49	58	61	61	47	50	40	70	43	42	37	40	48	46	46	46	45
67	46	52	46	69	61	71	59	50	62	63	60	56	47	54	51	39	42	52	46
49	52	33	38	66	71	59	56	50	60	54	50	61	49	40	50	45	50	41	28
56	54	38	47	56	57	61	47	56	54	59	43	61	45	31	36	62	42	44	28
47	52	54	39	63	54	55	55	62	61	62	43	41	52	50	48	42	41	47	42
63	63	36	56	72	60	49	53	53	67	54	57	41	48	50	50	40	38	37	37
66	55	44	58	64	56	59	52	44	53	46	47	36	66	38	41	30	39	51	29
41	47	40	55	58	54	56	54	63	60	63	57	45	51	46	43	30	36	50	45
64	33	58	39	57	64	67	52	57	52	59	54	48	51	49	45	47	43	36	45
47	48	59	59	54	53	67	64	52	46	60	57	45	41	57	41	50	52	36	37
62	46	43	32	58	56	67	52	48	59	42	44	48	30	53	38	48	45	44	41
38	58	58	46	67	58	70	62	73	47	51	42	45	57	51	50	36	46	42	39
41	49	56	51	61	64	48	60	50	44	46	51	40	54	49	52	56	42	37	38
42	59	52	43	59	60	66	47	60	56	45	54	42	43	44	47	41	32	44	38
53	55	44	51	50	60	51	56	52	59	42	41	44	47	49	44	52	39	47	51
49	55	42	38	61	60	46	47	62	50	55	51	43	50	39	50	39	43	50	40
51	42	33	52	55	59	68	56	42	57	48	43	56	45	55	39	40	48	39	39
47	37	46	57	65	62	56	66	54	51	50	45	51	54	39	44	36	38	40	33
51	44	50	53	59	64	53	56	58	47	47	47	54	53	42	54	44	31	50	44
61	54	57	60	71	61	49	56	51	46	43	56	42	45	41	33	45	38	43	39

由圖 13 中可看出 p_n 值的分布差距已不明顯，這是由於 0.8 代入後，只有第 1 次代入時，會發生類似前面以 0.3 代入時，數值遞增的情形，但 0.3 代入時，前 4 次的 k_n 可取最小值 1 ，到第 5 次時，k_n 才變爲其他的數值，停止遞增的規律。而 $p = 0.8$ 時，$k_1 = 8$，$p_1 = 0.8388608$，到了 n_2，已經不能用最小值 8，必須爲 9，而 $p_2 \fallingdotseq 0.8329303\ldots$，立刻就停止了遞增的規律。$p = 0.3$ 時，p_n 達到 0.7 以後，下一次代入所產生的 p_{n+1} 分布不均，集中在接近 p 值(0.3)的地方，但 $p = 0.8$ 時未發生此狀況，原因是曲線 $p_{n+1} = \dfrac{0.8^{k_{n+1}}}{1-p_n}$ 傾斜程度相當大，已經接近直線，導致彎曲情況不明顯，**也就是任意一個 p_n 值，下一次代入時，產生的 p_{n+1} 在(0.8, 1)中的分布接近均勻分布，可用來產生亂數。**

（六） $0.25 < p < 1$ 時，代入所得到的 p_n 值在亂數產生器(Random Number Generator)上的應用：

1. p 接近 1 時，p_n 在(p, 1)之間接近平均分布，所以如要產生 a ~ b 間的亂數 r，可利用線型函數 $r = a + \dfrac{b(p_n-p)}{1-p}$ ，把 p ~ 1 的數值轉換爲 a ~ b 間的數值即可，爲避免開始時產生的幾個數值不夠均勻，可考慮捨去最初的幾個 p_n 值不用。例如下表 10 爲 p = 0.9 代入 100 次的數值，欲產生 0 ~ 1 間的亂數，從 p_{10} 開始取用：
$r_1 = 0.7644915…$,　$r_2 = 0.5658556…$,　$r_3 = 0.7643012…$,
$r_4 = 0.5581310…$,　$r_5 = 0.5936046…$,　依此類推

表 10

1).984770902183612	2).970568520843624	3).945003096738492	4).951594729868443
5).973061132485150	6).929198491218439	7).912562740599305	8).912270622656761
9).909232978649651	10).976449153866438	11).956585568823122	12).976430120724319
13).955813107120308	14).959360468966092	15).938791405241909	16).950025682387580
17).942509857518431	18).910325858376755	19).988349367969318	20).924865160613559
21).955479499995319	22).952171657343451	23).984798643571927	24).972339738981343
25).904964527740656	26).932592631882362	27).958512707090255	28).919607903333041
29).992217473905698	30).908400486485661	31).967574802486815	32).953059865558774
33).903089835166414	34).914551960041752	35).933508166083718	36).971710585898658
37).983153586876528	38).974878340444946	39).996413274173846	40).942751712122974
41).914171676455297	42).929372027583961	43).914804951665213	44).936280272574271
45).912586091502338	46).912514317775938	47).911765686099131	48).904029731182470
49).923508731282576	50).938535861320232	51).946075846047458	52).970525441137113
53).943621890110441	54).928281622162509	55).900896267323153	56).993676903568007
57).905634914572590	58).939217940566622	59).956692444203495	60).978839777391401
61).958192168649778	62).912557317984198	63).912214049684090	64).908647030530762
65).970186100259932	66).932881615181347	67).962639656285516	68).919071796162667
69).985644550288110	70).926672094171494	71).979024260382969	72).966619530973894
73).925785505244521	74).967328538794436	75).945876102332696	76).966943726492773
77).934865037823350	78).991952965552336	79).976156777054526	80).944855466748701
81).949047171443835	82).924409660997379	83).949721878700612	84).936814776590232
85).920305949160431	86).900817539237712	87).992888152416208	88).994066648203076
89).965123440007528	90).984524798613644	91).955133479197681	92).944828293283331
93).948579741632754	94).916006423695162	95).949673136757217	96).935907464473602
97).907277837043756	98).955859725126879	99).960373681339030	100).962795529256995

2. p 較小，p_n 分布較不均勻時的使用方式：

下表 11 爲 p = 0.3 代入 25 次的數值。由於 $p = \frac{3}{10}$ 爲有理數，故 p_n 也都是有理數，亦即有限小數或循環小數。若某個 p_n 爲有限小數，則可採用此數的所有位數；若爲循環小數，則可採用一個循環節的數字。爲避免產生的數值分布不均勻，每次產生數值的前幾位小數可先刪去，又因最初幾個 p_n 值的分子與分母較小，可能使循環節太短，故剛開始的幾個 p_n 值可不採計，若循環節過長，可採用其中一部分，例如從 p_5 起，每個數值從第 3 位小數開始採用：

p_5 爲混循環小數 $0.4\overline{846153}$，可取用 46153，

p_6 的循環節較長 0.58208955…，可取用整個循環節除了最前面兩位數 58 以外的部分，或取 10 位數 2089552238，

p_7 爲混循環小數 $0.717\overline{857142}$，可取用 7857142，

全部串聯後得到 4615320895522387857142…，可作爲亂數之用。

表 11

1)	3	/	7	=	0.4285714285714285
2)	21	/	40	=	0.5250000000000000
3)	12	/	19	=	0.6315789473684210
4)	57	/	70	=	0.8142857142857143
5)	63	/	130	=	0.4846153846153846
6)	39	/	67	=	0.5820895522388060
7)	201	/	280	=	0.7178571428571429
8)	126	/	395	=	0.3189873417721519
9)	237	/	538	=	0.4405204460966543
10)	807	/	1,505	=	0.5362126245847176
11)	903	/	1,396	=	0.6468481375358166
12)	2,094	/	2,465	=	0.8494929006085192
13)	4,437	/	7,420	=	0.5979784366576819
14)	2,226	/	2,983	=	0.7462286288970835
15)	26,847	/	75,700	=	0.3546499339498019
16)	22,710	/	48,853	=	0.4648639796941846
17)	146,559	/	261,430	=	0.5606051333052825
18)	78,429	/	114,871	=	0.6827571797929852
19)	344,613	/	364,420	=	0.9456478788211404
20)	491,967	/	990,350	=	0.4967607411521179

三、研究結果與討論

（一）$p < 0.25$ 的研究結果：

p_n 將收斂到 $p_\infty = \dfrac{1-\sqrt{1-4p}}{2}$ 。

（二）$0.25 < p < 1$ 的研究結果：

1. 若 p 為 $\frac{1}{2}$ 和 $\frac{1}{3}$ 以外的有理數，則 $p\in I_\infty$，並且沒有兩個相異的 a 和 b，使得 $I_a=I_b$。

2. $\frac{1}{2}\in I_1$, $\frac{1}{3}\in I_3$, $\frac{2-\sqrt{2}}{2}\in I_5$, $\frac{5-\sqrt{5}}{10}\in I_7$, $2-\sqrt{3}\in I_9$。

3. 對於任意自然數 n，存在 $\dfrac{1}{4\cos^2\frac{\pi}{n+3}}\in I_n$，故每個 I_n 均不為空集合。

（三）當 $k_n = 1$，把 p_n 用 p 表示時，得到的分數為 $\dfrac{p\,a_{n-1}(p)}{a_n(p)}$ ，
其中多項式 $a_n(p)$ 的遞迴關係式為 $a_{n+1}(p) = a_n(p) - p\,a_{n-1}(p)$,
$a_0(p) = 1$, $a_1(p) = p$。

若要使 $p_n = 1$，則此分數之分子和分母必須相等：

$a_n(p) - p\,a_{n-1}(p) = 0$,

亦即 $a_{n+1}(p) = 0$。
此多項式可用 Fibonacci 多項式推算出，與 Fibonacci 多項式的差別如下表 12：

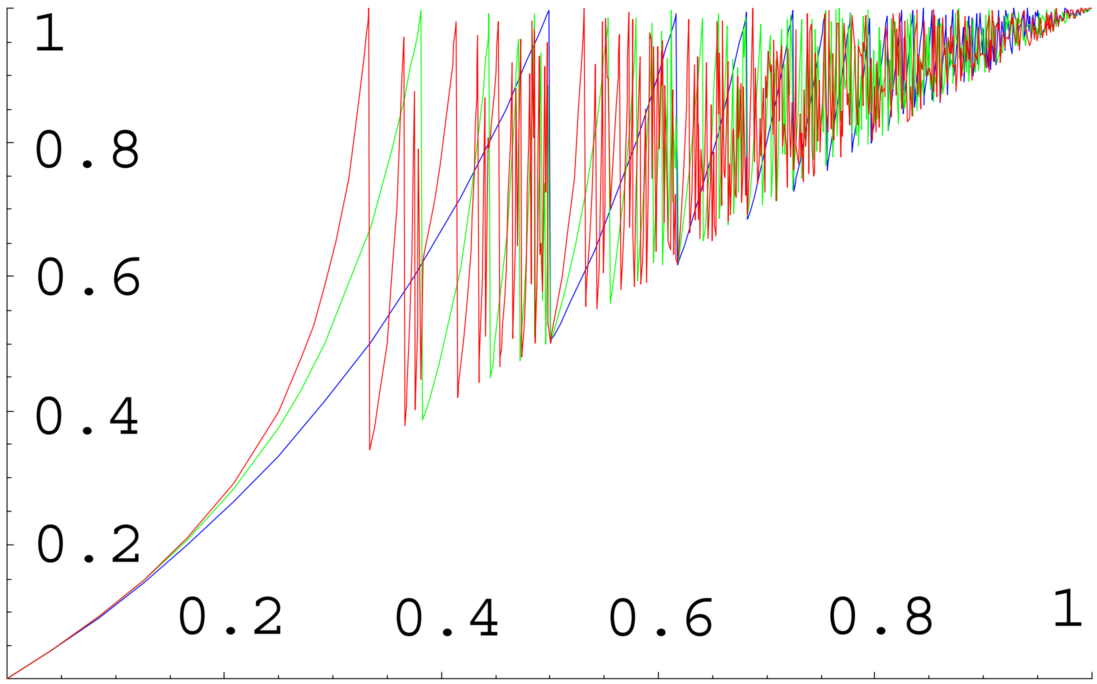
表 12

項目	p_n的分母減去分子 $a_{n+1}(p)$	Fibonacci 多項式 $f_n(x)$	令兩種多項式=0，求得的根之差異
相鄰兩項的次數差	1 次	2 次	令 $p = x^2$, 則欲求 $a_{n+1}(p) = 0$ 的解，須把 $f_n(x) = 0$ 的解平方
最低次項	常數項(0 次)	n 為奇數時，最低次項為常數；n 為偶數時，最低次項為 1 次項	當 Fabonacci 多項式的最低次項為 1 次項時，把 x 提出，不影響 $a_{n+1}(p)=0$ 之解，因為此時產生的解 x=0，無法推算出 $a_{n+1}(p)=0$ 的解。
係數之正負	常數項為正，正負相間	全部為正	$f_n(x) = 0$ 的解平方後，還需乘以–1。
係數順序	常數項必為 1，次序與 Fibonacci 多項式相反	最高次項為 1	因係數順序相反，所以上面得到的結果，需要取倒數。

（四）下圖為 0~1 間的 p 值代入 3 次所得到的結果：

（藍色線表示 p_1；綠色線表示 p_2； 紅色線表示 p_3）

圖 14



從圖 14 中可看出：

當 p 從 0 開始增加時藍色線 p_1 從 0 遞增到 1 以後，又回到與 p 相同的值，再開始增加。此線分為無限多段，因 k_n 的值可以從 1 一直到∞。每段的頂端所對應的 p 值，就屬於 I_1。p 值越接近 1，屬於 I_1 的數出現頻率就越高。

綠色線 p_2 分段的情形則是在 p_1 圖形的每一段中，再分為更多段。每段代表的是第 2 次代入時所取的 k_n 值。由此屬於 I_2 的數也有無限多個。

同理 I_3 以後的情況也如此，故此圖為碎形，因此把圖形任意放大後，仍可看出如同上圖 14 的情形。

（五）當 $k_n=1$ 時，p_n 用 p 表示的結果，得到的分子與分母與 Fibonacci 多項式相關。如果令 $k_n = 2$，則：

$$p_1 = \frac{p^2}{1+p} ,$$

$$p_2 = \frac{p^2-p^3}{1-p-p^2} \text{，}$$

$$p_3 = \frac{p^2-p^3-p^4}{1-p-2p^2+p^3} \text{，}$$

$$p_4 = \frac{p^2-p^3-2p^4+p^5}{1-p-3p^2+2p^3+p^4} \text{，}$$

$$p_5 = \frac{p^2-p^3-3p^4+2p^5+p^6}{1-p-4p^2+3p^3+3p^4-p^5} \text{，}$$

$$p_6 = \frac{p^2-p^3-4p^4+3p^5+3p^6-p^7}{1-p-5p^2+4p^3+6p^4-3p^5-p^6} \text{，}$$

$$p_7 = \frac{p^2-p^3-5p^4+4p^5+6p^6-3p^7-p^8}{1-p-6p^2+5p^3+10p^4-6p^5-4p^6+p^7} \text{。}$$

令 p_n 的分子爲 $p^2\ b_{n-1}(p)$，分母爲 $b_n(p)$，則 $b_n(p)$各項係數爲：

表 13

	常數項	p	p^2	p^3	p^4	p^5	p^6	p^7	p^8
$b_0(p)$	1								
$b_1(p)$	1	-1							
$b_2(p)$	1	-1	-1						
$b_3(p)$	1	-1	-2	+1					
$b_4(p)$	1	-1	-3	+2	+1				
$b_5(p)$	1	-1	-4	+3	+3	-1			
$b_6(p)$	1	-1	-5	+4	+6	-3	-1		
$b_7(p)$	1	-1	-6	+5	+10	-6	-4	+1	
$b_8(p)$	1	-1	-7	+6	+15	-10	-10	+4	+1

與表 7 中 $a_n(p)$的差異有：

1. k 每增加 1，$b_n(p)$的次數就增加 1，而前面的 $a_n(p)$則是 k 每增加 2，次數才增加 1。

2. 常數項與一次項的係數固定爲 1 和-1，且符號是按照“-, -, +, +”，每 4 項循環一次。

3. 上表 13 中從常數項開始算起每兩直欄一組，則每組兩欄除正負號相反外，左邊一欄某一格所出現的數字，在右邊一欄的下一格也將出現。如果把每一組的右邊一欄刪除，則剩下的部分將與 $a_n(p)$相同。

此一部分尚在研究中。

（六）當 $p > 0.25$ 時，會發生渾沌現象：

p_n 在 k_n 能取最小值時遞增，例如 $p = 0.3$ 時，k_n 最小值為 1，從 p_1 至 p_5，代入時都可取 $k_n = 1$，所以開始時連續 5 個 p_n 都是遞增的；$p = 0.8$ 時，k_n 最小須取 8，但只有第 1 次代入時，k_1 可用 8，k_2 則必須用 9，所以只有 p_0 至 p_1 是遞增的。

當某個 p_n 代入時，k_{n+1} 不能再取最小值時，就停止遞增的規律，而產生的 p_{n+1} 則必須視 p_n 而定，在 p 值較小時，所得到的 p_{n+1} 較小（接近 p）的情況較多，但在 p 接近 1 時，則接近均勻分布。P_n 接近 1 時，只要 p_n 有些微的差距，就可能使得出的 p_{n+1} 相差甚遠。

重複此過程多次後，p_n 值就會變得難以預測，故產生渾沌的狀況。當 p 較小時，p_n 呈現遞增，到不能遞增時，下一個值又經常回到 p 附近，故反覆代入多次後，可觀察到 p_n 在 $p_0, p_1, p_2, \ldots$等剛開始的幾個 p 值附近出現較密集。P 較大時，遞增的現象不易發生，且在不能遞增時，下一次代入所產生的值接近均勻分布在(p, 1)中，所以經過多次代入後，p_n 接近均勻分布於(p, 1)之中。

每次產生的 p_n 值可作為亂數使用，有兩種方式：

1. 當 p_n 均勻分布在(p, 1)之間時，可直接把各 p_n 經由線型函數 $r = a + \frac{b(p_n-p)}{1-p}$ 當作在區間(a, b)所產生的亂數來使用。

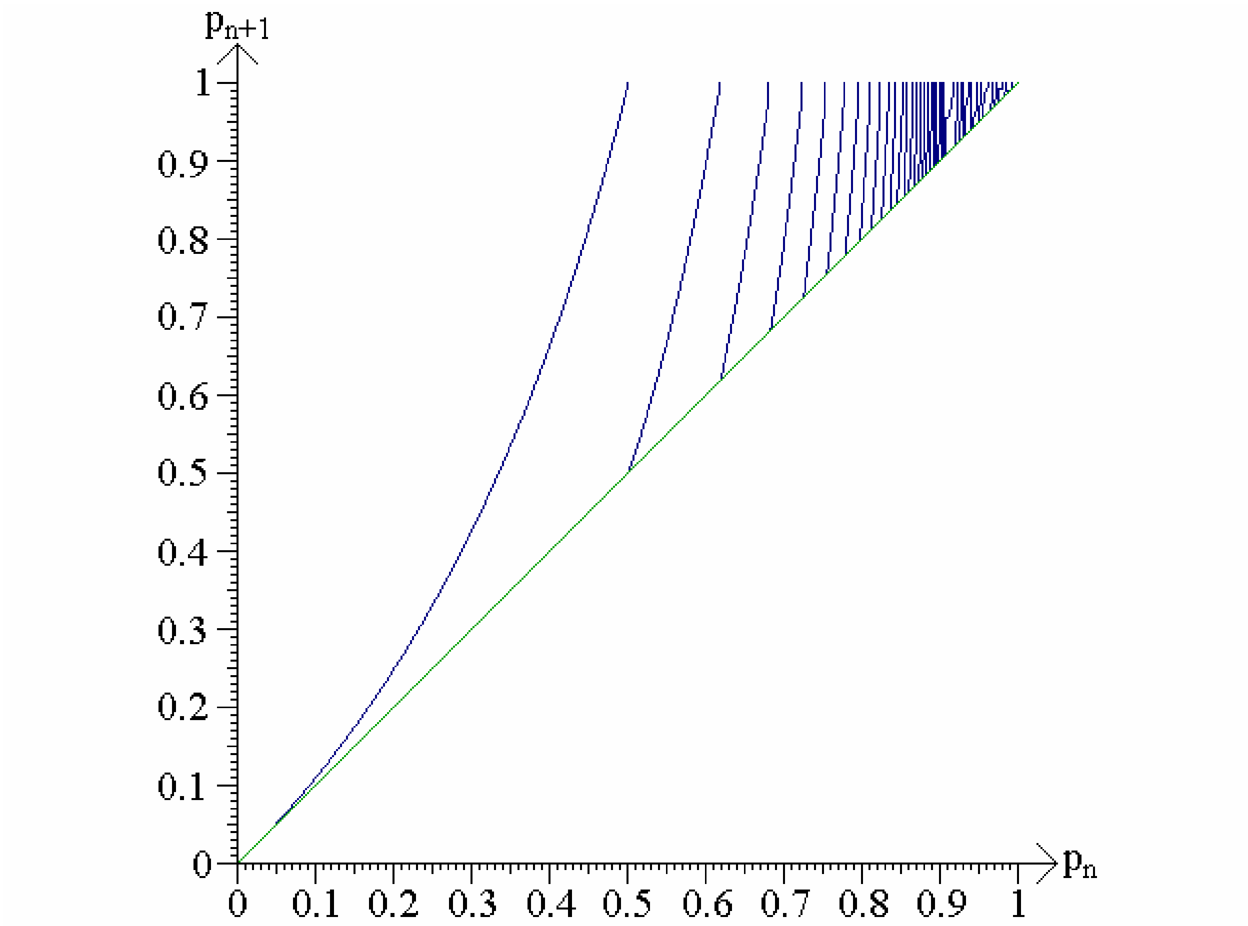
2. 當 p 為有理數，但 p_n 不為均勻分布時，可先把 p_n 的前幾位小數刪去，再依以下方法處理：
若 p 為有限小數，則取所有的位數；
若 p 為循環小數（純循環或混循環），可取到第一個循環節

結束爲止。爲避免發生如 1/7=0.142857…，142+857=999；1/17=0.0588235294117467，05882352+94117467=99999999 等循環節的前半段與後半段之和爲一串 9 的情況，可改爲取到第一個循環節之一半爲止。

（七）如果將遞迴式改爲 $p_{n+1}=\frac{p_n^{k_n}}{1-p_n}$ ，即分子部分由p^{k_n} 改爲$p_n^{k_n}$ ，則與原來遞迴式的差異爲：

原本 p_{n+1} 值受到 p 與 p_n 兩個值的影響，而現在 p_{n+1} 只由 p_n 來決定，因此無論原來的 p 值爲何，代入 p_n 所求得 p_{n+1} 的函數圖形就只有一種：

圖 15



四、結論與應用

（一）當 $p < 0.25$ 時，$p \in I_\infty$，且 p_n 收斂於 $p_\infty = \dfrac{1-\sqrt{1-4p}}{2}$ 。

當 $0.25 < p < 1$ 且 p 爲除了 $\frac{1}{2}$ 與 $\frac{1}{3}$ 以外的有理數時，$p \in I_\infty$，且對於相異的兩數 a 與 b，$p_a \neq p_b$。

對於任意自然數 n，在集合 I_n 中至少有一個元素 $\dfrac{1}{4\cos^2\frac{\pi}{n+3}}$ 。

（二）遞迴式 $p_{n+1} = \dfrac{p^{k_n}}{1-p_n}$ 與 Fibonacci 多項式的關聯：

1. 當 $k_n = 1$，把 p_n 表示成分數就相當於展開連分數 $\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\dots}}}$ （共 n 層），將此連分數化簡成分母、分子均爲多項式的形式後，則分母、分子均可由 Fibonacci 多項式演變而得：
先把 Fibonacci 多項式 $f_n(x)$相鄰兩項的次數差由 2 次改爲 1 次，如果最低次項爲 1 次項而非常數項，則把各項都降低 1 次。再將各項的符號改爲正負相間，其中最高次項爲正。最後把係數倒轉，即可得到多項式 $a_n(x)$，而 $p_n = \dfrac{p\, a_n(p)}{a_{n+1}(p)}$ 。

要求出 $p_n=1$ 的解，就令 $p\, a_n(p)=a_{n+1}(p)$，由於此遞迴式爲

$a_{n+1}(p) = a_n(p) + a_{n-1}(p)$，故 $a_{n+2}(p) = 0$ 的解即爲 $p = 1$ 的解。

已知 Fibonacci 多項式的解爲 $x \in \{2i \cos\frac{j\pi}{n}, j = 1, 2, \dots, n-1\}$，

故可推出 $p_n = 1$ 即 $a_{n+1}(p) = 0$ 的解爲

$p \in \{\frac{1}{4\cos^2\frac{j\pi}{n+3}}, j = 1, 2, \ldots, [\frac{n+2}{2}]\}$，但只有$\frac{1}{4\cos^2\frac{\pi}{n+3}}$ 在合理範圍內，因此$\frac{1}{4\cos^2\frac{\pi}{n+3}} \in I_n$。

2. n= 0 ~ 7 時多項式 $a_n(p)$分別為：

表 14

	常數	p	p^2	p^3	p^4
$a_0(p)$	1				
$a_1(p)$	1	-1			
$a_2(p)$	1	-2			
$a_3(p)$	1	-3	+1		
$a_4(p)$	1	-4	+3		
$a_5(p)$	1	-5	+6	-1	
$a_6(p)$	1	-6	+10	-4	
$a_7(p)$	1	-7	+15	-10	+1

與巴斯卡三角形對照：

表 15

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$a_n(p)$每項的係數，在巴斯卡三角形中沿左下到右上方向的斜線出現。

3. 如果把 k_n 改為 2，則 $p_n = \frac{p^2 b_n(p)}{b_{n+1}(p)}$，n=0~8 時的多項式 $b_n(p)$

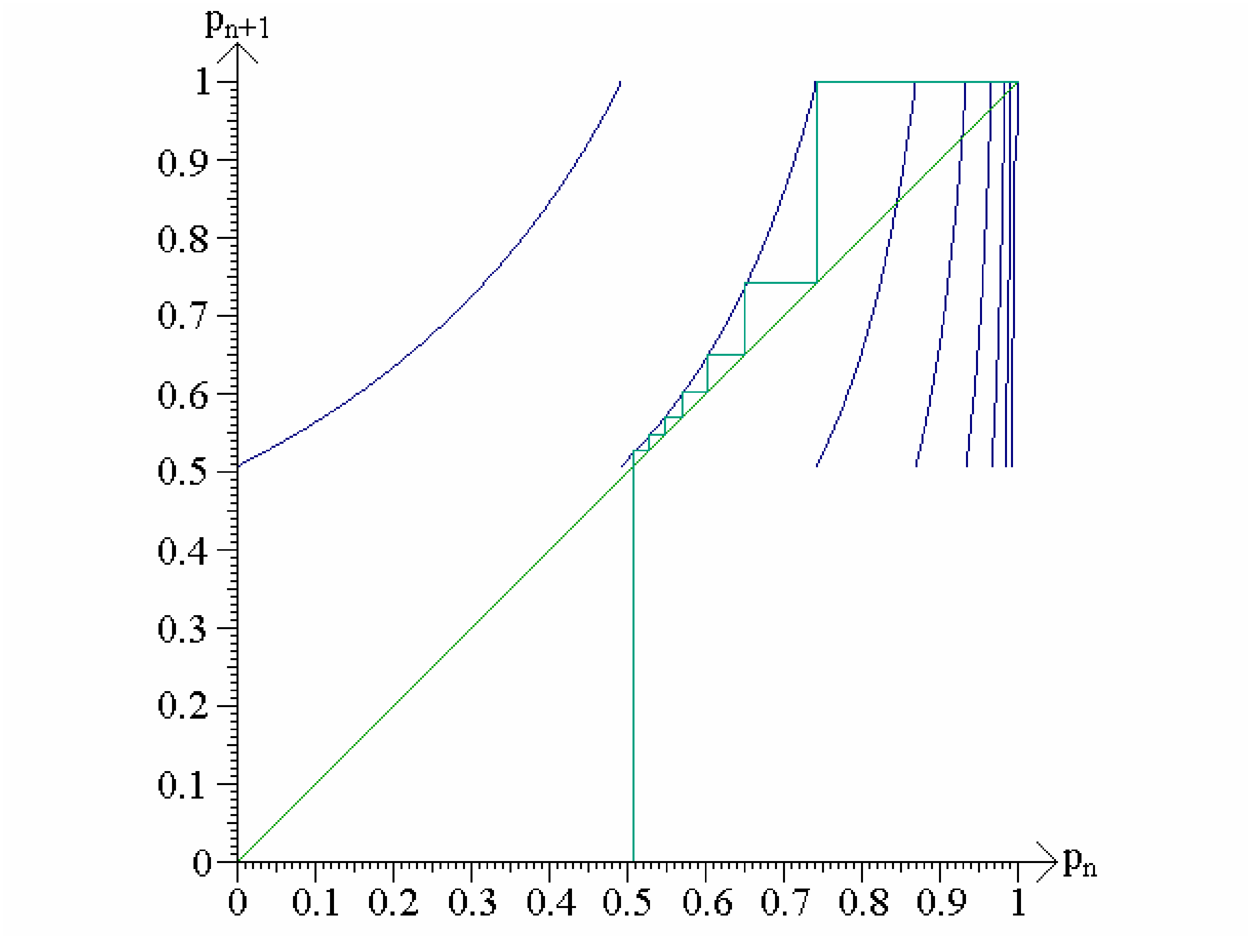
分別爲：

表 16

	常數項	p	p^2	p^3	P^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}
$b_0(p)$	1											
$b_1(p)$	1	-1										
$b_2(p)$	1	-1	-1									
$b_3(p)$	1	-1	-2	+1								
$b_4(p)$	1	-1	-3	+2	+1							
$b_5(p)$	1	-1	-4	+3	+3	-1						
$b_6(p)$	1	-1	-5	+4	+6	-3	-1					
$b_7(p)$	1	-1	-6	+5	+10	-6	-4	+1				
$b_8(p)$	1	-1	-7	+6	+15	-10	-10	+4	+1			
$b_9(p)$	1	-1	-8	+7	+21	-15	-20	+10	+5	-1		
$b_{10}(p)$	1	-1	-9	+8	+28	-21	-35	+20	+15	-5	-1	
$b_{11}(p)$	1	-1	-10	+9	+36	-28	-56	+35	+35	-15	-6	+1

以求 $p_7 = 1$ 的解爲例，利用 Mathematica 計算
$b_8(p) = 1-p-7p^2+6p^3+15p^4-10p^5-10p^6+4p^7+p^8 = 0$ 的解，
得到 $b_8(p) = 0$ 有 8 個解，其中唯一符合的解爲
$p = 0.50866\ 09187\ 58394\ldots$，把此 p 值實際代入檢驗，過程如下圖 16 所示：

圖 16



從圖 16 中可看出從 p_0 至 p_7 的值都位於深藍色線由左算起的第 2 段 $p_{n+1} = \dfrac{p^2}{1-p_n}$ 與綠色線 $p_{n+1} = p_n$ 之間。這情況類似 $0.25 < p < 0.5$ 且當 $k_n=1$ 時屬於 I_n 集合的數 $\dfrac{1}{4\cos^2\dfrac{\pi}{n+3}}$ 代入時的圖形，但差別爲 $p_0, p_1, \ldots$ 等數位於圖 5 中深藍色線的第 1 段 $p_{n+1} = \dfrac{p}{1-p_n}$ 與綠色線 $p_{n+1} = p_n$ 之間。

當 $p = 0.25$ 時，$p_{n+1} = p_n$ 與 $p_{n+1} = \dfrac{p}{1-p_n}$ 相切於點(0.5, 0.5)；$p < 0.25$ 時，兩線則相交於兩點。這導致 p_n 收斂到兩線相交處。

但當 $k_n = 1$ 時，$p_{n+1} = \dfrac{p^{k_n}}{1-p_n}$ 的斜率太大，所以無法與 $p_{n+1} = p_n$ 相切於一點或相交於兩點，故不會發生收斂的情形。

（三）$0.25 < p < 1$ 時產生渾沌現象，在亂數產生器(Random Number Generator)上的應用之探討：

p_n 需要取到足夠接近 1 時，利用線型函數轉換成 a ~ b 間的亂數時，才不會失真。因爲當 p_n 不夠大時，會發生連續幾個 p_n 值遞增的情形，接下來再代入一次，回到 p 附近的可能性太大，會造成 p_n 在 $p_0, p_1, p_2,\ldots$等剛開始的幾個 p 值附近出現的頻率太高，而不能均勻分布。P 值較大時，可避免連續遞增以及遞增結束後，下一 p_n 值位置分布不均的情況發生，而能產生較均勻的亂數。

如果 p_n 太小，可用以下的方式代替：去掉前面幾位數，只採用後面一段。但 p 爲有理數時，p_n 也都是有理數，亦即有限小數或是循環小數。若 p_n 爲有限小數，則採用所有的小數位數除去前面幾位；若爲循環小數，無論是純循環或是混循環，都可採用到第一個循環節的一半爲止。

五、參考文獻

（一）Alexandru Lupaş (1999). A Guide of Fibonacci and Lucas Polynomials. Octogon, Math. Magazine, vol.7, No.1, 2-12.

（二）James Gleick (1991). Chaos: Making a New Science. 天下文化。

（三）http://www.research.att.com/~njas/sequences/, The On-Line Encyclopedia Integer Sequences, A011973, Triangle of numbers $\{C_{k}^{n-k}, n \geqq 0, 0 \leqq k \leqq [\frac{n}{2}]\}$; or, triangle of coefficients of Fibonacci polynomials.

（四）吳振奎，世界數學名題欣賞叢書(2)斐波那契數列。九章數學。

六、附錄

程式總表

程式名稱	用途
Chaos.bas	把輸入的 p 值代入 10,000 次，以 0.002 爲組距畫出 p_1~$p_{10,000}$ 在各組出現次數直方圖，並以 0.00001 爲單位印出每次代入的結果。圖形上方顯示 0~0.998(以 0.002 爲單位)分別代入 p_n 值計算的結果 p_{n+1}(最小單位 0.01)。
Chaos2.bas	把輸入的 p 值代入 80 次，以 15 位小數列出每次計算結果。
Chaos3.bas	用 3 種顏色在坐標圖上顯示出 p=0~399/400(以 1/400 爲單位)時，p_1~p_3 的值。
Chaos4.bas	修改 Chaos3.bas， p 值的間隔改爲 1/800，因輸出的圖表過大，故分左下、左上和右上等三部分輸出。(圖表右下沒有內容)
Chaos5.bas	設定一段 0~1 之間的範圍，在此範圍中，每隔一定的間隔，就取一個數當作 p 值，代入指定的次數，輸出得到的結果。
Chaos6.bas	改進 Chaos.bas，當直方圖長度超出 140 像素時，會自動縮小圖形的比例爲原本的一半，以避免超出範圍外。
Chaos7.bas	輸入正整數的分子及分母作爲 p 值，計算 p_1~p_{20} 的值，並分別以最簡分數和小數表示。
Chaos8.bas	改進 Chaos6.bas，p<0.7 時代入 25,000 次，p≧0.7 只代入 5,000 次(p 愈大時，計算花費時間愈長，且 p_n 出現的範圍也較小，只在 p 與 1 之間，故代入太多次，直方圖會太長)，圖形完成後，列出長條圖 500 根直條分別表示的數字(限於螢幕空間，每個數最多只能顯示 4 位數字，超過 9,999 就會溢位)。

Chaos9.bas	改進 Chaos8.bas，可改變觀察範圍，輸入 0~1 之間的任何區間作爲觀察範圍，即以此範圍的 1/500 作爲組距，統計 p 值代入特定次數($p<0.7$ 次數同 Chaos8.bas，$p\geqq 0.7$ 增加爲 10,000 次以利觀察)後每次的結果在這 500 組中出現的次數，並以觀察範圍的 1/100,000 爲最小單位，在方格上描出每次出現的數。最後與 Chaos8.bas 一樣列出長條圖各直條的確實數字，若數字超過 9,999，顯示不下時，即輸出 “E”和前 3 位數，例如 11,850 即爲 “E118”。
Chaos10.bas	改進 Chaos2.bas，先輸入 p 值與 p_0 值(可以不相同)，再計算 p_1~p_{80} 的值。
Chaos11.bas	輸入 p 值和代入次數，畫出 $0 < p_n < 1$ 之間 $p_{n+1} = \frac{p^{k_n}}{p_n}$ 與 $p_{n+1} = p_n$ 的函數圖形，並用淺藍色折線表示 p 代入時的情形(直線表示由 p_n 得到 p_{n+1} 的運算，橫線表示把 p_{n+1} 由縱軸移到橫軸，以作爲新的 p_n，進行下一次運算)，解析度爲 400×400。
Chaos12.bas	改進 Chaos11.bas，可以設定座標圖中橫軸與縱軸的觀察範圍。
Chaos13.bas	將 Chaos11.bas 輸出圖形的解析度改爲 500×400，以配合 Chaos9.bas 的圖形觀察。
Chaos14.bas	把遞迴式改爲 $p_{n+1} = \frac{{p_n}^{k_n}}{p_n}$ 的函數圖形，此爲固定的圖形，不需輸入 p 值。
Chaos15.bas	輸入 p 值，代入 200 次，畫出圖形的橫軸爲代入次數 n，縱軸爲 p_n。

以下是 Chaos7.bas, Chaos9.bas, Chaos15.bas 的程式碼，謹供參考：

Chaos7.bas：

```
CLS
1 INPUT P, Q
IF Q = 0 OR P > INT(P) OR Q > INT(Q) THEN 1
IF (P / Q) <= 0 OR (P / Q) >= 1 THEN 1
M# = P: N# = Q
GOSUB SUB1
P = M#: Q = N#
P0# = P / Q
P0P# = P: P0Q# = Q
FOR I = 1 TO 20
    A = 0
    DO
        A = A + 1
        P1P# = P0Q# * P ^ A
        P1Q# = (P0Q# - P0P#) * Q ^ A
        P1# = P1P# / P1Q#
        M# = P1P#: N# = P1Q#
        GOSUB SUB1
        P1P# = M#: P1Q# = N#
        P1# = P1P# / P1Q#
    LOOP UNTIL P1# < 1.00000000001#
    PRINT USING "##)    #####################,    /    #####################,    = 
#.################"; I; P1P#; P1Q#; P1#
    IF P1# > .99999999999# THEN END
    P0P# = P1P#: P0Q# = P1Q#
NEXT I
END

SUB1:
S# = M#: T# = N#
DO
    T# = T# - S# * INT(T# / S#)
    IF T# = 0 THEN R# = S#: EXIT DO
    S# = S# - T# * INT(S# / T#)
    IF S# = 0 THEN R# = T#: EXIT DO
LOOP
M# = M# / R#: N# = N# / R#
RETURN
```

Chaos9.bas：

```
CLS
SCREEN 12
LINE (0, 0)-(639, 479), 15, BF
DO
    INPUT "P0="; P#
LOOP WHILE P# <= 0 OR P# > .98
DO
    INPUT "Enter two numbers between 0 and 1 as the range"; B, E
    IF B < E AND B >= 0 AND E <= 1 THEN EXIT DO
LOOP
B# = B
E# = E
C# = (E# - B#) / 500
LINE (0, 0)-(640, 480), 15, BF
LINE (70, 100)-(570, 100), 6
FOR I = 0 TO 100
    IF I MOD 10 = 0 THEN
        LINE (70 + I * 5, 90)-(70 + I * 5, 99), 6
        LINE (70 + I * 5, 234)-(70 + I * 5, 243), 0
        LINE (70 + I * 5, 245)-(70 + I * 5, 444), 5
    ELSE
        IF I MOD 5 = 0 THEN
            LINE (70 + I * 5, 94)-(70 + I * 5, 99), 6
            LINE (70 + I * 5, 238)-(70 + I * 5, 243), 0
            LINE (70 + I * 5, 245)-(70 + I * 5, 444), 12
        ELSE
            LINE (70 + I * 5, 97)-(70 + I * 5, 99), 6
            LINE (70 + I * 5, 241)-(70 + I * 5, 243), 0
            LINE (70 + I * 5, 245)-(70 + I * 5, 444), 13
        END IF
    END IF
NEXT I
FOR I = 245 TO 440 STEP 5
    IF (I + 5) MOD 50 = 0 THEN
        LINE (59, I)-(68, I), 0
        LINE (70, I)-(570, I), 9
    ELSE
        IF (I + 5) MOD 25 = 0 THEN
            LINE (63, I)-(68, I), 0
            LINE (70, I)-(570, I), 2
        ELSE
            LINE (66, I)-(68, I), 0
            LINE (70, I)-(570, I), 11
        END IF
    END IF
NEXT I
LINE (69, 244)-(570, 445), 0, B
LOCATE 1, 1
PRINT "P0="; P#
```

```
PRINT "Range:"; B; "to"; E
LOCATE 6, 8
PRINT B
LOCATE 6, 71
PRINT E
DIM A(500)
LOCATE 1, 1
P0# = P#
IF P# < .8 THEN T = 25000 ELSE T = 10000
R = 1

FOR I = 1 TO T
    A = 0
    DO
        A = A + 1
        P1# = P# ^ A / (1 - P0#)
    LOOP UNTIL P1# <= 1
    D = (P1# - B#) / C#
    IF 0 <= D AND D < 500 THEN
        A(INT(D)) = A(INT(D)) + 1
        IF A(INT(D)) = 133 / R THEN GOSUB SUB3 ELSE GOSUB SUB1
    END IF
    IF I MOD 100 = 0 THEN
        LOCATE 3, 1
        PRINT USING "Times: ######,"; I;
        LOCATE 4, 1
        PRINT USING "    P    :.############"; P1#;
    END IF
    IF P1# >= .9999999899999999# THEN S = I: LOCATE 25, 41: PRINT P1#: EXIT FOR
    P0# = P1#
NEXT I
S = I - 1
LOCATE 3, 1
PRINT USING "Times: ######,"; S;
FOR I = 0 TO 499
    A = 0
    DO
        A = A + 1
        P1# = P# ^ A / (1 - (B# + C# * I))
    LOOP UNTIL P1# <= 1
    PSET ((70 + I), (100 - P1# * 100)), 12
NEXT I
IF P# - B# > -.0000001# AND P# < E# THEN LINE (70 + (P# - B#) / C#, 0)-(70 + (P# - B#) /
C#, 100), 3
LINE (70, 100 - B# * 100)-(570, 100 - (B# + C# * 500) * 100), 1
LOCATE 29, 10
PRINT "Press any key to continue";
DO
LOOP WHILE INKEY$ = ""
CLS
```

```
PRINT "P0="; P#
FOR I = 0 TO 24
    FOR J = 0 TO 19
        LOCATE I + 2, J * 4 + 1
        IF P# - (B + (J * 25 + I + 1) * C#) >= -.0000001 THEN
            PRINT "     -"
        ELSE
            IF A(I + J * 25) < 10000 THEN
                PRINT USING "####"; A(I + J * 25);
            ELSE
                PRINT "E"; MID$(STR$(A(I + J * 25)), 2, 3)
            END IF
        END IF
    NEXT J
NEXT I
END

SUB1:
PSET ((70 + INT(D)), (100 + A(INT(D)) * R)), 2
PSET ((70 + INT(D)), 245 + ((P1# - B#) / C# - INT((P1# - B#) / C#)) * 200), 0
RETURN

SUB3:
LINE (70, 101)-(570, 233), 15, BF
R = R / 2
FOR X = 0 TO 499
    IF A(X) > 0 THEN LINE (70 + X, 100)-(70 + X, 100 + A(X) * R), 2
NEXT X
LOCATE 5, 1
PRINT "Ratio: 1 :"; 1 / R
RETURN
```

Dense15.bas：

```
SCREEN 12
CLS
LINE (0, 0)-(639, 479), 15, BF
DO
    INPUT "P="; P#
LOOP UNTIL 0 < P# AND P# < 1
CLS
LINE (0, 0)-(639, 479), 15, BF
PRINT "P="; P#
COLOR 0
LINE (15, 20)-(15, 440)
LINE (15, 440)-(635, 440)
FOR I = 0 TO 100
    IF I MOD 10 = 0 THEN
        LINE (5, I * 4 + 40)-(14, I * 4 + 40)
        LINE (I * 3 + 15, 441)-(I * 3 + 15, 450)
        LINE (I * 3 + 315, 441)-(I * 3 + 315, 450)
    ELSE
        IF I MOD 5 = 0 THEN
            LINE (10, I * 4 + 40)-(14, I * 4 + 40)
            LINE (I * 3 + 15, 441)-(I * 3 + 15, 445)
            LINE (I * 3 + 315, 441)-(I * 3 + 315, 445)
        ELSE
            LINE (12, I * 4 + 40)-(14, I * 4 + 40)
            LINE (I * 3 + 15, 441)-(I * 3 + 15, 443)
            LINE (I * 3 + 315, 441)-(I * 3 + 315, 443)
        END IF
    END IF
NEXT I
LINE (15, 20)-(5, 30)
LINE (15, 20)-(25, 30)
LINE (625, 430)-(635, 440)
LINE (635, 440)-(625, 450)
P0# = P#
LINE (14, 440 - P0# * 400 - 1)-(16, 440 - P0# * 400 + 1), 0, BF
FOR I = 1 TO 200
    K = 0
    DO
        K = K + 1
        P1# = P# ^ K / (1 - P0#)
    LOOP UNTIL P1# <= 1
    LINE (15 + (I - 1) * 3, 440 - P0# * 400)-(15 + I * 3, 440 - P1# * 400), 0
    LINE (15 + I * 3 - 1, 440 - P1# * 400 - 1)-(15 + I * 3 + 1, 440 - P1# * 400 + 1), , BF
    IF P1# > .999999999999# THEN EXIT FOR
    P0# = P1#
NEXT I
END
```

Recursive sequences and chaos phenomena

Abstract

For a given fixed number $p \in (0,1)$, let $n_0=0$, $p_0=p$,

$$k_1 = \inf\{k \geq 1: \frac{p^k}{1-p} \leq 1\},$$

and

$$p_1 := \frac{p^{k_1}}{1-p}.$$

By induction, for given p_{n-1}, we define

$$k_n = \inf\{k \geq 1: \frac{p^k}{1-p_{n-1}} \leq 1\},$$

and

$$p_n := \frac{p^{k_n}}{1-p_{n-1}}.$$

Then, $\{p_n, n \geq 0\}$ is a sequence of numbers belongs to (p,∞). For each $n=1,2,\ldots$, let $I_n := \{p \in (0,1);$ there exists k_n such that $p^{k_n}/(1-p_{n-1}) = 1\}$, and denote $I_\infty = \{p \in (0,1);\ p^{k_n}/(1-p_{n-1}) < 1$ for all k and $k_n\}$. It is noted that the open interval (0,1) can be decomposed as a countably union of $I_1, I_2, \ldots$, and I_∞.

Motivated by the problem of random number generation in computer simulation, in this article, we study the behavior of the recursive sequence $\{p_n, n \geq 0\}$ for various $p \in (0,1)$. In particular, we explore the relationship between p and I_n, for $n=1,2,\ldots,\infty$. Our main results shows that

1. When $0 < p \leq 1/4$, then $p \in I_\infty$ and the sequence $\{p_n, n \geq 0\}$ is monotone increasing with limit $(1-\sqrt{1-4p})/2$.

2. When $1/4 < p < 1$ and is a rational number except 1/3 and 1/2, then $p \in I_\infty$ and p_n are distinct from each other for different n. That is, $p_i \neq p_j$ for $i \neq j$.

3. For each n=1,2,…,I_n is not empty. We first find explicit element in each I_n for small n, for instance, $\frac{1}{2} \in I_1$, $\frac{1}{3} \in I_3$, $\frac{2-\sqrt{2}}{2} \in I_5$, $\frac{5-\sqrt{5}}{10} \in I_7$, $2-\sqrt{3} \in I_9$.

4. In general, we let k_n=1, and expand the ratio of p_n to get recursive $a_0(p) = 1$, $a_1(p) = 1-p$, $a_{n+1}(p) = a_n(p) - p\, a_{n-1}(p)$ $(n\geq 1)$,

$$p_n = \frac{p\, a_{n-1}(p)}{a_n(p)} .$$

In order to make $p_n = 1$, we need $a_n(p) - p\, a_{n-1}(p) = 0$, that is $a_{n+1}(p) = 0$. Then the polynomial $a_n(p)$ has relation with the Fibonacci polynomial $f_0(x) = 1$, $f_1(x) = x$, $f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x)$ $(n \geqq 1)$. A simple transformation leads that

$$\frac{1}{4\cos^2 \frac{\pi}{n+3}} \in I_n, \text{ for } n=1,2,\ldots.$$

5. By using the result form part 2, we find that for a suitable initial point $p \in (0,1)$, the recursive sequence $\{p_n, n \geq 0\}$ can provide an example of chaotic systems. To be more precise, when $.25 < p < 1$, a chaotic phenomena happens and it approaches to uniform distribution as p approaches to 1. Then the sequence $\{p_n, n \geq 0\}$ can be used to produce random numbers.

Keywords. chaos, Fibonacci polynomial, recursive sequence.

I. Research motive：

With a majority of random numbers found in day-to-day living being a series of random number values, it is more than often that a computer simulator is needed in search. A Random Number Generator allows random numeral values be generated using the recursive functions of recurring decimals. To which, I have defined p value to be $0<p<1$ and that $p_0 = p$, $p_n = \frac{p^{k_n}}{1-p_{n-1}}$, in which k_n being the smallest positive number that produces $p_n \leqq 1$ with which to observe the numeral values generated in each simulation and how they are distributed within the given parameter of p and 1.

II. Research process:

In computer simulation, the reason for adopting the reversal function $p_n= \frac{p^{k_n}}{1-p_{n-1}}$ lies in that k_n being the smallest positive number that produces $p_n \leqq 1$ can help observe the value distribution of such function.

(I) Some phenomena concluded from the computer simulation show that when p is smaller than or equals to 0.25, p_n will gradually increase, but p_∞ will always equal to a value less than or equals to 0.5. This serves to validate that,

1. No validation by mathematical inductive means would prove p_n to be greater than 0.5

   Given n =1, equation $p_1 = \frac{p}{1-p_0} \leqq \frac{0.25}{1-0.25} = \frac{1}{3} < 0.5$，will withstand,

   Given n = k is to withstand，which indicates $p_k \leqq 0.5$， Given n = k+1, equation $p_{k+1} = \frac{p}{1-p_k} \leqq \frac{0.25}{1-0.5} = 0.5$ would therefore also withstand

2. This also validates that $p_n > p_{n-1}$

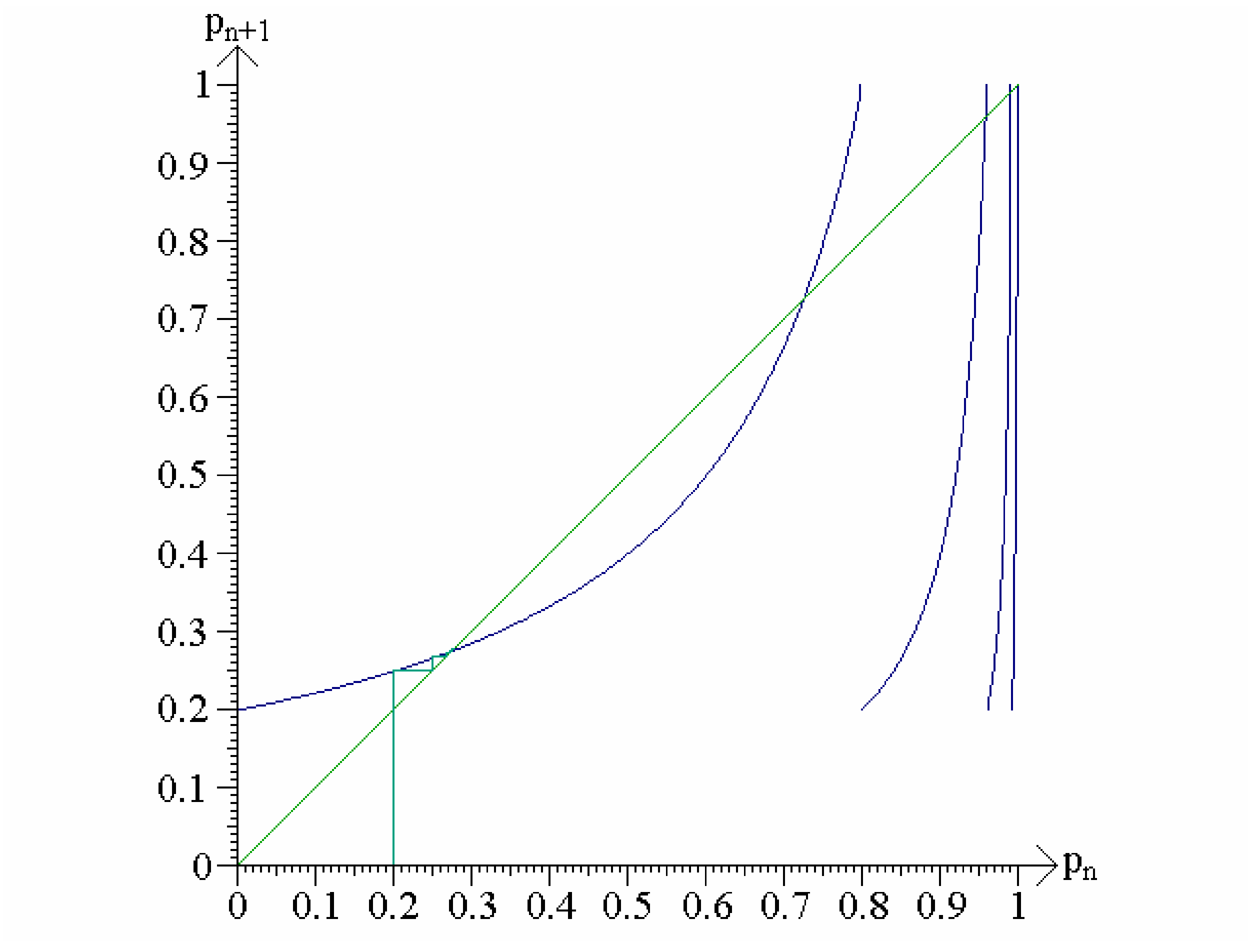
   Given n = 1, $\frac{p}{1-p_0} > p = p_0$ is to withstand,

   Given n = k is to withstand, which indicates $p_k > p_{k-1}$，

   Therefore given n = k + 1, equation $p_{k+1} = \frac{p}{1-p_k} > \frac{p}{1-p_{k-1}} = p_k$ would also withstand.
3. At which, p_n will coverage to a numeral value p_∞ that is $\leqq 0.5$;

   For instance, given p=0.2：

Graph 1



In the foregoing chart where the horizontal axe being p_n，the vertical axes being p_{n+1}，the blue line being $p_{n+1} = \frac{p^{k_{n+1}}}{1-p_n}$ （where k_{n+1}bieng the smallest positive number that ensures $p_{n+1} \leqq 1$）, the green line being $p_{n+1}=p_n$, the point where the two lines intersect, $\frac{p^{k_{n+1}}}{1-p_n} = p_n$, would be the value of p_∞.

In which, the blue line is not only noticeably of an incomplete curve but segments of its graph tend to become vertical when p_n approaches 1 as its denominator becomes very small when p_n n nears, hence the rate of k_n in $\frac{p^{k_n}}{1-p_n}$ will ac accelerate .

4. Calculating the p_∞ value:

$\because p, p_n < 0.5$, as long as $p < 1 - p_n$ and k_n equals to 1, equation $\frac{p^{k_n}}{1-p_n} < 1$ will withstand.

Given $x = p_\infty$，then $\frac{p}{1-x} = x$，

$p = x(1-x)$,

$p = x-x^2$,

$x^2-x+p = 0$,

$x = \frac{1 \pm \sqrt{1-4p}}{2}$,but $\frac{1+\sqrt{1-4p}}{2} \geqq \frac{1}{2}$ will arise to contradict

but $\frac{1-\sqrt{1-4p}}{2}$ will conform，

$\therefore p_\infty = \frac{1-\sqrt{1-4p}}{2}$ 。

(III) Given 0.25<p<1, attempts are made to validate that all rational numbers, except $\frac{1}{3}$ and $\frac{1}{2}$., would belong to I_∞ , and that none will be identical to I_a and I_b:

1. Given P being a rational number and its denominator other

than 1:

(1) Taking to the example of p= $\frac{4}{11}$ and p= $\frac{3}{7}$ a logical order can be observed by repeatedly replaced the both:

① Where p= $\frac{4}{11}$

Table 1

p_0	4/11
p_1	4/7
p_2	28/33
p_3	48/55
p_4	320/847
p_5	308/527
p_6	2108/2409
p_7	14016/36421
p_8	13244/22405
p_9	89620/100771
p_{10}	586304/1349271

Given that the numerator of p= $\frac{4}{11}$ an even number, the value of numerator derived in each replacement will also produce an even number.

② But an odd number to the numerator of p does not always produce an odd number to the numerator of p_n,

For example, when p = $\frac{5}{19}$:

Table 2

p_0	5/19
p_1	5/14
p_2	70/171
p_3	45/101
p_4	505/1064
p_5	280/559
p_6	2795/5301
p_7	1395/2506
p_8	12530/21109
p_9	5555/8579
p_{10}	42895/57456

In spite not all numerators are in odd number but rather multiplications of 5, this can be deduced to the fact that the number of p being 5.

③ The numerator is then modified to 3 to observe whether the numerators of p_n are multiplications of 3, where p= $\frac{3}{7}$

Table 3

p_0	3/7
p_1	3/4
p_2	36/49
p_3	9/13
p_4	117/196
p_5	36/79
p_6	237/301
p_7	387/448
p_8	1728/2989
p_9	549/1261
p_{10}	3783/4984

Upon replacing 10 times, it has been found that the numerators are consistently multiplications of 3 in each validation.

Under the premises that the numerators of p are multiplications to a quantifiable number m, validation is made that the numerators of p_n will contain said quantifiable number m:

(2) Upon validating p as a rational number using the inductive method and converting to a simple divider, where its numerators contains primary factors and its denominators other than the multiplications of m, set to mb_0+c_0, this indicates that $p=\frac{ma_0}{mb_0+c_0}$, where none of its numerators of p_n would be multiplications to m, and its denominators multiplications other than that of m:

① Given n=0, $p_0 = p = \frac{ma_0}{mb_0+c_0}$ would withstand.

② Given $n=\ell$, meaning $p_\ell = \frac{ma_\ell}{mb_\ell+c_\ell}$, then when , $n=\ell+1$

$$p_{\ell+1}= \frac{p^{k_{\ell+1}}}{1-p_\ell} = \frac{\left(\frac{ma_0}{mb_0+c_0}\right)^{k_{\ell+1}}}{1-\frac{ma_\ell}{mb_\ell+c_\ell}} = \frac{mb_\ell+c_\ell}{m(b_\ell-a_\ell)+c_\ell} \times \frac{(ma_0)^{k_{\ell+1}}}{(mb_0+c_0)^{k_{\ell+1}}},$$

Where neither the numerators or denominators of $\frac{mb_\ell+c_\ell}{m(b_\ell-a_\ell)+c_\ell}$ to left were multiplications of m were multiplications of m, and the numerator, mb_0+c_0 , the power of ma_0, which are multiplications of m, it can be concluded that the numerators of $n=\ell+1$ are multiplications of m, but not the denominators. And suppose the fraction can be further subdivided, the m factor in the numerator cannot be weeded out for none of its denominators contain m factor, leaving the simplified numerator fraction as multiplications of m, but not the denominators, hence $n=\ell+1$ will also withstand.

③ As validated through ① and ② , the equation is to withstand irrespective whatever the number n is.

(3) The above experiment indicate that although a host of p_n values concluded from the same p varies as both the denominators and numerators go up, despite occasionally reducing through subdivision, the velocity of their expansion will accelerate, and that no identical p_n values will be repeated.

But suppose a given p_a be identical to p_b that previously occurred, but p_{a-1} and p_{b-1} being dissimilar, this indicates that replacing the both numbers would arrive at the same numeral value.

To demonstrate using the example of $p = \frac{2}{5}$, it is intended to calculate whether there are any other p_n that allows $p_{n+1} = \frac{2}{3}$ except $p_1 = \frac{2}{3}$,

$$\frac{(\frac{2}{5})^{k_n}}{1-p_n} = \frac{2}{3}$$, where k_n being a positive number,

$$\frac{1-p_n}{(\frac{2}{5})^{k_n}} = \frac{3}{2} ,$$

$$1-p_n = \frac{3}{2} \times (\frac{2}{5})^{k_n} ,$$

Given $k_n = 1, 2, 3, \ldots,$

$$p_n = \frac{2}{5} , \frac{19}{25} , \frac{113}{125} , \frac{601}{625} , \ldots,$$

Yet bound by the criterion that the numerator of P must be an odd number, this concludes that only $\frac{2}{5}$ meets the requirement.

This indicates that only an optimal k_n value derived can the numerator of p_n and p contain the identical primary factor as further validated below:

(4) Given $p = \frac{ma_0}{mb_0+c_0}$,all probable values of p_n can be deduced from a known p_{n+1} , which is set to $\frac{ma_{n+1}}{mb_{n+1}+c_{n+1}}$,

For $p_{n+1}= \frac{p^{k_{n+1}}}{1-p_n}$,

$1 - p_n = \frac{p^{k_{n+1}}}{p_{n+1}}$,

This will conclude $p_n = 1 - \frac{p^{k_{n+1}}}{p_{n+1}}$,

meaning $p_n = 1 - \frac{(\frac{ma_0}{mb_0+c_0})^{k_{n+1}}}{\frac{ma_{n+1}}{mb_{n+1}+c_{n+1}}}$

$$= 1 - \frac{mb_{n+1}+c_{n+1}}{ma_{n+1}} \times \frac{(ma_0)^{k_{n+1}}}{(mb_0+c_0)^{k_{n+1}}}$$

Given the numerator of p_n must be a multiplication of m and its denominator not; therefore,

$$1 - p_n = 1 - \frac{ma_{n+1}}{mb_{n+1}+c_{n+1}} = \frac{m(b_{n+1}-a_{n+1})+c_{n+1}}{mb_{n+1}+c_{n+1}}$$

$$= \frac{mb_{n+1}+c_{n+1}}{ma_{n+1}} \times \frac{(ma_0)^{k_{n+1}}}{(mb_0+c_0)^{k_{n+1}}}$$

Where only neither the simplified numerator nor the denominator are multiplications of m, and neither $mb_{n+1}+c_{n+1}$ and $(mb_0+c_0)^{n_{k+1}}$ are multiplications of m can they cancel each other after simplification being that the number of m factors under $(ma_0)^{k_{n+1}}$

and ma_{n+1} must be identical. Suppose a_{n+1} contains a "d" number of 2 factors, and a_0 contains an "e" number of 2 factors, and that k_n must equal to $\frac{d+1}{e+1}$, and p, p_{n+1}, k_{n+1} have all been established, this will allow the one and only p_n be concluded, meaning that only one specific p_n value can be used to produce a specific p_{n+1} .

(5) Assuming two numbers amongst p_0, p_1, p_2,… have been repeated and the first two repeated numbers being p_a and p_b (b>a) , this can be concluded from the preceding (2) to conclude that and should be identical, a fact that contradicts with the initial hypothesis assuming that p_a and p_b will be the first to repeat to rule out all probabilities that any two numbers will recur.

2. Suppose under the scenario where p being a rational number and its numerator being 1:

Of all tangible fractions that are greater than 0.25, only $\frac{1}{3}$ and $\frac{1}{2}$ produce numerator of 1.

(1) $p = \frac{1}{2}$, $p_1 = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$, thus $\frac{1}{2} \in I_1$ 。

(2) $p = \frac{1}{3}$, $p_1 = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$, $p_2 = \dfrac{\frac{1}{3}}{1-\frac{1}{2}} = \frac{2}{3}$, $p_3 = \dfrac{\frac{1}{3}}{1-\frac{2}{3}} = 1$,

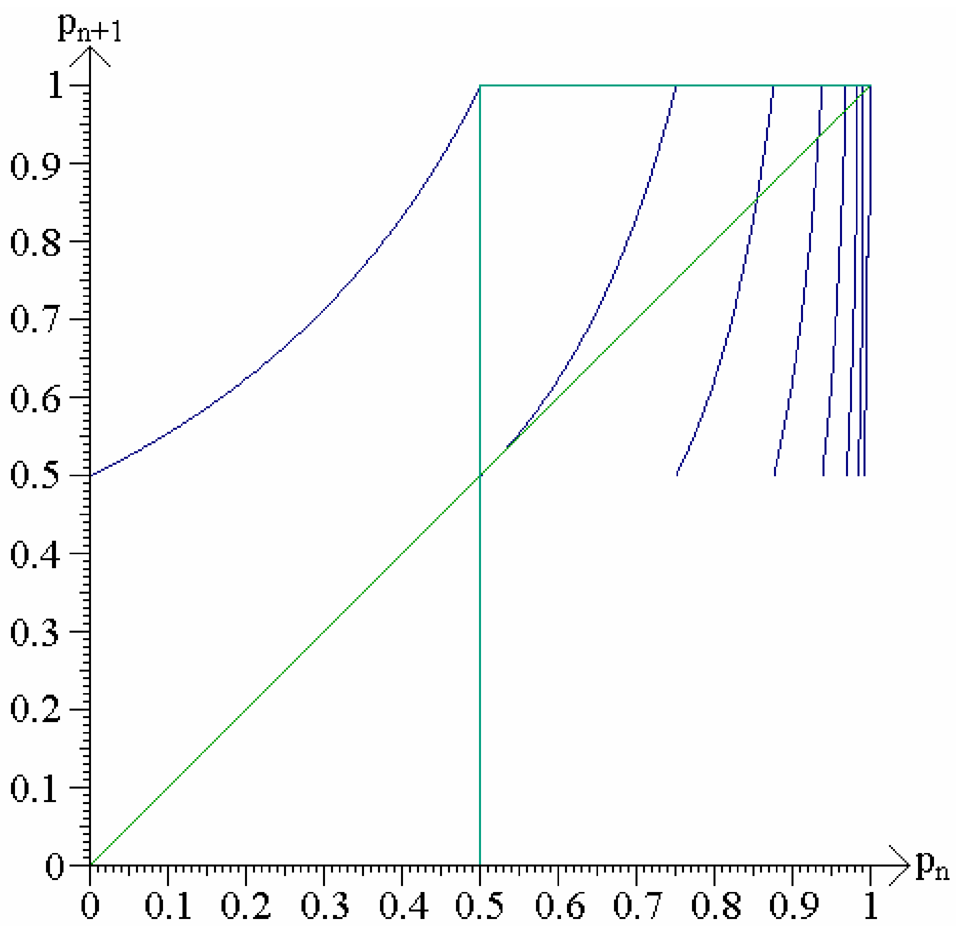
thus $\frac{1}{3} \in I_3$ 。

(III) Given $0.25 < p \leqq 0.5$, some of the p values, upon undergoing conditional computations, can be concluded to 1:

1. For example taking to the foregoing $\frac{1}{2}$ and $\frac{1}{3}$, which can be graphed as indicated below,

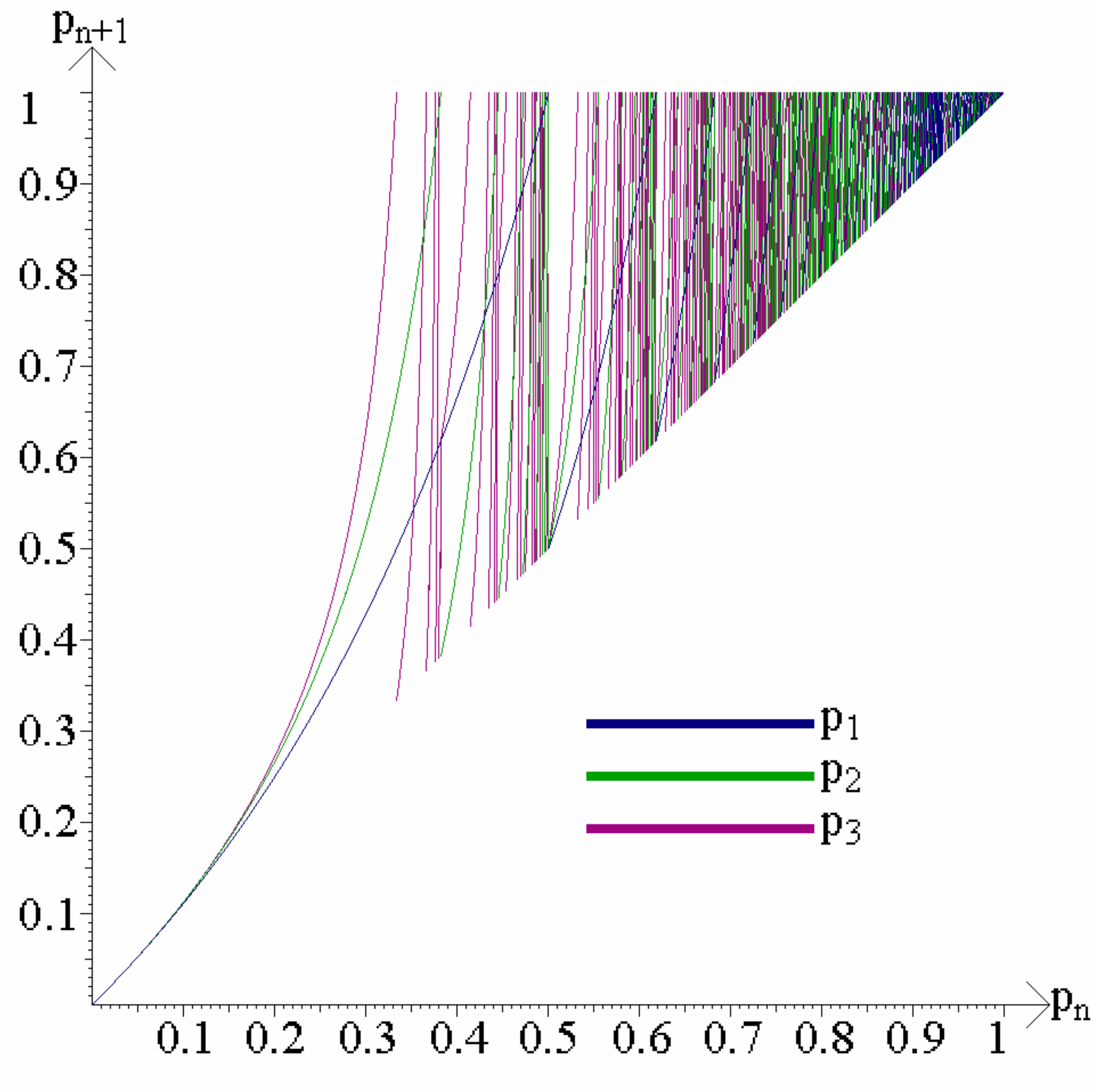
Where p = $\frac{1}{2}$

Graph 2



Where p = $\frac{1}{3}$

Graph 3



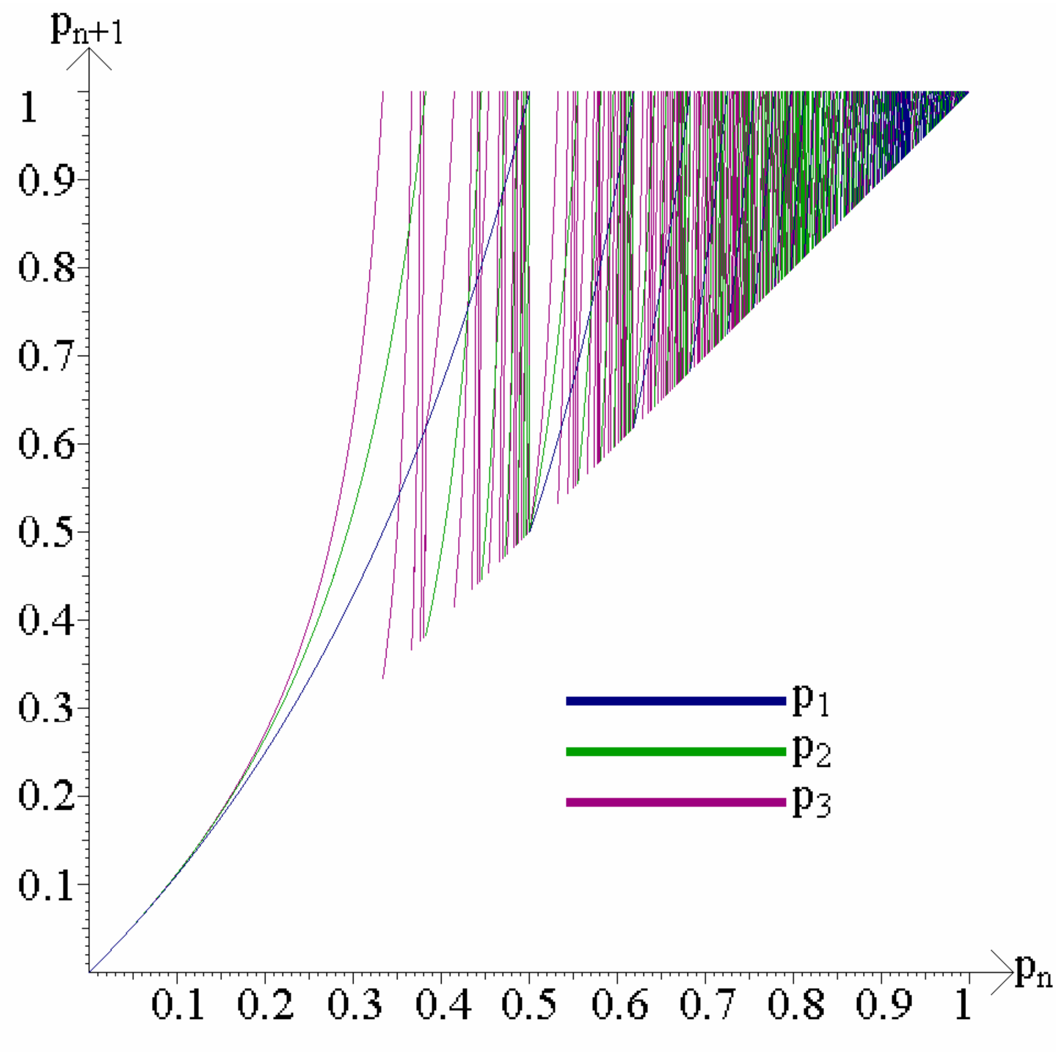
2. Validation is then performed to locate what other figures would produce 1 as they are plotted in:

(1) Validation method:

An n value, where (n=1,2,3,…) is chosen to condition $p_n=1$, whereby p is gradually increased from 0.25 as p_n value also increases, and keep adjusting the p until it reaches $p_n=1$ to locate $p \in I_n$ 。

Graph 4 below has the p value plotted in with p_1, p_2 and p_3 concluded, where it indicates that as p increases from 0, p_n will also increase with it; however, when p_n reaches 1 and p continues to increase, p_n will diminish to become closer to p, a method that deploys examining the value of p to condition p_n equals to 1.

Graph 4



(2) Experiment scenario where $p_7=1$：

① By increasing P from 0.25 to stop at $p_7 = 1$, the value of p_7 will increase alongside P before reaching one, but it will require diminished the p value as p_7 becomes closer to that of p when exceeding 1.

Table 4

p	p_7	Over or underrated P value
0.25	0.44444 44444 44444	Underrated
0.26	0.52087 17671 73351	Underrated
0.27	0.68191 59224 35899	Underrated
0.275	0.89008 27805 49313	Underrated
0.28	0.48331 38401 55949	Overrated
0.276	0.96475 80974 21005	Underrated
0.277	0.29445 79839 85269	Overrated
0.2765	0.27934 10174 99825	Overrated
0.2764	0.27657 81470 72783	Overrated
0.2763	0.99128 91275 48333	Underrated
0.27635	0.99593 32506 04733	Underrated
0.27637	0.99780 96069 24777	Underrated
0.27638	0.99875 18499 75209	Underrated
0.27639	0.99969 68190 50126	Underrated

② This indicates that given $0.27639 < p < 0.27640$ if the experiment continues, it can be established that p = 0.27639 32022 50021…。

A scenario by repeatedly replacing the p value seven times shows,

Table 5

p_0	0.27639 32022 50021
p_1	0.38196 60112 50105
p_2	0.44721 35954 99958
p_3	0.50000 00000 00000
p_4	0.55278 64045 00042
p_5	0.61803 39887 49895
p_6	0.72360 67977 49978
p_7	0.99999 99999 99998

③ A phenomenon observed from the above table 5:
The last p_7 that has not fallen on 1 has been the insufficient number of decimal points taken.

Also when p_3 equals to $\frac{1}{2}$, p_5 should be $\frac{\sqrt{5}-1}{2}$, and p_1 the value of 1 - p_6 , meaning $\frac{3-\sqrt{5}}{2}$.

④ Calculating p value using $p_1 = \frac{3-\sqrt{5}}{2}$:

Suppose $\frac{p}{1-p} = p_1 = \frac{3-\sqrt{5}}{2}$,

Then $p=\frac{3-\sqrt{5}}{2}(1-p)$, $p + \frac{3-\sqrt{5}}{2} p = \frac{3-\sqrt{5}}{2}$,

$\frac{5-\sqrt{5}}{2} p = \frac{3-\sqrt{5}}{2}$, $p = \frac{3-\sqrt{5}}{5-\sqrt{5}} = \frac{5-\sqrt{5}}{10}$ 。

⑤ The p value is then plotted in seven times for validation：

$p_1 = \frac{3-\sqrt{5}}{2}$, $p_2 = \frac{\sqrt{5}}{5}$, $p_3 = \frac{1}{2}$, $p_4 = \frac{5-\sqrt{5}}{5}$,

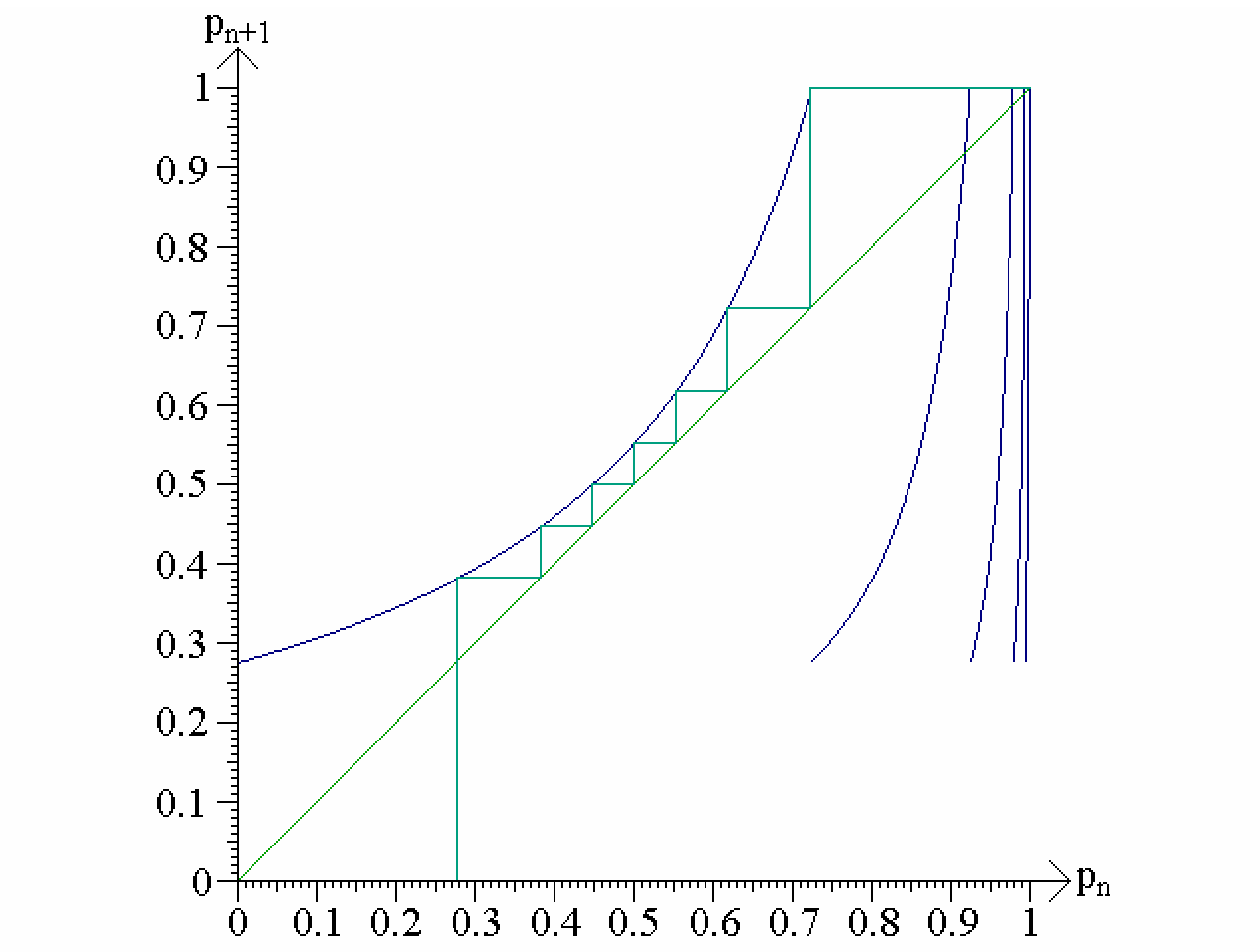
$p_5 = \frac{\sqrt{5}-1}{2}$, $p_6 = \frac{5+\sqrt{5}}{10}$, $p_7 = 1$ 。

Where p_7 has indeed been validated as 1，hence $\frac{5-\sqrt{5}}{10} \in I_7$ 。

It has also been observed when p = 0.2, the value of p_∞，$\frac{1-\sqrt{0.2}}{2} = \frac{5-\sqrt{5}}{10}$, will also fall on the this value.

Graph 5 below depicts how $\frac{5-\sqrt{5}}{10}$ is plotted in seven times to achieve the result of one:

Graph 5



3. Scenarios where $p_5 = 1$ and $p_9 = 1$:
   (1) Two p values concluded from the experiment, which are plotted in five and nine times respectively, would produce

results that are in close proximity to 1:

Table 6

p_0	0.29289 32188 13452	0.26794 91924 31122
p_1	0.41421 35623 73094	0.36602 54037 84437
p_2	0.49999 99999 99998	0.42264 97308 10372
p_3	0.58578 64376 26902	0.46410 16151 37752
p_4	0.70710 67811 86451	0.49999 99999 99996
p_5	0.99999 99999 99977	0.53589 83848 62240
p_6		0.57735 02691 89617
p_7		0.63397 45962 15546
p_8		0.73205 08075 68845
p_9		0.99999 99999 99877

(2) The process in which $p - I_5$ shown to left is plotted can be observed that,

$$p_1 \doteqdot \sqrt{2}-1,\ p_2 \doteqdot \frac{1}{2},\ p_4 \doteqdot \frac{\sqrt{2}}{2},$$

$$\because \frac{p}{1-p_1} = p_2, \quad \therefore \frac{p}{2-\sqrt{2}} = \frac{1}{2},$$

$$p = \frac{2-\sqrt{2}}{2} \circ$$

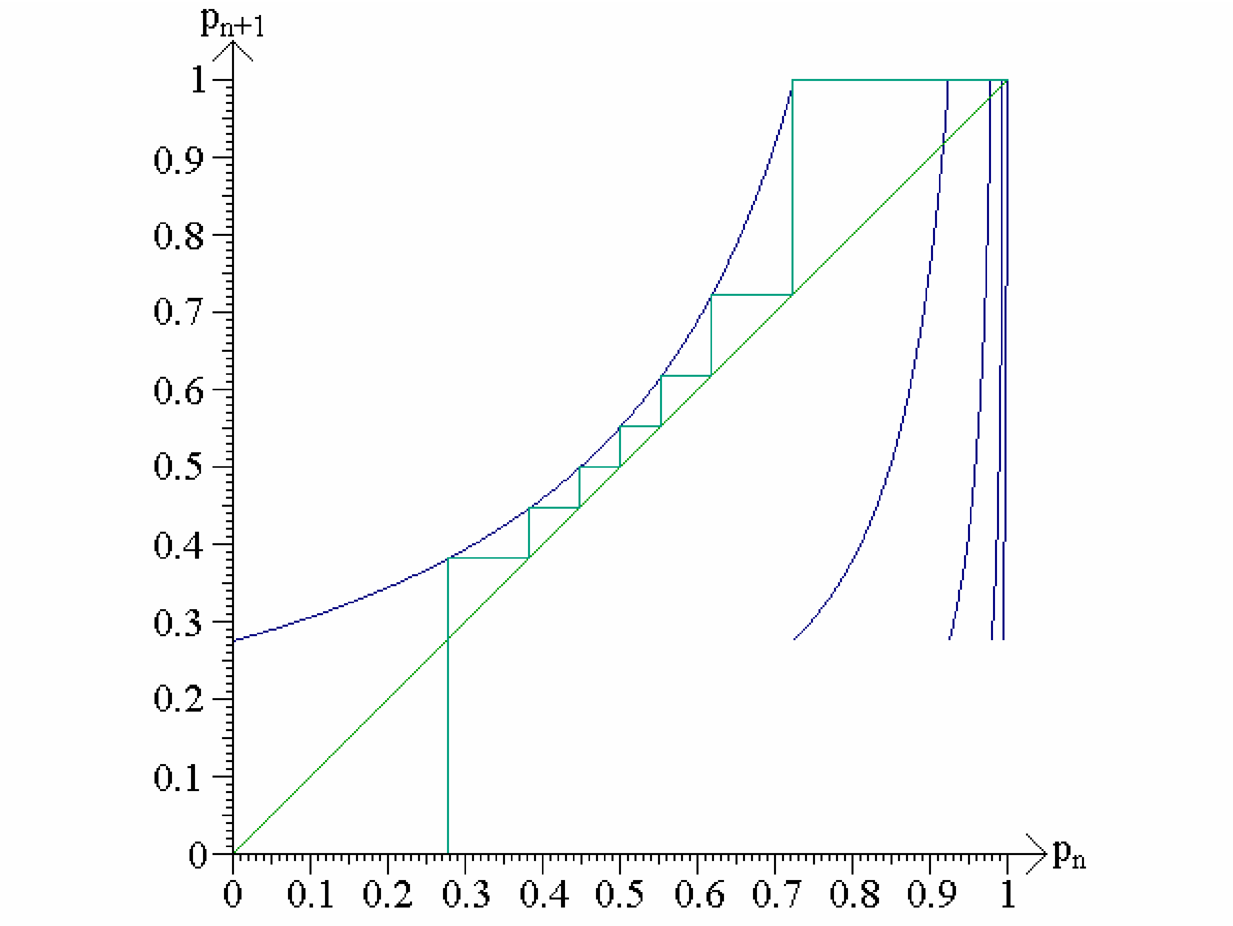
Scenario where $p = \frac{2-\sqrt{2}}{2}$ is plotted in:

$$p_1 = \sqrt{2}-1,\quad p_2 = \frac{1}{2},\quad p_3 = 2-\sqrt{2},\quad p_4 = \frac{\sqrt{2}}{2},\quad p_5=1$$

which validates $\frac{2-\sqrt{2}}{2} \in I_5$ ∘

Graph 6 below depicts a scenario when $p = \frac{2-\sqrt{2}}{2}$ is plotted in:

Graph 6



(3) The process in which p value - I_9 is plotted in as shown to the right column of Table 6 on page 16, which belong to p.16 can be observed that：

$p_3 \fallingdotseq 2\sqrt{3}-3$ ， $p_4 \fallingdotseq \frac{1}{2}$ ， $p_8 \fallingdotseq \sqrt{3}-1$ ，

$\because \frac{p}{1-p_3} = p_4$ ，

$\therefore \frac{p}{4-2\sqrt{3}} = \frac{1}{2}$ ，

$p = 2-\sqrt{3}$

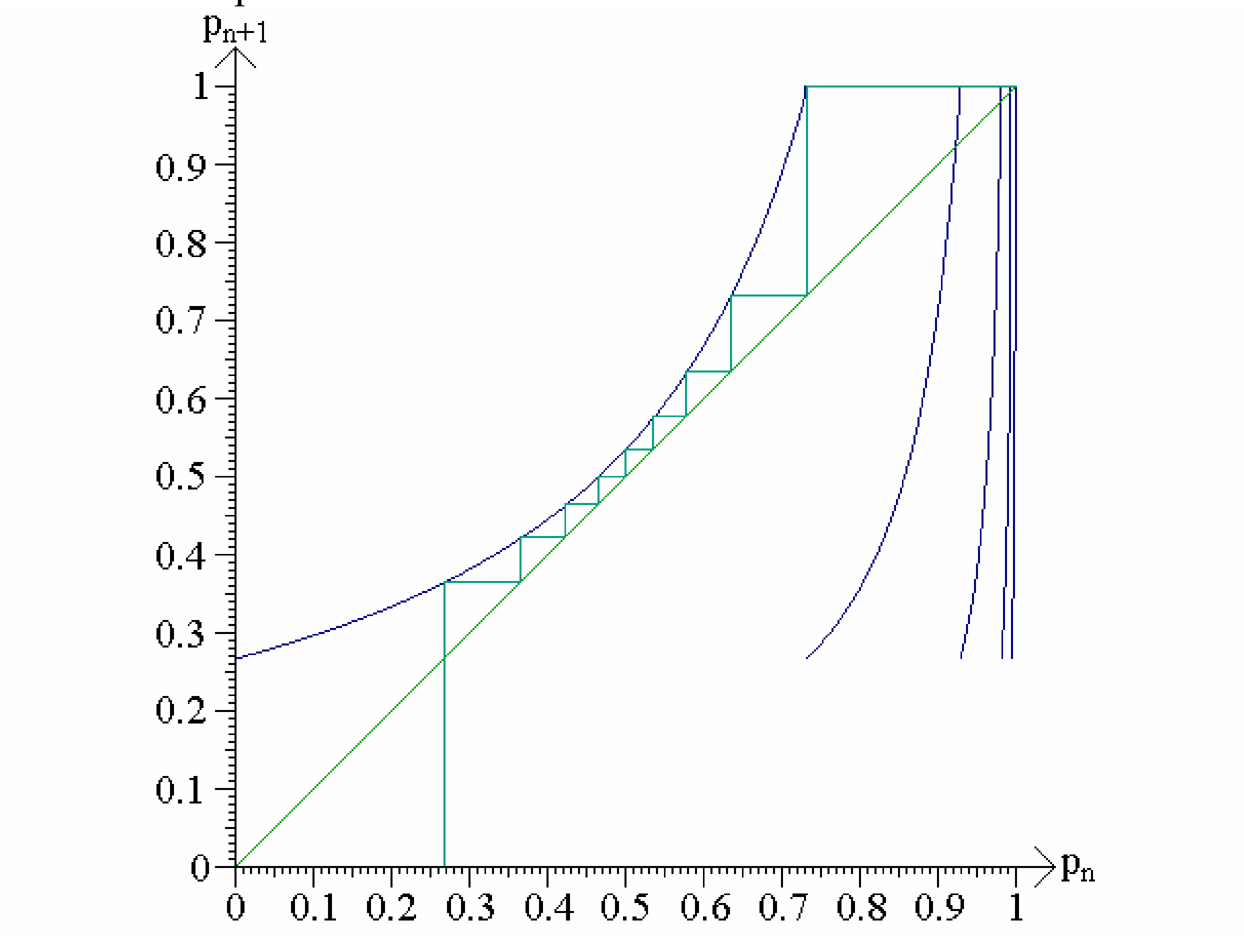
A scenario that depicts p being replaced with $2-\sqrt{3}$,

$p_1 = \frac{\sqrt{3}-1}{2}$ ， $p_2 = \frac{3-\sqrt{3}}{3}$ ， $p_3 = 2\sqrt{3}-3$ ， $p_4 = \frac{1}{2}$ ，

$p_5 = 4\text{-}2\sqrt{3}$, $p_6 = \dfrac{\sqrt{3}}{3}$, $p_7 = \dfrac{3\text{-}\sqrt{3}}{2}$, $p_8 = \sqrt{3}\text{-}1$, $p_9 =$ 1。

Which validates $2\text{-}\sqrt{3} \in I_9$。

Graph 7



(IV) While it may be unfeasible to validate through experiment whether I_n consists of all tangible elements being that the number of a set, I_n, can be infinite. In light of which, the premise of $k_n = 1$ has been taken in an attempt to ascertain a p value that produces $p_n = 1$:

1. Assuming $k_n = 1$, p_0, p_1, p_2,… are expressed in p:

   $p_0 = p$,

   $p_1 = \dfrac{p}{1\text{-}p_0} = \dfrac{p}{1\text{-}p}$,

$p_2 = \dfrac{p}{1-p_1} = \dfrac{p-p^2}{1-2p}$,

$p_3 = \dfrac{p}{1-p_2} = \dfrac{p-2p^2}{1-3p+p^2}$,

$p_4 = \dfrac{p}{1-p_3} = \dfrac{p-3p^2+p^3}{1-4p+3p^2}$,

$p_5 = \dfrac{p}{1-p_4} = \dfrac{p-4p^2+3p^3}{1-5p+6p^2-p^3}$,

$p_6 = \dfrac{p}{1-p_5} = \dfrac{p-5p^2+6p^3-p^4}{1-6p+10p^2-4p^3}$ 。

2. Assuming $p_n = \dfrac{a(p)}{b(p)}$ ，then $p_{n+1} = \dfrac{p}{1-p_n} = \dfrac{p\,b(p)}{b(p)-a(p)}$ ，in which the numerator being the denominator of p_n times p, and the denominator's a(p) being the denominator of p_{n-1} times p, the denominator of p_{n+1} will equal to the denominator of p_n minus the denominator of p_{n-1} times p.

   Thus for any random natural number n, its p_n can be expressed as $\dfrac{p\,a_{n-1}(p)}{a_n(p)}$ ，in which $a_0(p) = 1$, $a_1(p) = 1 - p$, $a_{n+1}(p) = a_n(p) - p\,a_{n-1}(p)$，as indicated in the table blow：

   Table 7

	Constant number	P	p^2	p^3	p^4	p^5
$a_0(p)$	1					
$a_1(p)$	1	-1				
$a_2(p)$	1	-2				
$a_3(p)$	1	-3	+1			
$a_4(p)$	1	-4	+3			
$a_5(p)$	1	-5	+6	-1		
$a_6(p)$	1	-6	+10	-4		
$a_7(p)$	1	-7	+15	-10	+1	
$a_8(p)$	1	-8	+21	-20	+5	
$a_9(p)$	1	-9	+28	-35	+15	-1
$a_{10}(p)$	1	-10	+36	-56	+35	-6

3. All polynomials above have been derived from the Fibonacci polynomial:

(1)The Fibonacci polynomial's inductive equation is defined as $f_{n+1}(x) = x\ f_n(x) + f_{n-1}(x)$, where $a_{n+1}(p) = a_n(p) - p\ a_{n-1}(p)$ times p being a_{n-1} instead of a_n , hence the coefficients produced will be the opposite to the sequence of Fibonacci polynomial, meaning the sequential ranking will be exactly the opposite. While the "Minus" that appeared the computation of $a_n(p)$ and $a_{n-1}(p)$ would result in a conditional value to $a_n(p)$, thus producing constants within to cover both positive and negative integrals.

(2) While the varied integrals in the Fibonacci polynomial, where $f_1 = 1$, $f_2 = x$, would produce polynomials with a deviation of 2 amongst every two adjacent polynomials, meaning there are omission in between every two, there will not be any omission with $a_n(p)$.

4. To conclude a solution, where $p_n = 1$：

(1) To condition $p_n = \dfrac{p\ a_{n-1}(p)}{a_n(p)} = 1$，it is necessary that $p\ a_{n-1}(p) = a_n(p)$，in which the conversion allows $a_n(p) - p\ a_{n-1}(p) = 0$ be substituted with $a_{n+1}(p) = 0$。

(2) The solution for Fibonacci polynomial using $f_n(x) = 0$ will conclude

$$x \in \{2i \cos \frac{j\pi}{n}, j = 1, 2, \ldots, n-1\}$$。

(3) Suppose the x^2 in the Fibonacci polynomials are substituted with p, where any odd number is simplified with an x, a polynomial, $g_n(p)$, can therefore be concluded, and the solution to polynomials $g_n(p) = 0$ wil always be the square root to the solution where $g_n(p) = 0$：

$$p \in \{-4 \cos^2 \frac{j\pi}{n}, j = 1, 2, \ldots, n-1\}$$。

Moreover, for $\cos\frac{j\pi}{n} = \cos\frac{(n-j)\pi}{n}$ ， one half of the solutions will be repetitive, hence the dissimilar solutions would be:

$p \in \{-4\cos^2\frac{j\pi}{n}, j = 1, 2, \ldots, [\frac{n}{2}]\}$

For example：$f_6(x) = x^5 + 4x^3 + 3x$, then $g_6(p) = p^2 + 4p + 3$, The solution where $g_6(p) = 0$ will conclude -3, -1, 0。

(4) By converting all elements under $g_n(p)$ to fall within the positive and negative integral and a leading coefficient as positive will conclude $h_n(p)$, and the solution to $h_n(p)$ will be the exact reserve number of that for $g_n(p)$:

$p \in \{4\cos^2\frac{j\pi}{n}, j = 1, 2, \ldots, [\frac{n}{2}]\}$。

For example: The solution to $h_6(p) = p^2 - 4p + 3$, $h_6(p) = 0$ will be 3, 1, 0。

(5) Finally, all $h_n(p)$ coefficients are sequentially reserved to arrive at the initial $a_{n-2}(p)$ for $h_2(p)$ corresponds to $a_0(p)$, and the solution to where $a_{n-2}(p) = 0$ will be the reverse number of the solution for $h_n(p)$:

$p \in \{\frac{1}{4\cos^2\frac{j\pi}{n}}, j = 1, 2, \ldots, [\frac{n-1}{2}]\}$。Where $j = \frac{n}{2}$ emerges as incoherent since $\cos\frac{\pi}{2} = 0$，thus rendering $p = \frac{1}{0}$ meaningless.

For example：The solution to $a_4(p) = 3p^2 - 4p + 1$，$a_4(p) = 0$ will be $\frac{1}{3}$, 1。

(6) And the solution to $p_n = 1$ will be that of the solution of $a_{n+1}(p) = 0$, meaning,

$$p \in \{\frac{1}{4\cos^2 \frac{j\pi}{n+3}}, j = 1, 2, \ldots, [\frac{n+2}{2}]\}$$ 。

5. Validation is then made to certain that only $\frac{1}{4\cos^2 \frac{\pi}{n+3}}$ of all solutions found where $a_{n+1} = 0$, meaning j = 1, will produce the optimal result:

   (1) The solution for where $p_{n+1} = 0$ must be smaller than the solution where $p_n = 0$ for it is necessary to reduce the value of p_n in order to delay the occurrence of where $p_n = 1$ since p_n does increase alongside increasing p_0 before reaching 1.

   (2) When the mathematical recursive method is applied:
   Given n = 1, only one solution where $a_2(p) = 0$ can be obtained, which is $\frac{1}{2}$, and j = 1is to withstand.
   Given where n = k is to withstand, this indicates a rational solution will be $\frac{1}{4\cos^2 \frac{\pi}{k+3}}$,
   Then when n = k + 1 and j being two or higher, since $\frac{2\pi}{k+4} > \frac{\pi}{k+3}$,
   $\cos \frac{2\pi}{k+4} < \cos \frac{\pi}{k+3}$,
   $\frac{1}{4\cos^2 \frac{2\pi}{k+4}} > \frac{1}{4\cos^2 \frac{\pi}{k+3}}$,
   would only arise to contradict would only allow j be set at 1.

(3) At which， $\frac{1}{4\cos^2\frac{\pi}{n+3}} \in I_n$ 。It is feasible that at least one p value that befores to I_n can be located in any random set of natural number n.

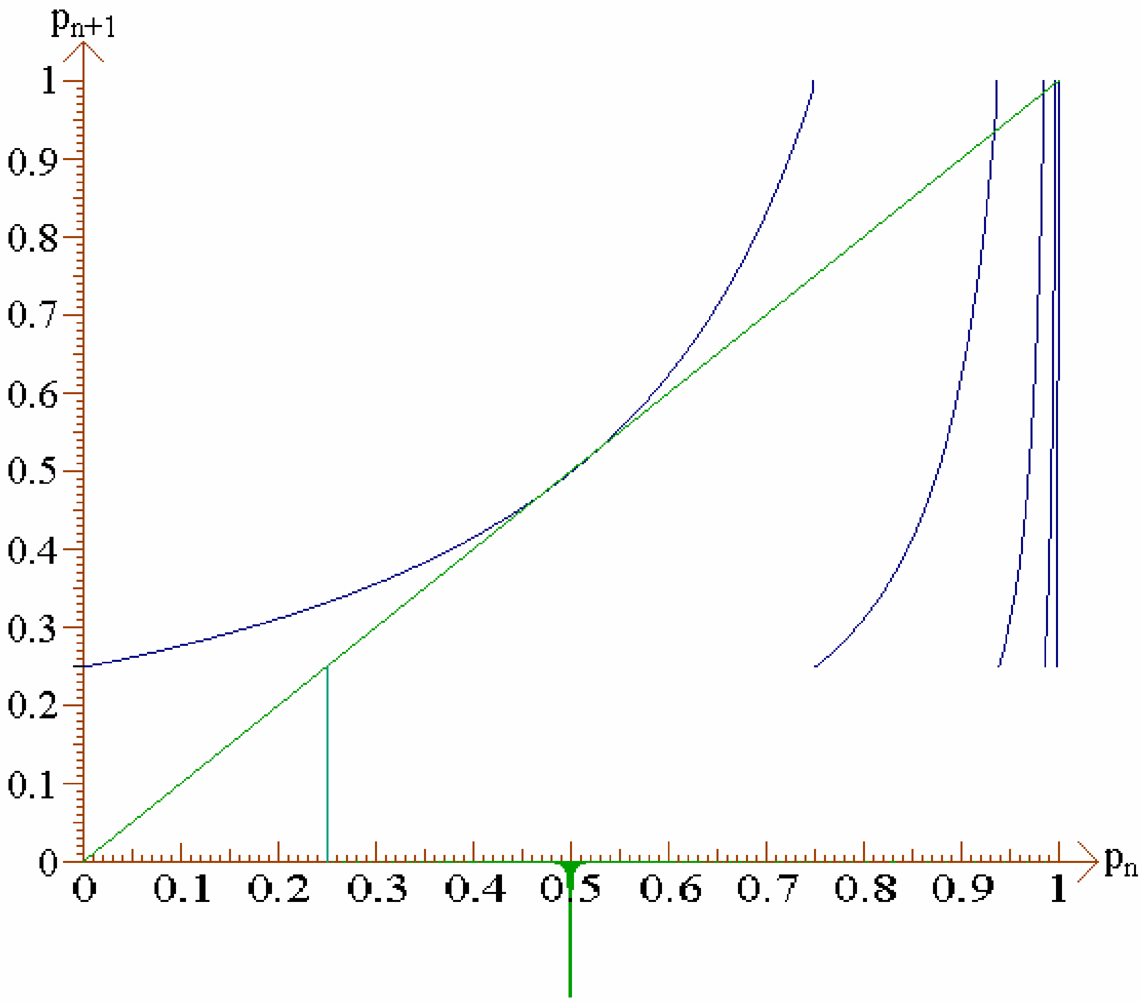
(V) Observation on findings concluded from replacing certain p values:

1. The p values close to 0.25, which are repeatedly plotted in do not produce a consistent distribution of p_n values but rather occurred around the proximity of 0.5:

(1) Distribution of p_n when p=0.250001：

In Fig. 8 below where the deep blue curve being where $p_{n+1} = p_n$, the blue diagonal line being, in between the light blue parameter are the axial position of p, and the square at the center has had 500 grids divided to cover from 0 to 1, this has been made to depict the distribution of p_n in various segments when p is plotted in 25,000 times. The grids below are subdivision of each segment in 200 shares, in which a corresponding black dot is placed when a p_n value falls on a certain point within a segment.

Graph 8



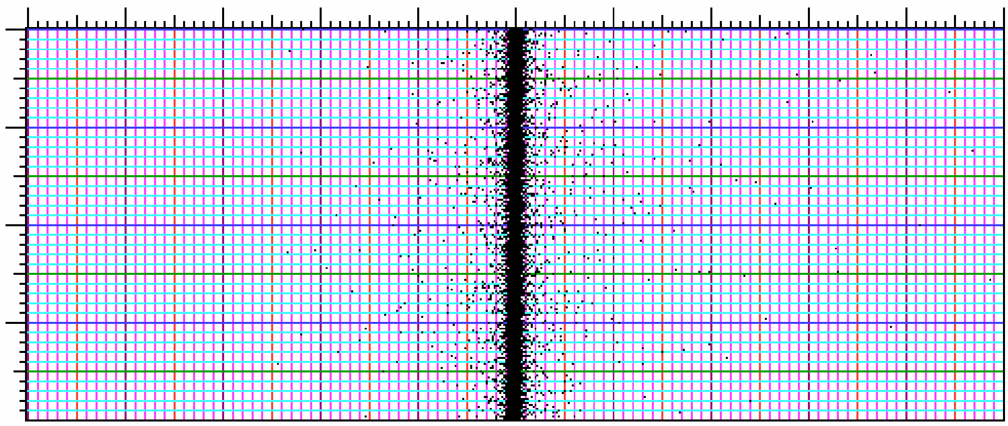


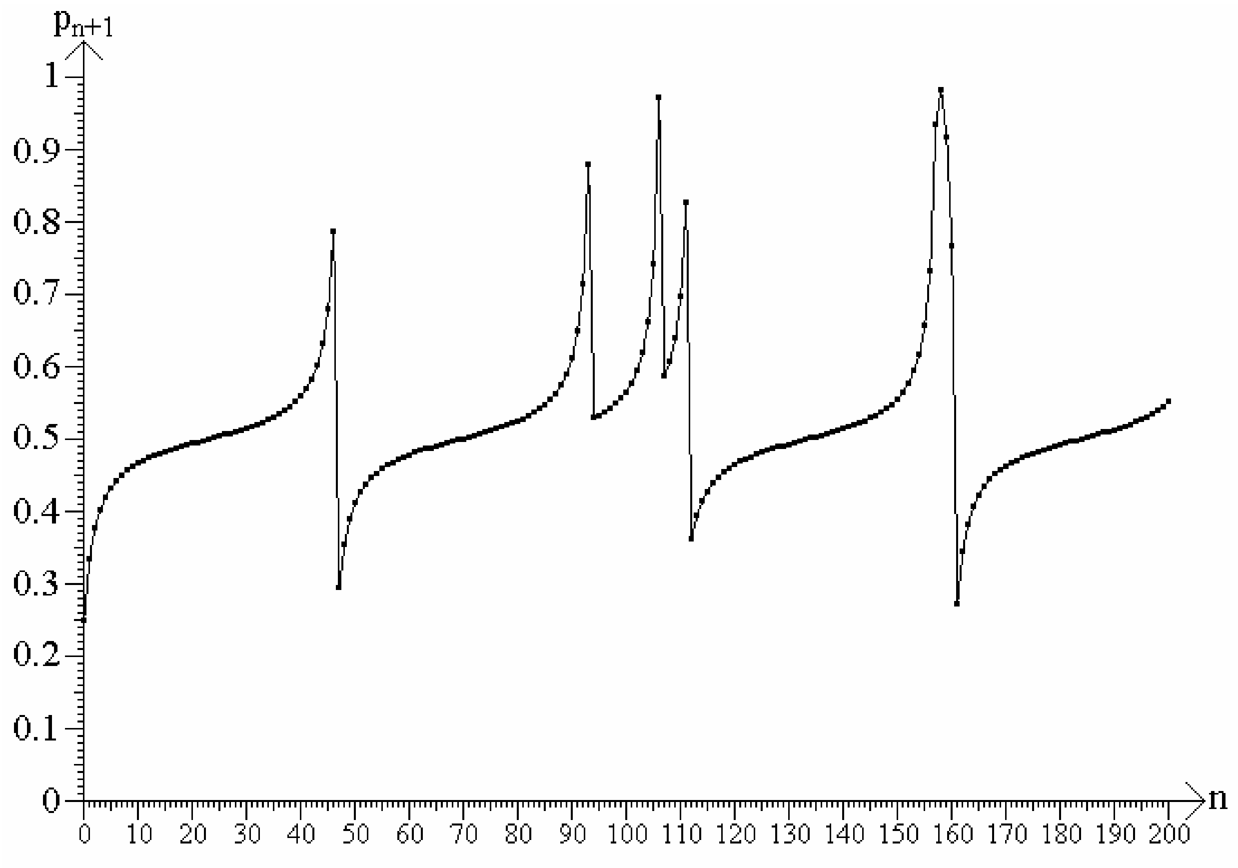
Table 8 below represents the numeral figures shown in the vertical grid in the center portion of Graph 8, where it consists of 20 rows and 20 columns, in an up-down and left-right configuration, where numbers that fall below P to result in no p_n value within the parameter are shown with a strike.

Table 8

P0= .250001

-	-	-	-	-	0	0	0	2	7	8856	8	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	0	2	6	1752	6	2	0	0	0	1	0	0	0
-	-	-	-	-	0	0	1	3	8	637	4	3	0	0	0	1	0	0	0
-	-	-	-	-	1	1	0	0	6	327	7	2	1	0	1	0	0	0	0
-	-	-	-	-	0	0	0	1	9	196	5	0	0	0	0	0	0	0	0
-	-	-	-	-	0	0	1	1	10	133	3	3	0	1	0	0	0	0	0
-	-	-	-	-	0	0	0	3	11	94	4	0	1	1	0	0	0	0	0
-	-	-	-	-	0	0	1	2	11	72	5	2	1	0	0	0	1	1	0
-	-	-	-	-	1	1	2	2	15	55	5	1	0	1	2	0	0	0	0
-	-	-	-	-	1	1	0	2	15	44	2	1	1	0	0	0	1	0	1
-	-	-	-	-	0	0	1	3	18	37	2	0	0	0	0	0	0	0	0
-	-	-	-	-	0	0	1	1	21	29	5	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	1	3	26	27	3	1	1	0	0	0	0	0	0
-	-	-	-	-	0	0	1	3	29	21	3	1	0	1	0	0	0	0	0
-	-	-	-	-	0	0	2	2	37	19	3	2	2	0	2	1	0	0	0
-	-	-	-	-	2	0	0	5	45	18	1	2	0	0	0	1	0	0	0
-	-	-	-	-	1	1	3	1	55	14	2	1	1	0	0	1	0	0	0
-	-	-	-	-	0	0	0	4	71	12	3	0	0	1	0	0	1	0	0
-	-	-	-	-	0	1	1	4	95	13	3	2	0	0	0	0	0	0	1
-	-	-	-	-	0	0	0	5	132	12	2	0	2	0	1	0	0	0	0
-	-	-	-	-	0	2	1	2	198	7	3	0	0	1	0	0	0	0	0
-	-	-	-	-	0	0	0	5	326	10	2	3	0	0	0	0	0	0	0
-	-	-	-	-	1	0	3	4	639	8	1	0	0	0	0	0	0	1	0
-	-	-	-	-	0	2	1	5	1682	8	1	0	0	1	0	0	0	0	0
-	-	-	-	-	0	1	1	5	8858	6	2	2	3	0	0	0	0	0	0

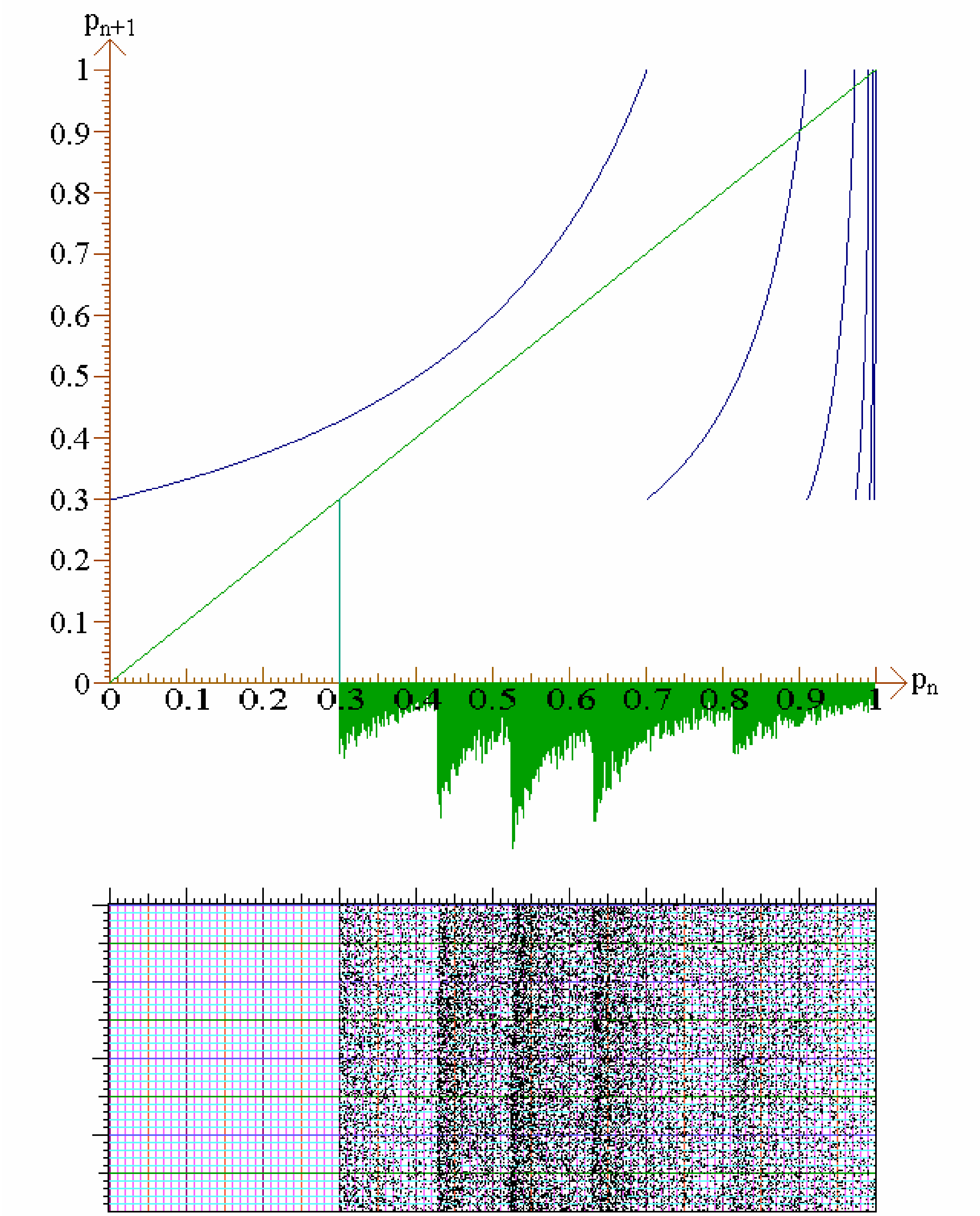
(2) Condition where p = 0.251 is plotted in 200 times and the corresponding p_n values are mapped into a curve:
Graph 9



Graph 9 above indicates that as p_n gradually diminishes from 0.25 as it approaches 0.5, its rate of increase tapers and picks up again as it approaches 1, and that thereafter has no specific pattern and can be anywhere between p and 1, depending on the previously p_n. What follows is the same repetitive process, and the slowest point of increase in p_n is still at around 0.5, being that as p approaches 0.25 and p_n approaches 0.5, $\frac{p}{1-p_n} \doteqdot \frac{0.25}{1-0.5} = 0.5$, regulating the p_{n+1} and p_n produced to contain little variation.

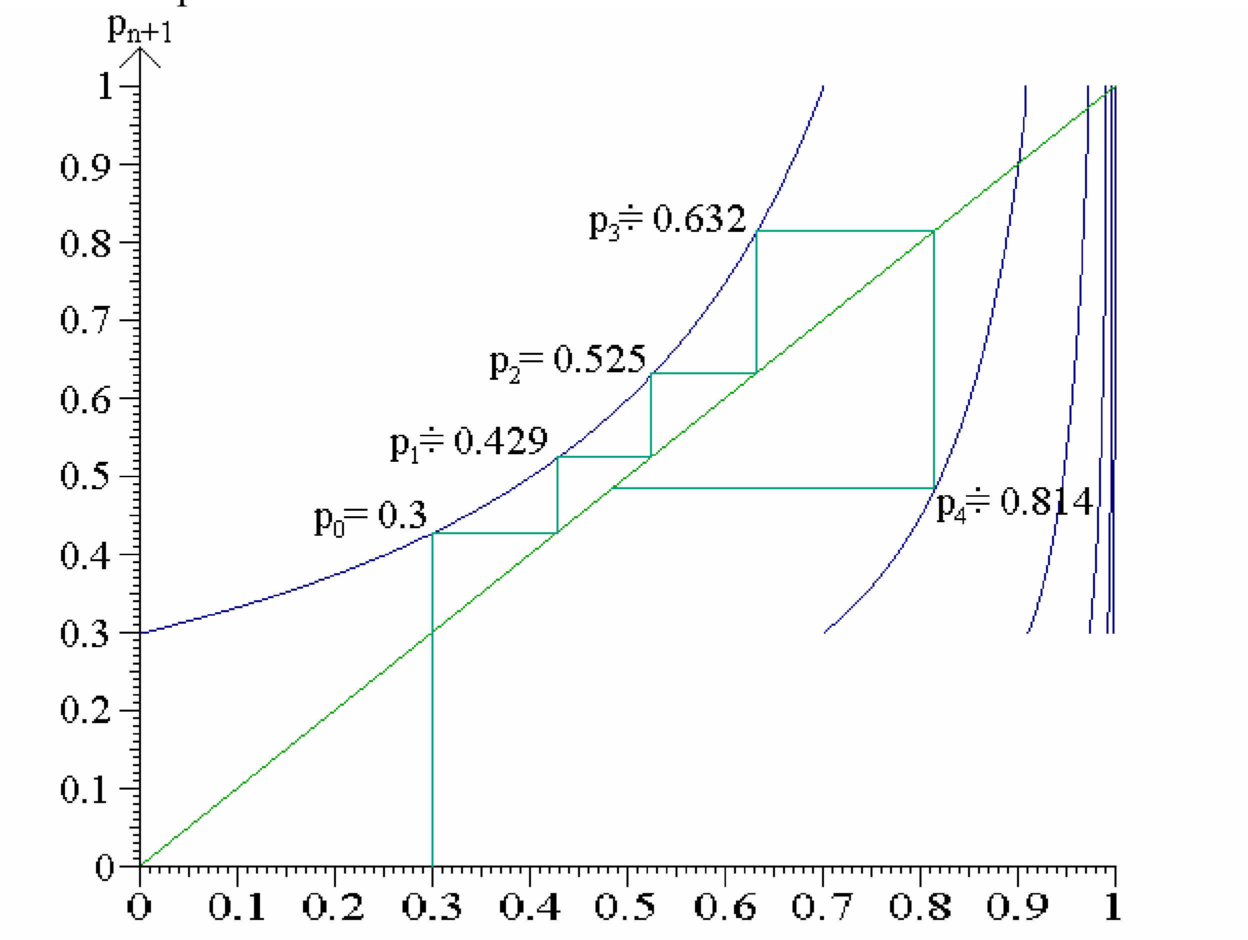
This also indicates that despite p_n values are largely concentrated at around 0.5 when p approaches 0.25 but the distribution has not been an even one.

2. Under the scenario where 0.25<p<0.5 ：
   assuming p = 0.3 for instance,
   A fluctuation curve using p=0.3 as shown in Fig. 8, page 21 indicates:
   Graph 10



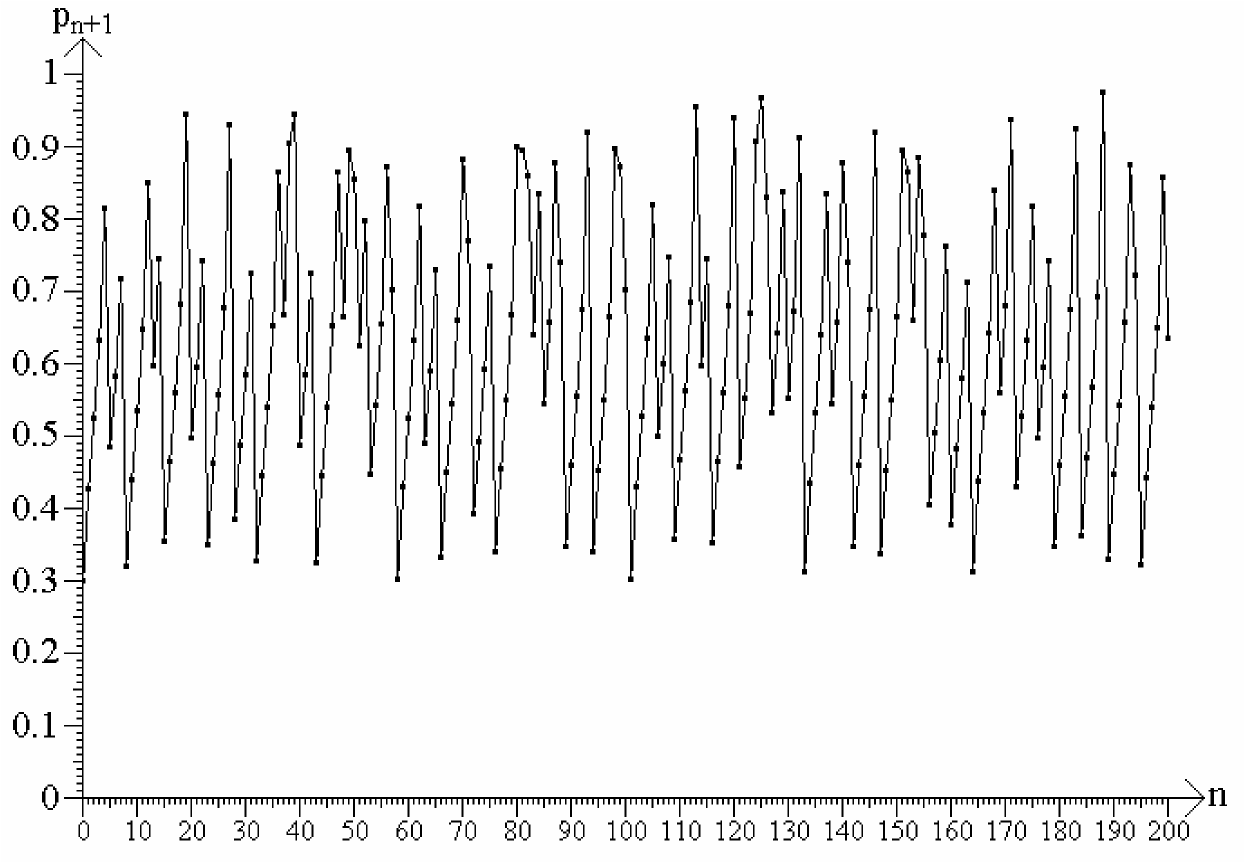
Graph 11 below depicts a flowchart mapping the positions of p_1 through p_5 five times each, where p = 0.3:

Graph 11



Graph 10 from the previous page shows p_n has a exceptionally high occurrence at roughly five spots of 0.300, 0.428, 0524, 0.642, .814, which are also roughly coincide with that appears between p_0 through p_4 where p=0.3 as depicted in Graph 11. A more in-depth observation is now made through replacing p 200 times and connecting all p_n values produced into a flowchart format.

Graph 12



Graph 12 above provides a few observations as follows：

(1) At the starting point where p=0.3, p_n has begun incrementally increasing and has already reached 0.8 or higher before replacing the fourth time, and it is evident, when compared with that in Graph 11 of the previous page, that the value of p_5 is tied to p_4 and k_5 when fifth replacement takes place, instead of continuing to increase.

(2) The process is then repeated – Where 2 when $p_n < 0.7$ and k_{n+1} is defined as 1 until a certain value arises to indicate $p_{n+m} > 0.7$, it can be concluded from Graph 10 that when $p_{n+m} > 0$, p_{n+m+1} can be any number between p and 1 but most probably a smaller value being that the cure appears milder when p_{n+1} in Graph 10 is relatively smaller, hence a greater area of p_n is being projected onto a relatively smaller p_{n+1} .

(3) Following the same method and upon repeated replacement, where a given replacement that reaches 0.7 or higher tends to produce a p value equals to 0.3 but less frequently on other figures. While continuing to plot in these figures four more times and the results will become closer to p_1, p_2, p_3 and p_4; however, upon the fifth replacement, the figures may not always come near p_5, which is because when $p_4 > 7$, it can be observed from Graph 10 that when p_n value increases, any minute difference can result in a greater disparity on p_{n+1}. Which explains why there is a concentration of p_n in the parameter encircled by p_0, p_1, p_2, p_3 and p_4 as observed in the center block in Graph 10.

(4) Yet noticeably is that once the sequence of replacement reaches 0.7, the ones that follow tend to become less orderly and are scattered between p and 1, and the smaller the numbers, the higher the frequency that they appear. While 0.3 being a rational number, no p_n generated will be repeated, thus the chaos phenomenon after repeatedly replacing p will make p_n values generated even more unpredictable.

The chaos phenomenon, taking to the characteristics of certain graphic fragments, has had such characteristics largely came from the accumulation of a particular repetition. And the ability to ascertain a set of rules with which to randomly execute the replacement would allow the grappling of certain configuration traits simply by repeating these rules as the scale gradually goes down. At which, it can be concluded that the more fragmented a configuration, the simpler the rules would become.

2. Scenario where $p > 0.5$：
The greater that p is, the distribution of p_n will become more evenly as can be demonstrated when $p = 0.8$,

The graphic below depicts a scenario on sequence of replacement where p=0.8, and the graph has been proportionally increased for easier reading, indicating only the range between 0.8 and 1 since $p_n \geqq 0.8$.

The upper half of the coordinate shown in the graph depicts the values of p_{n+1} when p_n is repeatedly replaced. In which, the area between 0.7 and 1, subdivided into 500 grids, indicates a square block, in which the number of p_n in each grid is tallied as p is repeatedly replaced 25,000 times. The square below is a subdivision of 200 from each grid, where a dot is placed wherever it corresponds to a certain p_n value.

Graph 13

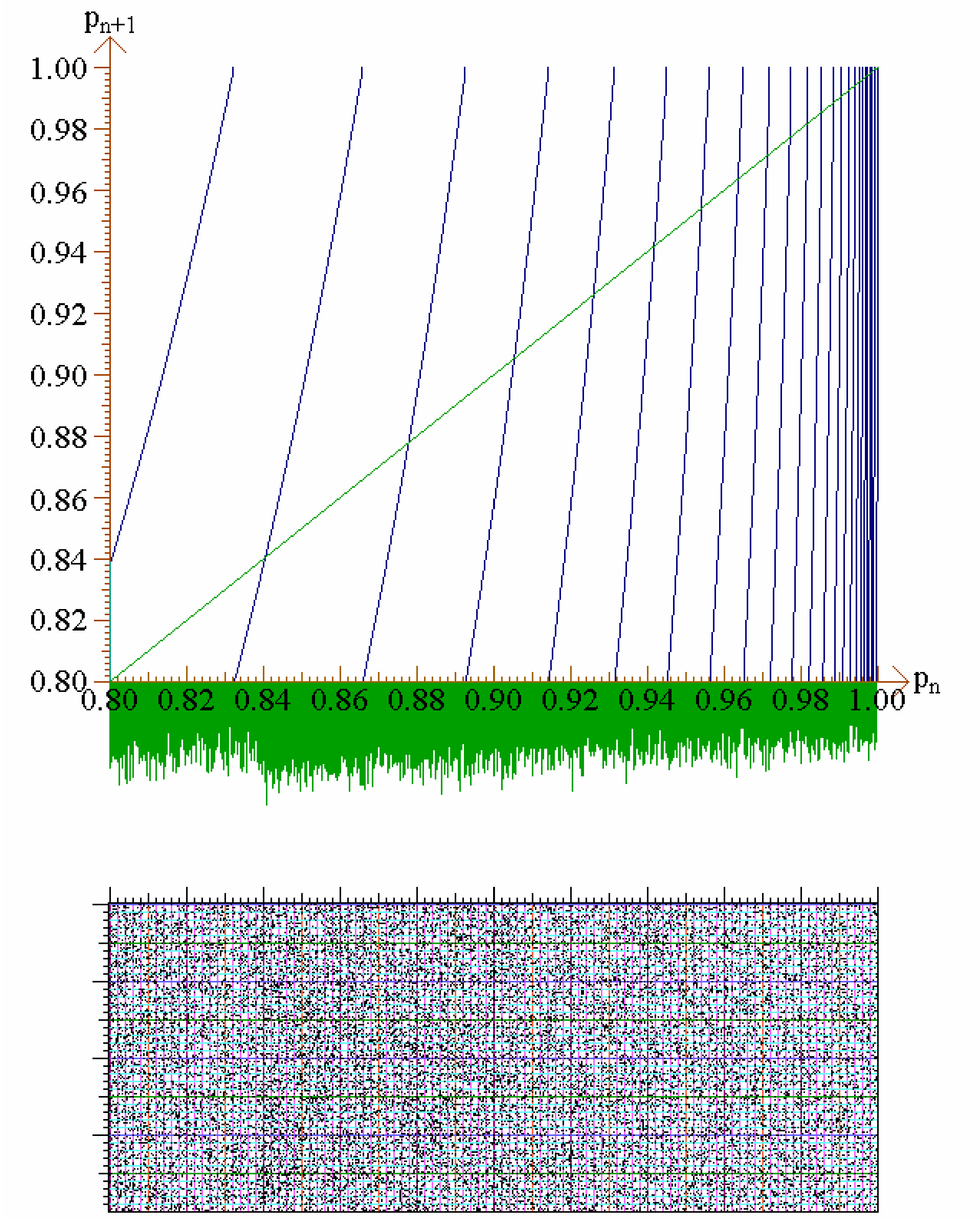

p_{n+1}
1.00
0.98
0.96
0.94
0.92
0.90
0.88
0.86
0.84
0.82
0.80
0.80 0.82 0.84 0.86 0.88 0.90 0.92 0.94 0.96 0.98 1.00
p_n

Table 9 below depicts the numbers that represent the 500 straight lines appeared in the center block in Graph 13, where it consists of 25 rows and 20 columns in an up-down, left-right configuration.

Table 9

P0= .8

56	45	47	45	51	56	59	42	57	49	61	61	49	47	47	52	50	54	53	41
52	64	48	54	65	52	48	54	54	50	43	43	71	47	34	36	52	41	38	45
48	50	44	61	80	65	52	55	62	64	58	53	44	44	53	43	56	58	40	39
50	48	53	29	57	68	50	54	52	71	45	52	46	56	50	38	35	37	35	52
50	50	41	47	60	51	55	46	67	44	49	45	49	44	47	55	56	57	42	38
52	64	41	49	58	61	61	47	50	40	70	43	42	37	40	48	46	46	46	45
67	46	52	46	69	61	71	59	50	62	63	60	56	47	54	51	39	42	52	46
49	52	33	38	66	71	59	56	50	60	54	50	61	49	40	50	45	50	41	28
56	54	38	47	56	57	61	47	56	54	59	43	61	45	31	36	62	42	44	28
47	52	54	39	63	54	55	55	62	61	62	43	41	52	50	48	42	41	47	42
63	63	36	56	72	60	49	53	53	67	54	57	41	48	50	50	40	38	37	37
66	55	44	58	64	56	59	52	44	53	46	47	36	66	38	41	30	39	51	29
41	47	40	55	58	54	56	54	63	60	63	57	45	51	46	43	30	36	50	45
64	33	58	39	57	64	67	52	57	52	59	54	48	51	49	45	47	43	36	45
47	48	59	59	54	53	67	64	52	46	60	57	45	41	57	41	50	52	36	37
62	46	43	32	58	56	67	52	48	59	42	44	48	30	53	38	48	45	44	41
38	58	58	46	67	58	70	62	73	47	51	42	45	57	51	50	36	46	42	39
41	49	56	51	61	64	48	60	50	44	46	51	40	54	49	52	56	42	37	38
42	59	52	43	59	60	66	47	60	56	45	54	42	43	44	47	41	32	44	38
53	55	44	51	50	60	51	56	52	59	42	41	44	47	49	44	52	39	47	51
49	55	42	38	61	60	46	47	62	50	55	51	43	50	39	50	39	43	50	40
51	42	33	52	55	59	68	56	42	57	48	43	56	45	55	39	40	48	39	39
47	37	46	57	65	62	56	66	54	51	50	45	51	54	39	44	36	38	40	33
51	44	50	53	59	64	53	56	58	47	47	47	54	53	42	54	44	31	50	44
61	54	57	60	71	61	49	56	51	46	43	56	42	45	41	33	45	38	43	39

What can be seen in Graph 13 is that the disparity amongst the distribution of p_n has become less ominous, which stems from that only the incrementally increasing figures would occur when the 8^{th} replacement first takes place, similar to what occurred on the third replacement, where the first four K_n can be taken to the smallest value of 1 and that only the fifth placement would K_n become some other numbers as the incrementally increasing rhythm stops. Whereas when p=0.8 and k_1=, p_1 =0.8388608, and when repeated up to n_2, the maximum sequence of replacement of 8 no longer applies but rather by 9, under which the incrementally increasing rhythm will immediately come to a halt as soon as $p_2 \fallingdotseq 0.8329303\ldots$, which has stemmed from the relatively larger gradient of $p_{n+1} = \dfrac{0.8^{k_{n+1}}}{1-p_n}$ that is approaching a straight line that no longer represents a curve, which means any p_{n+1} , m between 0.8 and 1, derived from p_n used in the next sequence of replacement process will produce a nearly even distribution, which can be used for simulating random numbers.

(VI) Applications of p_n values derived from replacing the p values, where $0.25 < p < 1$, using a random number generator:

1. Given that p_n generated when p approaches one would appear an indirect even distribution, linear function, where r = $a + \frac{b(p_n-p)}{1-p}$, can be used to convert the value of p to 1 into that of between a and b for generating a set of random numbers between given a and b. To weed out the first few numbers that may not have been as evenly normalized, the first few p_n values generated may be scrapped as demonstrated in Table 10 below, where random figures derived from replacing the P values, where p=0.9, have had the random figures begin at P_{10}.

$r_1 = 0.7644915\ldots$, $r_2 = 0.5658556\ldots$, $r_3 = 0.7643012\ldots$,
$r_4 = 0.5581310\ldots$, $r_5 = 0.5936046\ldots$, so on and so forth

Table 10

1).984770902183612	2).970568520843624	3).945003096738492	4).951594729868443
5).973061132485150	6).929198491218439	7).912562740599305	8).912270622656761
9).909232978649651	10).976449153866438	11).956585568823122	12).976430120724319
13).955813107120308	14).959360468966092	15).938791405241909	16).950025682387580
17).942509857518431	18).910325858376755	19).988349367969318	20).924865160613559
21).955479499995319	22).952171657343451	23).984798643571927	24).972339738981343
25).904964527740656	26).932592631882362	27).958512707090255	28).919607903333041
29).992217473905698	30).908400486485661	31).967574802486815	32).953059865558774
33).903089835166414	34).914551960041752	35).933508166083718	36).971710585898658
37).983153586876528	38).974878340444946	39).996413274173846	40).942751712122974
41).914171676455297	42).929372027583961	43).914804951665213	44).936280272574271
45).912586091502338	46).912514317775938	47).911765686099131	48).904029731182470
49).923508731282576	50).938535861320232	51).946075846047458	52).970525441137113
53).943621890110441	54).928281622162509	55).900896267323153	56).993676903568007
57).905634914572590	58).939217940566622	59).956692444203495	60).978839777391401
61).958192168649778	62).912557317984198	63).912214049684090	64).908647030530762
65).970186100259932	66).932881615181347	67).962639656285516	68).919071796162667
69).985644550288110	70).926672094171494	71).979024260382969	72).966619530973894
73).925785505244521	74).967328538794436	75).945876102332696	76).966943726492773
77).934865037823350	78).991952965552336	79).976156777054526	80).944855466748701
81).949047171443835	82).924409660997379	83).949721878700612	84).936814776590232
85).920305949160431	86).900817539237712	87).992888152416208	88).994066648203076
89).965123440007528	90).984524798613644	91).955133479197681	92).944828293283331
93).948579741632754	94).916006423695162	95).949673136757217	96).935907464473602
97).907277837043756	98).955859725126879	99).960373681339030	100).962795529256995

2. Means for applying smaller p when p_n values appear unevenly distributed:
   Table 11 below indicates figures concluding from replacing p 25 times, where p=0.3.
   Given that $p=\frac{3}{10}$ is of a rational number, and hat all p_n values are of rational number , meaning of limited decimal points or repetitive decimal numbers, a certain p_n values that are in limited decimal points may simple have all the decimal points of such a number taken, whereas those taking to repetitive decimal numbers may allow the number of a repetitive cycle be taken. To weed out the numbers generated from being unevenly normalized, the first few decimal points of the figure may be dropped, and that the first few p_n values may also be dropped since the smaller numerator and denominator may risk shortening the repetitive cycle, or have a portion taken, such as that from p_5, or that from the third decimal point, if a repetitive cycle should become excessive.
   P_5, being a repetitive decimal of $0.4\overline{846153}$ would allow 46153 be taken.
   P_6, being having an extended repetitive cycle of 0.58208966…, may allow that except the first two numbers 58 be taken, or simple the 10 digits after that.
   P_7, being a mixed repetitive decimal of $0.717\overline{857142}$, may allow 7857142 be taken.
   And the final figure blending the above findings would conclude 461532089552238785714 2…, which can be used for generating random numbers.

Table 11

1)	3	/	7	=	0.4285714285714285
2)	21	/	40	=	0.5250000000000000
3)	12	/	19	=	0.6315789473684210
4)	57	/	70	=	0.8142857142857143
5)	63	/	130	=	0.4846153846153846
6)	39	/	67	=	0.5820895522388060
7)	201	/	280	=	0.7178571428571429
8)	126	/	395	=	0.3189873417721519
9)	237	/	538	=	0.4405204460966543
10)	807	/	1,505	=	0.5362126245847176
11)	903	/	1,396	=	0.6468481375358166
12)	2,094	/	2,465	=	0.8494929006085192
13)	4,437	/	7,420	=	0.5979784366576819
14)	2,226	/	2,983	=	0.7462286288970835
15)	26,847	/	75,700	=	0.3546499339498019
16)	22,710	/	48,853	=	0.4648639796941846
17)	146,559	/	261,430	=	0.5606051333052825
18)	78,429	/	114,871	=	0.6827571797929852
19)	344,613	/	364,420	=	0.9456478788211404
20)	491,967	/	990,350	=	0.4967607411521179

III. Research finding & Discussion:

(I) Research findings on the hypothesis that $p < 0.25$:

p_n will converge to $p_\infty = \dfrac{1-\sqrt{1-4p}}{2}$.

(II) Research findings on the hypothesis that $0.25 < p < 1$ ：

1. Suppose p being a rational number other than $\frac{1}{2}$ and $\frac{1}{3}$, then $p \in I_\infty$, and that no two dissimilar a and be would conclude $I_a = I_b$.

2. $\frac{1}{2} \in I_1$, $\frac{1}{3} \in I_3$, $\frac{2-\sqrt{2}}{2} \in I_5$, $\frac{5-\sqrt{5}}{10} \in I_7$, $2-\sqrt{3} \in I_9$ 。

3. Given that any natural number n exists within $\dfrac{1}{4\cos^2 \frac{\pi}{n+3}} \in I_n$, thus none of the I_n values would be an empty set.

(III) Given $k_n = 1$, and that p_n is expressed as p, the fraction concluded would be $\dfrac{p\, a_{n-1}(p)}{a_n(p)}$, in which the recursive equation of polynomial , , would be $a_{n+1}(p) = a_n(p) - p\, a_{n-1}(p)$, $a_0(p) = 1$, $a_1(p) = p$.

To condition $p_n = 1$, the numerator and denominator of this fraction would need to be identical:

$a_n(p) - p\, a_{n-1}(p) = 0$,

meaning $a_{n+1}(p) = 0$ 。

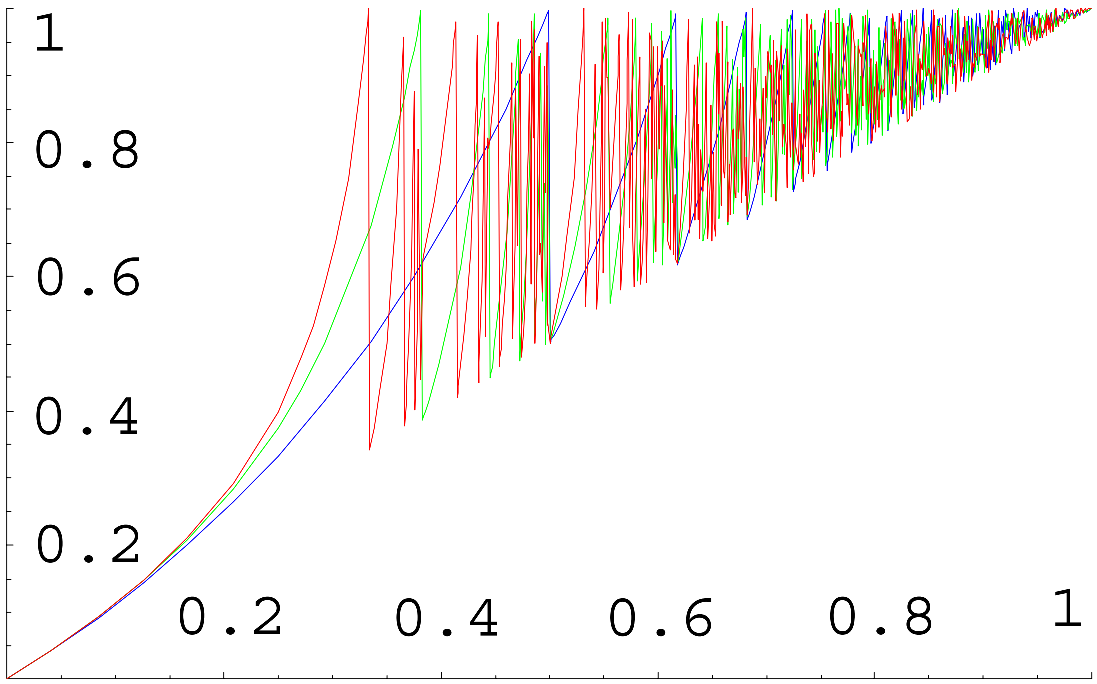
The polynomial can be deduced from the Fibonacci polynomial, and its differences to that of a straight Fibonacci polynomial are excerpted in Table 12:

Table 12

Heading	The denominator of p_n minus the numerator, $a_{n+1}(p)$	Fibonacci polynomial, $f_n(x)$	The differential of square roots, where both polynomials equal to zero
Sequential order of two adjacent numbers	Once	Twice	Given $p = x^2$, concluding the solution, where $a_{n+1}(p) = 0$, would require taking the square root, where $f_n(x) = 0$.
Minimum replacement	Constant number of zero times	The minimum replacement would be constant if n is of an odd number, whereas it would be at 1 when n is of an odd number	Given that extracting x bears no relation to the solution of a Fabonacci polynomial, $a_{n+1}(p)=0$, for the solution generated hypothesizing x=0 cannot be used for concluding a solution, where $a_{n+1}(p)=0$ 。
Positive and negative coefficients	The constant numbers are intermittently positive, positive and negative	All positive	Where upon taking the square root assuming $f_n(x) = 0$ ， the figures are then multiplied with –1 。
Coefficient sequence	The constant numbers must be 1, and in an order that is exactly the opposite of the Fibonacci polynomial	The maximum replacement being 1	The findings concluded as stated above will need to be reserved being that its coefficient sequence is reversed

(IV) The graph below depicts the result of p value, of between 0 and 1, being replaced three times, where blue indicates p_1, green p_2, and red p_3.

Graph 14



What can be concluded from Graph 14 are that:

When p increases from 0, the blue line, p_1, will incrementally increase from 0 t o1 before reverting to values identical to p, where the line can be broken down to infinite sections, hence the values of k_n can be from 1 to infinity, and the values of p matched at the topside will belong to I_1 . The closer p values approach 1, the more likely that numbers belonging to I_1 will occur.

The subdivision of green line p_2 further divides each segment of graph p_1, where each segment represents a K_n value in the second replacement, one that belongs to I_2 also contains an infinite number.

Similarly, all conditions after I_3 will be identical, thus the graph will be come fragmented, and a randomly enlarged graphic would indicate the same conditions as depicted in Graph 14.

(V) Where given $k_n=1$, findings on the numerators and denominators concluded, shown in p_n and p, will be related to the Fibonacci polynomials; however, given $k_n=2$, then,

$$p_1 = \frac{p^2}{1+p} ,$$

$$p_2 = \frac{p^2-p^3}{1-p-p^2} ,$$

$$p_3 = \frac{p^2-p^3-p^4}{1-p-2p^2+p^3} ,$$

$$p_4 = \frac{p^2-p^3-2p^4+p^5}{1-p-3p^2+2p^3+p^4} ,$$

$$p_5 = \frac{p^2-p^3-3p^4+2p^5+p^6}{1-p-4p^2+3p^3+3p^4-p^5} ,$$

$$p_6 = \frac{p^2-p^3-4p^4+3p^5+3p^6-p^7}{1-p-5p^2+4p^3+6p^4-3p^5-p^6} ,$$

$$p_7 = \frac{p^2-p^3-5p^4+4p^5+6p^6-3p^7-p^8}{1-p-6p^2+5p^3+10p^4-6p^5-4p^6+p^7} 。$$

Given the numerator of p_n being $p^2\, b_{n-1}(p)$，and its denominator being $b_n(p)$，the coefficients of $b_n(p)$ will be:

Table 13

	Constant	p	p^2	p^3	p^4	p^5	p^6	p^7	p^8
$b_0(p)$	1								
$b_1(p)$	1	-1							
$b_2(p)$	1	-1	-1						
$b_3(p)$	1	-1	-2	+1					
$b_4(p)$	1	-1	-3	+2	+1				
$b_5(p)$	1	-1	-4	+3	+3	-1			
$b_6(p)$	1	-1	-5	+4	+6	-3	-1		
$b_7(p)$	1	-1	-6	+5	+10	-6	-4	+1	
$b_8(p)$	1	-1	-7	+6	+15	-10	-10	+4	+1

What come to deviate from $a_n(p)$ in Table 7 are:

1. For every one increased to k, the number of replacement to $b_n(p)$ will also increase by one, whereas only two increases to k in the previous $a_n(p)$ would the number of replacement be increased by once.

2. The coefficient of the constant and the first derivative has been constant at 1 and –1, and the positive-negative sequence is repeated as "-,-,+,+" every fourth time.

3. By grouping the vertical columns in a group of two starting from the constant column in Table 13, the number appeared to the left of a certain column can be found to the next column to its left and right except in opposite integral, and the rest will be identical to that of $a_n(p)$ if the numbers to the left of each grouping were deleted.

   This portion is still under study.

(VI) When $p > 0.25$, the chaos phenomenon will occur:
p_n will begin to incrementally increase when k_n is taken to the smallest value; for example, given p=0.3, and the smallest value of k_n being 1, replacing p_1 through p_5 would conclude k_n=1, indicating that the first p_n values taken are incrementally increasing. Whereas given p=9 and 8 taken as the smallest of k_n, only from p_0 to p_1 will incrementally increase since 8 has been taken for k_n and 9 is filled in for K_2.
When the replacement reaches a p_n tier, where no more smallest nubmer can be taken for k_{n+1} , the generation of p_{n+1} will depend on p_n, where a smaller p will tend to conclude a smaller p_{n+1} as it approaches p, whereas when p approaches 1, a more even distribution can be found. Whereas when p_n approaches 1, any slight variation to p_n may likely to conclude a p_{n+1} that is way off.

And by repeating the process a few more times, the values of p_n will become unpredictable, hence a chaos will occur. When P remains smaller, p_n will incrementally increase and return to the vicinity of p when it can no longer increase incrementally, which allows the observation of more concentrated occurrence of the first few p values when p_n is at p_0, p_1, $p_{2,}$ … Whereas as p grows larger and the incrementally increasing phenomenon becomes less feasible without any incremental increase, all values generated in the next tier will tend to distribute evenly between p and 1, hence the values of p_n will be evenly distributed between p and 1 after several replacements.

The values of p_n generated in each time can be used for generating random numbers , which take to two forms:

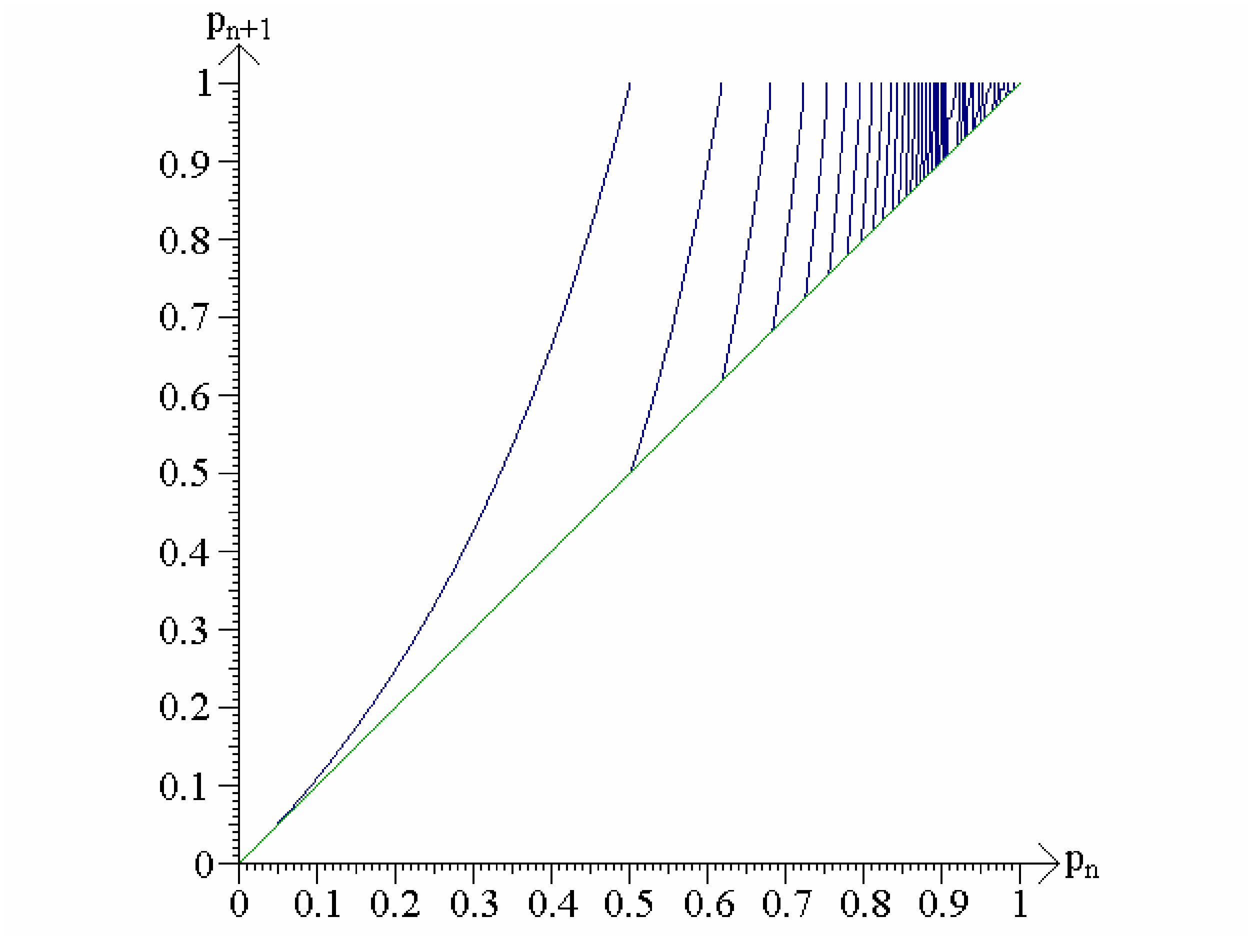
1. When p_n values are evenly distributed between p and 1, all values of p_n can be converted through linear function, where $r= a + \frac{b(p_n-p)}{1-p}$, as random numbers generated between the parameters a and b.
2. When p is of a rational number but its p_n being unevenly distributed, the first few decimal numbers of the p_n can be deleted and treated by the following:
   When p is of a limited decimal, all decimal points will be taken.
   When p is of a cyclical decimal, pure or mixed, the first complete repetitive cycle can be taken, or one half of the first repetitive cycle be taken to avoid a probable circumstance, where 1/7=0.142857…，142+857=999；
   1/17=0.0588235294117467，05882352+94117467=99999999 may result in a string of 9's .

(VII) Suppose the recursive equation is modified to $p_{n+1}=\frac{p_n^{k_n}}{1-p_n}$,

meaning the numerator is changed from p^{k_n} to , p^{k_n} the variations to the previous recursive equation are:

The value of p_{n+1}, which is once determined by the factors of p and p_n, will now has p_{n+1} determined by p_n, and the function graph of P_{n+1} derived from p_n will only be one irrespective what the previous p values are.

Graph 15



IV.Conclusion & Applications

(I) When p<0.25, not only $p \in I_\infty$ but p_n will converge to $p_\infty = \frac{1-\sqrt{1-4p}}{2}$.

Given $0.25 < p < 1$ and p being a rational number other than $\frac{1}{2}$ and $\frac{1}{3}$, no only $p \in I_\infty$ but $p_a \neq p_b$ as far as any two dissimilar numbers, a and b, are concerned.

To any natural number n, there is at least one element,

$$\frac{1}{4\cos^2\frac{\pi}{n+3}}$$, in a given set, I_n .

(II) The correlation of recursive equation, $p_{n+1} = \frac{p^{k_n}}{1-p_n}$, to the Fibonacci polynomial:

1. Given k_n=1, and the values of p_n expressed in fraction being the equivalent of an extended fraction, $\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\ldots}}}$, to n tier, a process in which denominators and numerators can be simplified from such a serial fractions would allow the denominator and numerator be concluded directly from the Fibonacci polynomial:

   First the sequential order of two adjacent sequences of a Fibonacci polynomial, $f_n(x)$, is reduced from 2 to 1, or an even reduction of 1 if the lower tiers are of single intervals instead of a constant. And the numeral values are conditioned to fall within positive and negative, where the highest order being positive. Lastly, the coefficient is then reversed to conclude a polynomial, $a_n(x)$, and $p_n = \frac{p\,a_n(p)}{a_{n+1}(p)}$. To conclude the solution where p=1, it only needs to define $p\,a_n(p)=a_{n+1}(p)$ being that its recursive equation is of $a_{n+1}(p) = a_n(p) + a_{n-1}(p)$, thus the solution of $a_{n+2}(p) = 0$ will be that of p=1.

   Given that the known solution of a Fibonacci polynomial being $x\in \{2i \cos \frac{j\pi}{n}, j = 1, 2, \ldots, n-1\}$, it can be deduced that when p_n=1, the solution under the hypothesis where $a_{n+1}(p) = 0$

will be $p \in \{\frac{1}{4\cos^2\frac{j\pi}{n+3}}$, $j = 1, 2, \ldots, [\frac{n+2}{2}]\}$, provided that $\frac{1}{4\cos^2\frac{\pi}{n+3}}$ falls within a rational range, hence $\frac{1}{4\cos^2\frac{\pi}{n+3}}$ $\in I_n$.

2. Given n= 0 ~ 7, the polynomials，$a_n(p)$, will be:

Table 14

	Constant	p	p^2	p^3	p^4
$a_0(p)$	1				
$a_1(p)$	1	-1			
$a_2(p)$	1	-2			
$a_3(p)$	1	-3	+1		
$a_4(p)$	1	-4	+3		
$a_5(p)$	1	-5	+6	-1	
$a_6(p)$	1	-6	+10	-4	
$a_7(p)$	1	-7	+15	-10	+1

A comparison to the Pascal triangle：

Table 15

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Each coefficient of $a_n(p)$ will appear along the diagonal line from lower left to upper right in a Pascal triangle.

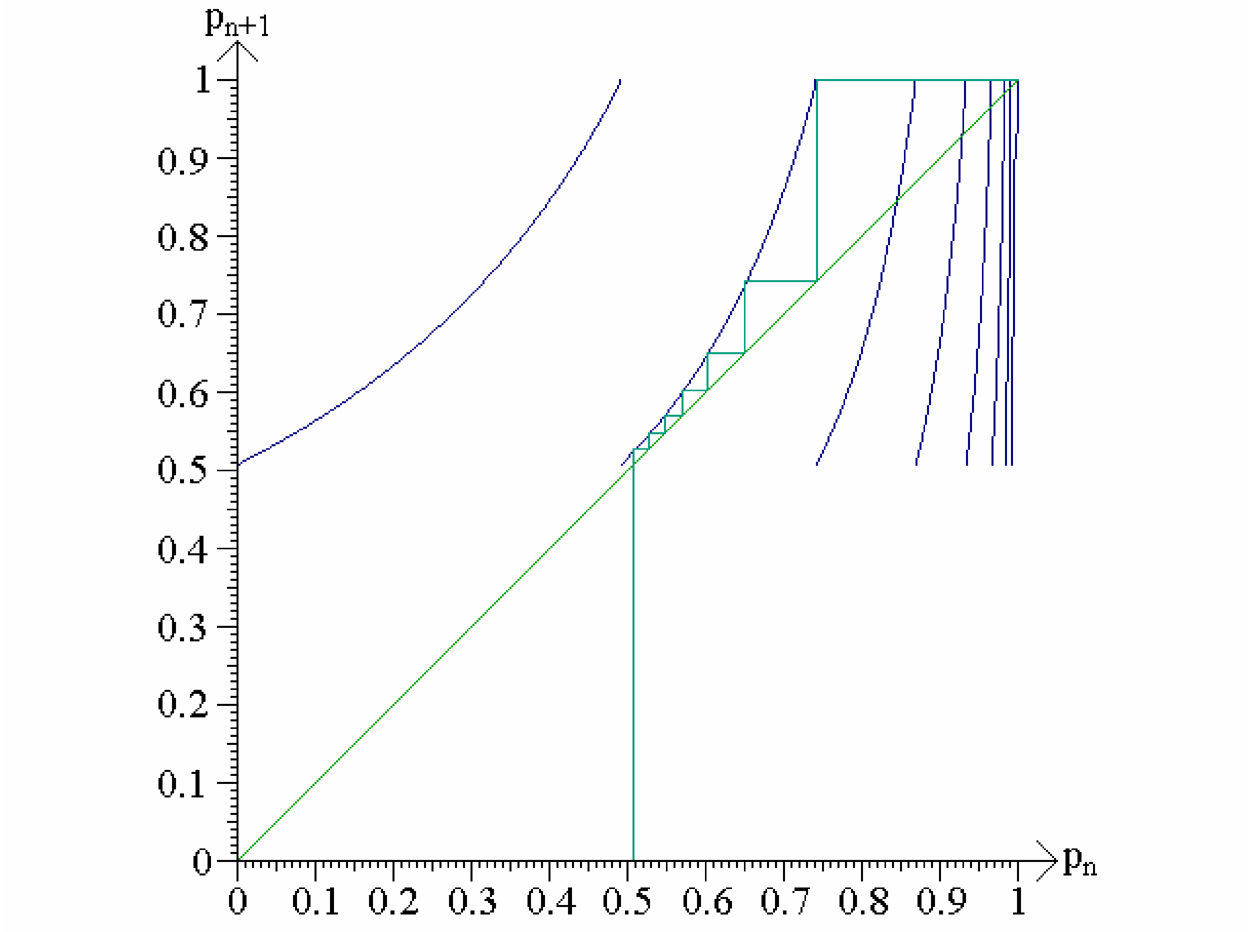
3. Suppose k_n is substituted by 2, and that $p_n = \dfrac{p^2\, b_n(p)}{b_{n+1}(p)}$, n=0~8 , the polynomials of $b_n(p)$ will be:

Table 16

	Constant	p	p^2	p^3	P^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}
$b_0(p)$	1											
$b_1(p)$	1	-1										
$b_2(p)$	1	-1	-1									
$b_3(p)$	1	-1	-2	+1								
$b_4(p)$	1	-1	-3	+2	+1							
$b_5(p)$	1	-1	-4	+3	+3	-1						
$b_6(p)$	1	-1	-5	+4	+6	-3	-1					
$b_7(p)$	1	-1	-6	+5	+10	-6	-4	+1				
$b_8(p)$	1	-1	-7	+6	+15	-10	-10	+4	+1			
$b_9(p)$	1	-1	-8	+7	+21	-15	-20	+10	+5	-1		
$b_{10}(p)$	1	-1	-9	+8	+28	-21	-35	+20	+15	-5	-1	
$b_{11}(p)$	1	-1	-10	+9	+36	-28	-56	+35	+35	-15	-6	+1

Taking to illustrate the solution based on the hypothesis where p_7=1 to be completed via Mathematica for concluding the solution of,
$b_8(p) = 1-p-7p^2+6p^3+15p^4-10p^5-10p^6+4p^7+p^8 = 0$, would conclude eight solutions where $b_8(p) = 0$, but the only satisfactory solution has been p= p = 0.50866 09187 58394…, which can be validated by replacing the p values as shown in Graph 16 below:

Graph 16



As indicted in Graph 16 are how the values of p_0 to p_7 fall between the second section of the blue line from left, where $p_{n+1} = \frac{p2}{1-pn}$, and the second section of green line, where $p_{n+1} = p_n$. A scenario that is fairly similar to the graph depicting an In set replaced by $\frac{1}{4\cos^2 \frac{\pi}{n+3}}$, where $0.25 < p < 0.5$, and k_n=1, except the only difference lies in that the numbers of p_0, p_1, … fall between the first section, where $p_{n+1} = \frac{p}{1-p_n}$, and that of the green line, where $p_{n+1} = p_n$.

Given that p=0.25, and $p_{n+1} = p_n$ and $p_{n+1} = \frac{p}{1-p_n}$ intersect at the point of 0.5 and 0.5, where p< 0.25, the two lines will intersect in two places, which will result in p_n being converged to where the two lines intersect.

Yet given $k_n=1$, no convergence will take place being that the excessive gradient of $p_{n+1} = \frac{p^{k_n}}{1-p_n}$ will prevent it from intersecting with $p_{n+1} = p_n$ at one or both points.

(III) Examining the application of chaos phenomenon through a random number generator when $0.25 < p < 1$:

Only p_n Values that are taken to as close to 1 can distortion be eliminated using linear function for converting random numbers to fall between a and b for when p_n is underrated, several incrementally increasing p_n values may result, preventing an even distribution as the first few p_n values of p_0, p_1, p_2, … tends to revert to where the p values are. Higher p_n values can help to avoid unevenly distributed p_n values during and after the incremental increase to generate more evenly distributed random numbers.

Whereas if the p_n values are underrated, the following means can be used for substitution: By dropping a few of the decimals in front and using only the rear. However, given p values are of a rational number, the values of p_n would also need to be in rational numbers, meaning they are either limited decimals or repetitive decimals. Suppose p_n being limited decimal and the decimal points taken have had the first few numbers dropped, this would allow the number be taken up to the first half of a repetitive cycle whether it be a pure or mixed repetition.

V. References

(I) Alexandru Lupaς (1999). A Guide of Fibonacci and Lucas Polynomials. Octogon, Math. Magazine, vol.7, No.1, 2-12.

(II) James Gleick (1991). Chaos: Making a New Science, Cosmos Cultural Publishing.

(III) http://www.research.att.com/~njas/sequences/, The On-Line Encyclopedia Integer Sequences, A011973, Triangle of numbers $\{C^{n-k}_{k}, n \geqq 0, 0 \leqq k \leqq [\frac{n}{2}]\}$; or, triangle of coefficients of Fibonacci polynomials.

(IV) Chen-Kuwei Wu, The World Famous Mathematical Quiz Collection – the Fibonacci Polynomials, Chapter IX Mathematics.

VI. Appendage

Summary of equations

Equation	Usage
Chaos.bas	The input p values are replaced 10,000 times and graphed in a 0.002 set interval of a p_1 to $p_{10,000}$ grid, and a printout is obtained using 0.0001 as the output unit, with which findings of p_{n+1} values, set to a minimum unit of 0.001, are concluded by replacing the p_n value, set to 0.002 per unit, are graphed to cover 0 to 0.0098.
Chaos2.bas	The input p values are replaced 80 times, and the findings of each computation printed out with 15 decimals.
Chaos3.bas	The values of p_1 through p_3 shown in three colors on a coordinate graph taken to 1/400 per unit, where p = 0 ~ 399/400.
Chaos4.bas	A modification of Chaos3.bas, where the interval of p values has been changed to 1/800, where the output is divided into three parts, i.e. lower left, upper left and upper right, for the oversized graph, and no entry for lower right of the graph.
Chaos5.bas	The results of output using numbers taken from a given interval between 0 and 1 as the values of p.
Chaos6.bas	An improvement of Chaos.bas that automatically compresses the size of a graph by one-half when the length of an upright graph should exceed 140 pixels to avoid going over the margin.
Chaos7.bas	The numerator and denominator of positive input numbers taken as p values for calculating the value of p_1 to p_{20}, and expressed in simplest fraction and decimal point.
Chaos8.bas	An improved Chaos6.bas that offers numeral expression indicating the 500 straight lines in an elongated graph and rounded to four decimal points as it overflows when reaching 9,999 being that it requires 25,000 replacement when $p < 0.7$, or 5000 times when $p \geqq 0.7$, where the values of p_n will become cluttered and the computation longer as p increases, besides the narrowly rated p_n values will infinitely extend the length of printout paper.

Chaos9.bas	An improved Chao8.bas that can be used to modify an observation parameter by keying in any area between 0 and 1 as an observation area, taking to the set interval of 1/500, whereby by the findings has the statistics of p values replaced remain identical as that of Choas8.bas when $p<0.7$, but up to 10,000 times when $p \geqq 0.7$, with which a recurrence can be charted from an observation area rated to 1/100,000th as the minimum unit. It then procedures the precise numeral numbers of a long graph and rounded by E and the first three decimals, i.e. 11,850 as E118 when exceeding 9,999.
Chaos10.bas	An improved Chao2.bas, which allows keying in varied p values and P_0 values before computing the values of p_1 to p80.
Chaos11.bas	It converts the input p values and number of replacements into a function graph for $p_{n+1} = \frac{p^{k_n}}{p_n}$ and $p_{n+1} = p_n$ on the premises that $0 < p_n < 1$, and shown in blue line indicating the state of p replacements, in which straight line indicates how p_n is converted to p_{n+1}, and the horizontal axes indicates how p_{n+1} is shifted from the vertical axes to the horizontal one as the new p_n values readying for the next computation, at a resolution of 400 x 400.
Chaos12.bas	An improved Chaos11.bas, which can be used to reset the parameters of vertical and horizontal axes for observation purposes.
Chaos13.bas	It allows modifying the resolution of Chaos11.bas to 500 x 400 in support of observation graphs produced by Chaos9.bas.
Chaos14.bas	It converts recursive equation into a $p_{n+1} = \frac{p_n^{k_n}}{p_n}$ function graph, which is a fixed graph that does not require inputting p values.
Chaos15.bas	It allows repeatedly replacing the input p value 200 times and map out a graph with its horizontal axle depicting the number of replacements and vertical axes the p_n values.

Below excerpts the equation of Chaos7.bas, Chaos9.bas, Chaos15.bas as general references:

Chaos7.bas：

```
CLS
1 INPUT P, Q
IF Q = 0 OR P > INT(P) OR Q > INT(Q) THEN 1
IF (P / Q) <= 0 OR (P / Q) >= 1 THEN 1
M# = P: N# = Q
GOSUB SUB1
P = M#: Q = N#
P0# = P / Q
P0P# = P: P0Q# = Q
FOR I = 1 TO 20
    A = 0
    DO
        A = A + 1
        P1P# = P0Q# * P ^ A
        P1Q# = (P0Q# - P0P#) * Q ^ A
        P1# = P1P# / P1Q#
        M# = P1P#: N# = P1Q#
        GOSUB SUB1
        P1P# = M#: P1Q# = N#
        P1# = P1P# / P1Q#
    LOOP UNTIL P1# < 1.00000000001#
    PRINT USING "##)    ######################,    /    ######################,    =
#.#################"; I; P1P#; P1Q#; P1#
    IF P1# > .99999999999# THEN END
    P0P# = P1P#: P0Q# = P1Q#
NEXT I
END

SUB1:
S# = M#: T# = N#
DO
    T# = T# - S# * INT(T# / S#)
    IF T# = 0 THEN R# = S#: EXIT DO
    S# = S# - T# * INT(S# / T#)
    IF S# = 0 THEN R# = T#: EXIT DO
LOOP
M# = M# / R#: N# = N# / R#
RETURN
```

Chaos9.bas：

```
CLS
SCREEN 12
LINE (0, 0)-(639, 479), 15, BF
DO
    INPUT "P0="; P#
LOOP WHILE P# <= 0 OR P# > .98
DO
    INPUT "Enter two numbers between 0 and 1 as the range"; B, E
    IF B < E AND B >= 0 AND E <= 1 THEN EXIT DO
LOOP
B# = B
E# = E
C# = (E# - B#) / 500
LINE (0, 0)-(640, 480), 15, BF
LINE (70, 100)-(570, 100), 6
FOR I = 0 TO 100
    IF I MOD 10 = 0 THEN
        LINE (70 + I * 5, 90)-(70 + I * 5, 99), 6
        LINE (70 + I * 5, 234)-(70 + I * 5, 243), 0
        LINE (70 + I * 5, 245)-(70 + I * 5, 444), 5
    ELSE
        IF I MOD 5 = 0 THEN
            LINE (70 + I * 5, 94)-(70 + I * 5, 99), 6
            LINE (70 + I * 5, 238)-(70 + I * 5, 243), 0
            LINE (70 + I * 5, 245)-(70 + I * 5, 444), 12
        ELSE
            LINE (70 + I * 5, 97)-(70 + I * 5, 99), 6
            LINE (70 + I * 5, 241)-(70 + I * 5, 243), 0
            LINE (70 + I * 5, 245)-(70 + I * 5, 444), 13
        END IF
    END IF
NEXT I
FOR I = 245 TO 440 STEP 5
    IF (I + 5) MOD 50 = 0 THEN
        LINE (59, I)-(68, I), 0
        LINE (70, I)-(570, I), 9
    ELSE
        IF (I + 5) MOD 25 = 0 THEN
            LINE (63, I)-(68, I), 0
            LINE (70, I)-(570, I), 2
        ELSE
            LINE (66, I)-(68, I), 0
            LINE (70, I)-(570, I), 11
        END IF
    END IF
NEXT I
LINE (69, 244)-(570, 445), 0, B
LOCATE 1, 1
PRINT "P0="; P#
```

```
PRINT "Range:"; B; "to"; E
LOCATE 6, 8
PRINT B
LOCATE 6, 71
PRINT E
DIM A(500)
LOCATE 1, 1
P0# = P#
IF P# < .8 THEN T = 25000 ELSE T = 10000
R = 1

FOR I = 1 TO T
    A = 0
    DO
        A = A + 1
        P1# = P# ^ A / (1 - P0#)
    LOOP UNTIL P1# <= 1
    D = (P1# - B#) / C#
    IF 0 <= D AND D < 500 THEN
        A(INT(D)) = A(INT(D)) + 1
        IF A(INT(D)) = 133 / R THEN GOSUB SUB3 ELSE GOSUB SUB1
    END IF
    IF I MOD 100 = 0 THEN
        LOCATE 3, 1
        PRINT USING "Times: ######,"; I;
        LOCATE 4, 1
        PRINT USING "    P    :.############"; P1#;
    END IF
    IF P1# >= .9999999899999999# THEN S = I: LOCATE 25, 41: PRINT P1#: EXIT FOR
    P0# = P1#
NEXT I
S = I - 1
LOCATE 3, 1
PRINT USING "Times: ######,"; S;
FOR I = 0 TO 499
    A = 0
    DO
        A = A + 1
        P1# = P# ^ A / (1 - (B# + C# * I))
    LOOP UNTIL P1# <= 1
    PSET ((70 + I), (100 - P1# * 100)), 12
NEXT I
IF P# - B# > -.0000001# AND P# < E# THEN LINE (70 + (P# - B#) / C#, 0)-(70 + (P# - B#) /
C#, 100), 3
LINE (70, 100 - B# * 100)-(570, 100 - (B# + C# * 500) * 100), 1
LOCATE 29, 10
PRINT "Press any key to continue";
DO
LOOP WHILE INKEY$ = ""
CLS
```

```
PRINT "P0="; P#
FOR I = 0 TO 24
    FOR J = 0 TO 19
        LOCATE I + 2, J * 4 + 1
        IF P# - (B + (J * 25 + I + 1) * C#) >= -.0000001 THEN
            PRINT "     -"
        ELSE
            IF A(I + J * 25) < 10000 THEN
                PRINT USING "####"; A(I + J * 25);
            ELSE
                PRINT "E"; MID$(STR$(A(I + J * 25)), 2, 3)
            END IF
        END IF
    NEXT J
NEXT I
END

SUB1:
PSET ((70 + INT(D)), (100 + A(INT(D)) * R)), 2
PSET ((70 + INT(D)), 245 + ((P1# - B#) / C# - INT((P1# - B#) / C#)) * 200), 0
RETURN

SUB3:
LINE (70, 101)-(570, 233), 15, BF
R = R / 2
FOR X = 0 TO 499
    IF A(X) > 0 THEN LINE (70 + X, 100)-(70 + X, 100 + A(X) * R), 2
NEXT X
LOCATE 5, 1
PRINT "Ratio: 1 :"; 1 / R
RETURN
```

Dense15.bas：

```
SCREEN 12
CLS
LINE (0, 0)-(639, 479), 15, BF
DO
    INPUT "P="; P#
LOOP UNTIL 0 < P# AND P# < 1
CLS
LINE (0, 0)-(639, 479), 15, BF
PRINT "P="; P#
COLOR 0
LINE (15, 20)-(15, 440)
LINE (15, 440)-(635, 440)
FOR I = 0 TO 100
    IF I MOD 10 = 0 THEN
        LINE (5, I * 4 + 40)-(14, I * 4 + 40)
        LINE (I * 3 + 15, 441)-(I * 3 + 15, 450)
        LINE (I * 3 + 315, 441)-(I * 3 + 315, 450)
    ELSE
        IF I MOD 5 = 0 THEN
            LINE (10, I * 4 + 40)-(14, I * 4 + 40)
            LINE (I * 3 + 15, 441)-(I * 3 + 15, 445)
            LINE (I * 3 + 315, 441)-(I * 3 + 315, 445)
        ELSE
            LINE (12, I * 4 + 40)-(14, I * 4 + 40)
            LINE (I * 3 + 15, 441)-(I * 3 + 15, 443)
            LINE (I * 3 + 315, 441)-(I * 3 + 315, 443)
        END IF
    END IF
NEXT I
LINE (15, 20)-(5, 30)
LINE (15, 20)-(25, 30)
LINE (625, 430)-(635, 440)
LINE (635, 440)-(625, 450)
P0# = P#
LINE (14, 440 - P0# * 400 - 1)-(16, 440 - P0# * 400 + 1), 0, BF
FOR I = 1 TO 200
    K = 0
    DO
        K = K + 1
        P1# = P# ^ K / (1 - P0#)
    LOOP UNTIL P1# <= 1
    LINE (15 + (I - 1) * 3, 440 - P0# * 400)-(15 + I * 3, 440 - P1# * 400), 0
    LINE (15 + I * 3 - 1, 440 - P1# * 400 - 1)-(15 + I * 3 + 1, 440 - P1# * 400 + 1), , BF
    IF P1# > .999999999999# THEN EXIT FOR
    P0# = P1#
NEXT I
END
```

評　　　語

（1）題材及內容十分充實、完整

（2）注意由多次遞迴數值計算而產生的誤差。

（3）是否將 random number generator 改為 pseudo random number generator。

（4）（p.17）an（p）是否為 weel-known 函數？