# 台灣二〇〇二年國際科學展覽會

科　　別：數學科

作品名稱：移動棋子問題的致勝策略

得獎獎項：數學科第一名
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# 作 者 簡 介


我叫葉均承，今年就讀北一女中一年級。爸爸是數學教授，媽媽是公務員，我喜歡音樂、歷史、童軍和數學。我的數學能力和興趣是從小和爸爸玩數學遊戲培養出來的。

這次我研究的題目是延續去年作的“如何移動棋子的問題”，去年參展時，評審教授和觀眾提出許多新而且難的問題，當時我並不會。參展後，我訂下希望達到的目標，繼續研究這些新的問題，很高興今年完成了我的目標。

## 壹、研究動機

去年我研究一個遊戲：有一列 n 個的方格中，由左至右依序編號爲 1,2,3,⋯,n。在 $X_1$ 個、第 $X_2$ 個、第 $X_3$ 個格子中各放置一個棋子。甲、乙二個人按照下列規則輪流移動棋子：

一、甲、乙兩個人每次只能動一個棋子(三個棋子中任選一個)。遊戲開始由甲先移動棋子。

二、甲、乙兩個人每次移動某一個棋子時，只能將這個棋子移至左邊最近的空格(若前面連續有 p 個棋時可以跳過前面的 p 個棋子而且只能跳一次)，而且每個方格中最多只能放一個棋子。

研究這個遊戲問題時，我討論四種不同“輸贏結果”的規定：甲、乙兩個人中，

A. **誰先將 3 個棋子中任意一個棋子移到第一個方格，誰就是贏家。**

B. **誰先將 3 個棋子中任意一個棋子移到第一個方格，誰就是輸家。**

C. **誰先不能再移動任何棋子，誰就是輸家。**

D. **誰先不能再移動任何棋子，誰就是贏家。**

當“輸贏結果” 的規定採用 A,B,C,D 時，我們稱爲遊戲 A,B,C,D。今年我將把這個遊戲問題中棋子的個數由 3 個推廣到一般 k 個情形之後，再繼續研究遊戲的致勝策略，同時也將研究遊戲 A,B,C,D 之間的關係。

## 貳、研究目的

去年我針對這問題的研究，是採用很多例子的樹狀圖去觀察現象，先猜測結果、經過分析、修正，再運用數學歸納法來證明; 同時我也研究遊戲 A,B,C 之間的關係。但是遊戲 D 與遊戲 A,B,C 之間的關係卻找不出來。在今年的研究中，希望藉著已知的結果和推導出的一些公式，能夠得到更多的結果;同時從過去我接觸過的一些遊戲中尋找一些相似之處，藉著一些蛛絲馬跡來猜測一般的結果; 再藉著一些例子樹狀圖形來驗證，進而得到正確的答案。也希望能解開遊戲 D 與遊戲 A,B,C 之間的關係之謎。這樣的研究過程可以培養我對數學研究的敏銳觀察力和耐心。

# 參、研究設備器材

圍棋、樸克牌、紙、筆及電腦。

# 肆、研究過程或方法

## 一、定義與符號

有一個$1\times n$的方格表，由左至右依序編號為$1,2,3,...,n$，在第$X_1$個、第$X_2$個、…、第$X_k$個格子中各放置一個棋子，其中$1<X_1<X_2<\ldots<X_k\le n$，放置這$k$個棋子方格的編號數(簡記為這 $k$ 個棋子的編號數)所成的集合記為$S=\{X_1,X_2,...,X_k\}$。當我們討論遊戲$t$時，$t\in\{A,B,C,D\}$，

1.如果存在有一個使甲必勝的策略，則我們定義函數$f_t(X_1,X_2,...,X_k)=0$; 如果存在有一個使乙必勝的策略，則我們定義函數$f_t(X_1,X_2,...,X_k)=1$。有時我們用$f_t(S)$來代替$f_t(X_1,X_2,...,X_k)$。

2.輪到某一個人移動棋子時，我們稱這個人為先手，另一個人為後手。我們用$\{X_1,X_2,...,X_k\}\xrightarrow{甲}$(或$\xrightarrow{乙}$)$\{X_1',X_2',...,X_k'\}$表示:經過甲(或乙)移動棋子後，這$k$個棋子的編號數所成的集合由原來的$\{X_1,X_2,...,X_k\}$變成為$\{X_1',X_2',...,X_k'\}$。而此時原來的先手變成後手，原來的後手反而變成先手。

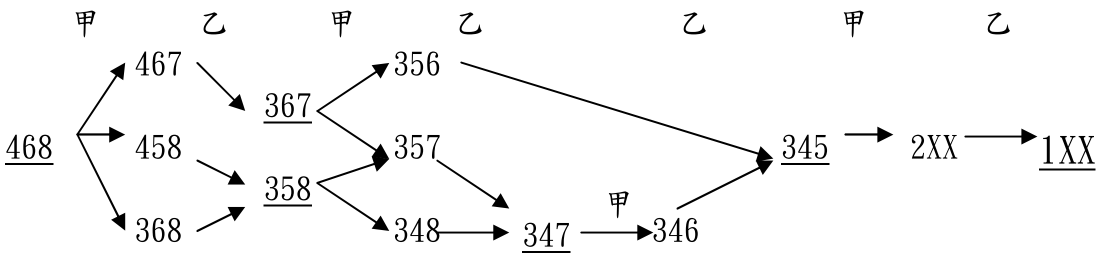
3. $S+r$ 表示集合$\{X_1+r,X_2+r,\ldots,X_k+r\}$。

4.我們定義函數$\chi(a)=\begin{cases}1 & 如果a是奇數;\\ 0 & 如果a是偶數。\end{cases}$

## 二、函數$f_A(X_1,X_2,...,X_k)$的值

### 1、樹狀圖形

經過分析遊戲A 之後，我們看到一個現象：如果$2\in S$，則$f_A(S)=0$。觀察圖一的樹狀圖形。我們不僅知道$f_A$(4,6,8)=1，而且我們也可以知道$f_A(3,6,7)=f_A(3,5,8)=f_A(3,4,7)=f_A(3,4,5)=1$和$f_A(4,6,7)=f_A(4,5,8)=f_A(3,6,8)=f_A(3,5,7)=f_A(3,5,6)=f_A(3,4,8)=f_A(3,4,6)=0$。




**圖一. $f_A(4,6,8)$的樹狀圖形**

## 2、函數$f_A(X_1,X_2,X_3)$的值

藉著很多的例子的觀察研究，我們發現：當$k$個棋子的編號數所成的集合為$S=\{X_1,X_2,...,X_k\}$，其中$1<X_1<X_2<\cdots<X_k\le n$。當輪到某個人移動棋子時，如果他把放在編號為$X_i$方格的棋子移到它左邊最近的空格，$1\le i\le k$，此時我們把移動後的$k$個棋子的編號數所成的集合記為$S_i$。當然每位先手總希望能贏，希望能找出必勝策略。由於移動棋子後，原來的先手變成後手，原來的後手反而變成先手，所以我們知道：如果存在有一個$S_i$使得$f_A(S_i)=1$，則先手會把編號為$X_i$方格的棋子移到它左邊最近的空格，因此$f_A(S)=0$；如果所有的$S_i$使得$f_A(S_i)=0$，也就是說：不管先手如何移動棋子，都會造成對手有必勝的策略，因此$f_A(S_i)=1$。所以我們知道：

$$f_A(S)=1-\max\{f_A(S_i)\,|\,1\le i\le k\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

藉著利用這個公式及上面例子，我們可以推導出更多的結果。當棋子的個數是3時，去年我們得到：

| |
|---|
| 如果$(a,b,c)\in L$,則 $f_A(a,b,c)=1$; 如果$(a,b,c)\in L'$,則$f_A(a,b,c)=0$. |

其中$L=A\cup B\cup C\cup D$和$L'=A'\cup B'\cup C'\cup D'$，

$A=\{(a,b,c)\,|\,3\le a<b-1<c-2, a+b+c\text{是偶數}\}$;

$B=\{(a,b,c)\,|\,3\le a=b-1<c-2, ac\text{是奇數}\}$;

$C=\{(a,b,c)\,|\,3\le a<b-1=c-2, ac\text{是奇數}\}$;

$D=\{(a,b,c)\,|\,3\le a=b-1=c-2, a\text{是奇數}\}$;

$A' = \{(a,b,c) \mid 3 \le a < b-1 < c-2, a+b+c$ 是奇數$\}$;

$B' = \{(a,b,c) \mid 3 \le a = b-1 < c-2, ac$ 是偶數$\}$;

$C' = \{(a,b,c) \mid 3 \le a < b-1 = c-2, ac$ 是偶數$\}$;

$D' = \{(a,b,c) \mid 3 \le a = b-1 = c-2, a$ 是偶數$\}$。

我們對 a+b+c 的值作數學歸納法，分別討論在上述不同的 8 個集合情形，我們證明：

$$f_A(a,b,c) = \begin{cases} 1-\chi(a+b+c) & \text{if } 2 \le a < b-1 < c-2; \\ \chi(ac) & \text{if } 2 \le a = b-1 < c-2, \text{ 或 } 2 \le a < b-1 = c \\ \chi(a) & \text{if } 2 \le a = b-1 = c-2. \end{cases}$$

## 3、函數 $f_A(X_1, X_2, ..., X_k), k = 4,5,6,$ 的值

由於輪到某一個人移動棋子，如果他看到編號爲 2 的方格有一個棋子，他可以把這個棋子移到編號爲 1 的方格，則他就是贏家。也就是說如果 $2 \in S$ ，則 $f_A(S) = 0$。所以甲、乙兩個人都避免把棋子移到編號爲 2 的方格，所以也就是說大家都只會把棋子移到編號 $\ge 3$ 的方格。如果編號爲 3 的方格已經放置一個棋子，誰移動這個編號爲 3 的棋子，誰就是輸家。所以甲、乙兩個人都可以把 $1 \times n$ 的方格表中編號爲 3 的方格忽略掉，所以我們知道：

$$f_A(3, X_1+1, X_2+1, ..., X_k+1) = f_A(X_1, X_2, ..., X_k) \cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

觀察函數 $f_A(X_1, X_2, X_3)$ 的結果，我們發現到相鄰的兩個棋子 $X_i$ 和 $X_{i+1}$ 的棋子編號是否差 1 扮演著重要的角色。當棋子的個數是 4 時，利用這個公式(1)和(2) 來得到很多例子的 $f_A(X_1, X_2, X_3, X_4)$ 值。我們經過分析後，我們先猜測答案，再用例子的結果來驗證，得到的答案都符合猜測的結果，再利用對 $X_1 + X_2 + X_3 + X_4$ 的值作數學歸納法來證明，類似棋子的個數是 3 的情形，需要分別討論 16 個集合情形，所以證明的過程更加繁瑣，經過一個月的努力，最後我們得到的答案是:

$$f_A(a,b,c,d)=\begin{cases}\chi(a+b+c+d-1) & if\ 3\le a<b-1<c-2<d-3\\ \chi(a(c+d)) & if\ 3\le a=b-1<c-2<d-3\\ \chi(c(a+d)) & if\ 3\le a<b-1=c-2<d-3\\ \chi(c(a+b)) & if\ 3\le a<b-1<c-2=d-3\\ \chi(a(d-1)) & if\ 3\le a=b-1=c-2<d-3\\ \chi(ac) & if\ 3\le a<b-1=c-2=d-3\\ \chi(a+c-1) & if\ 3\le a=b-1<c-2=d-3\\ \chi(a) & if\ 3\le a=b-1=c-2=d-3\end{cases}$$

又經過幾個月的努力分析，根據相鄰兩個棋子 $X_i$ 和 $X_{i+1}$ 的棋子編號是否差 1，需要分別討論 32 個集合和 64 個集合情形。我們終於證明了函數 $f_A(X_1,X_2,X_3,X_4,X_5)$ 和 $f_A(X_1,X_2,X_3,X_4,X_5,X_6)$ 的值:

**如果** $(a,b,c,d,e)$ **滿足下列 13 條件之一，則** $f_A\ (a,b,c,d,e)$**=1; 否則** $f_A\ (a,b,c,d,e,f)=0$**。**

1. $3\le a<b-1<c-2<d-3<e-4, a+b+c+d+e$ 是奇數;
2. $3\le a=b-1<c-2<d-3<e-4,\ a(c+d+e-1)$ 是奇數;
3. $3\le a<b-1=c-2<d-3<e-4,\ c(a+d+e-1)$ 是奇數;
4. $3\le a<b-1<c-2=d-3<e-4,\ c(a+b+e-1)$ 是奇數;
5. $3\le a<b-1<c-2<d-3=e-4,\ e(a+b+c-1)$ 是奇數;
6. $3\le a=b-1=c-2<d-3<e-4,\ a(d+e)$ 是奇數;
7. $3\le a<b-1=c-2=d-3<e-4,\ c(a+e-1)$ 是奇數;
8. $3\le a<b-1<c-2=d-3=e-4,\ c(a+b)$ 是奇數;
9. $3\le a=b-1<c-2=d-3<e-4,\ e(a+d)$ 是奇數;
10. $3\le a=b-1<c-2<d-3=e-4,\ c(a+d)$ 是奇數;
11. $3\le a<b-1=c-2<d-3=e-4,\ a(c+d)$ 是奇數;
12. $3\le a<b-1=c-2=d-3=e-4,$

    $3\le a=b-1<c-2=d-3=e-4, 3\le a=b-1=c-2<d-3=e-4,$

    $3\le a<b-1<c-2<d-3=e-4,\ ae$ 是奇數;
13. $3\le a=b-1=c-2=d-3=e-4,\ a$ 是奇數。

如果$(a,b,c,d,e,f)$滿足下列 25 條件之一，則 $f_A(a,b,c,d,e,f)$=1;

否則 $f_A(a,b,c,d,e,f)$=0。

1. $3 \le a < b-1 < c-2 < d-3 < e-4 < f-5, a+b+c+d+e+f$ 是奇數;
2. $3 \le a = b-1 < c-2 < d-3 < e-4 < f-5,\ a(c+d+e+f-1)$ 是奇數;
3. $3 \le a < b-1 = c-2 < d-3 < e-4 < f-5,\ c(a+d+e+f-1)$ 是奇數;
4. $3 \le a < b-1 < c-2 = d-3 < e-4 < f-5,\ c(a+b+e+f-1)$ 是奇數;
5. $3 \le a < b-1 < c-2 < d-3 = e-4 < f-5,\ e(a+b+c+f-1)$ 是奇數;
6. $3 \le a < b-1 < c-2 < d-3 < e-4 = f-5,\ e(a+b+c+d-1)$ 是奇數;
7. $3 \le a = b-1 = c-2 < d-3 < e-4 < f-5,\ a(d+e+f-1)$ 是奇數;
8. $3 \le a < b-1 = c-2 = d-3 < e-4 < f-5,\ c(a+e+f-1)$ 是奇數;
9. $3 \le a < b-1 < c-2 = d-3 = e-4 < f-5,\ c(a+b+f-1)$ 是奇數;
10. $3 \le a < b-1 < c-2 < d-3 = e-4 = f-5,\ e(a+b+c-1)$ 是奇數;
11. $3 \le a = b-1 < c-2 = d-3 < e-4 < f-5,\ (a+d)(e+f)$ 是奇數;
12. $3 \le a = b-1 < c-2 < d-3 = e-4 < f-5,$

    $3 \le a < b-1 = c-2 < d-3 < e-4 < f-5,\ (a+d)(c+f)$ 是奇數;
13. $3 \le a = b-1 < c-2 < d-3 < e-4 = f-5,$

    $3 \le a < b-1 = c-2 < d-3 = e-4 < f-5,\ (a+f)(c+d)$ 是奇數;
14. $3 \le a < b-1 < c-2 = d-3 < e-4 = f-5,\ (a+b)(c+f)$ 是奇數;
15. $3 \le a < b-1 < c-2 < d-3 = e-4 < f-5,\ (a+d)(e+f)$ 是奇數;
16. $3 \le a < b-1 < c-2 < d-3 < e-4 = f-5,\ (a+d)(e+f)$ 是奇數;
17. $3 \le a = b-1 = c-2 - 3 < e-4 < f-5,\ a(d+e+f-1)$ 是奇數;
18. $3 \le a < b-1 = c-2 = d-3 = e-4 < f-5,\ c(a+e+f-1)$ 是奇數;
19. $3 \le a < b-1 < c-2 = d-3 = e-4 = f-5,\ c(a+b+f-1)$ 是奇數;
20. $3 \le a = b-1 = c-2 < d-3 = e-4 < f-5,$

    $3 \le a = b-1 = c-2 < d-3 < e-4 = f-5,\ 3 \le a = b-1 < c-2 = d-3 = e-4 < f-5,$

    $b,d,f$ 全是奇數或全是偶數;
21. $3 \le a = b-1 < c-2 < d-3 = e-4 = f-5,\ \ 3 \le a < b-1 = c-2 = d-3 < e-4 = f-5,$

    $3 \le a < b-1 = c-2 < d-3 = e-4 = f-5,\ a,c,e$ 全是奇數或全是偶數;

$22. 3 \le a = b-1 < c-2 = d-3 < e-4 = f-5,\ (a+c+f)$ 是奇數;

$23. 3 \le a = b-1 = c-2 = d-3 = e-4 < f-5,$

$3 \le a = b-1 = c-2 = d-3 < e-4 = f-5,$

$3 \le a = b-1 < c-2 = d-3 = e-4 = f-5,$

$3 \le a < b-1 = c-2 = d-3 = e-4 = f-5,\ ae$ 是奇數;

$24. 3 \le a = b-1 = c-2 < d-3 = e-4 = f-5,\ (a+d)$ 是奇數;

$25. 3 \le a = b-1 = c-2 = d-3 = e-4 = f-5,\ a$ 是奇數;

## 4、函數 $f_A(X_1, X_2, ..., X_k)$ 的值

雖然利用這個公式(1)和(2)，得到很多例子的 $f_A(X_1, X_2, X_3, X_4)$ 值，但是再仔細分析，討論集合的個數，由 16 種，32 種，64 種，128 種，…。這樣的過程很難完成一般情形的分析。所以我們換一種思維，利用過去我接觸過的一些遊戲中尋找一些相似之處，藉著一些蛛絲馬跡來猜測一般的結果。

觀察函數 $f_A(X_1, X_2, ..., X_k)$，$k = 3,4,5,6,$ 的值時，我們注意到相鄰的兩個棋子的棋子編號是否差 1 扮演著重要的角色，我們試著把第 j 個棋子的棋子編號 $X_j$ 減去 j，$j = 1,2,...,k,$ 則相鄰的兩個棋子 $X_i$ 和 $X_{i+1}$ 分別減去 i 和 i+1 之後所得到的數相同。令 $Y_i = X_i$ -i，i=1,2,…k.當 $k = 3,4,5,6$，藉著 $(Y_1, Y_2, Y_3 ..., Y_k)$ 的值，我們重新整理函數 $f_A(X_1, X_2, ..., X_k)$ 的值，得到 $f_A(X_1, X_2, ..., X_k)$=1 的充要條件:

(1).當 $k = 3$ 時，$(Y_1, Y_2, Y_3)$ 中必須沒有相同的奇數，且恰好有偶數個奇數;

(2).當 $k = 4$ 時，$(Y_1, Y_2, Y_3, Y_4)$ 中必須沒有大於二個相同的奇數，且

a.如果有相同的奇數， 則恰好必須有兩對(不可以是四個相同的奇數)，

b.如果沒有相同的奇數，則恰好有偶數個奇數;

(3).當 $k = 5$ 時，$(Y_1, Y_2, Y_3, Y_4, Y_5)$ 中必須沒有大於二個相同的奇數，且

a.如果有相同的奇數， 則恰好必須有兩對(不可以是四個相同的奇數)，

b.如果沒有相同的奇數，則恰好有偶數個奇數;

(4).當 $k=6$ 時，$(Y_1,Y_2,Y_3,Y_4,Y_5,Y_6)$ 中必須沒有大於三個相同的奇數，且

a.如果有三個相同的奇數， 則必須是

(i)兩組三個相同的奇數，或

(ii)一組三個相同的奇數、一對相同的奇數和一個奇數;

b.如果沒有大於二個相同的奇數但是有二個相同的奇數，則必須是有兩對(不可以是四個相同的奇數);

c.如果沒有相同的奇數，則恰好有偶數個奇數。

當我們分析函數 $f_A(X_1,X_2,...,X_k),k=3,4,5,6,$ 的值，發現: 如果我們先去掉 $(Y_1,Y_2,Y_3,...,Y_k)$ 中的偶數，這種棋子遊戲的答案與 Nim 遊戲的答案有密切的關係，其中 Nim 遊戲是指“假設共有 r 堆棋子，每堆棋子的個數為 $Z_1,Z_2,Z_3,...,Z_r$ ，甲、乙二個人輪流拿走棋子，每次每個人只能在同一堆棋子中拿走一些棋子(至少一個，也可以拿走整堆)，輸贏結果的規定是誰拿到最後的一個棋子，誰就是贏家”，仿照 Nim 遊戲的證明再去對一般情形函數 $f_A(X_1,X_2,...,X_k)$ 的值作分析，再藉著一些例子樹狀圖來驗證，得到的答案都符合猜測的結果，最後我們證明而得到一般情形的答案。

假設 $k$ 個棋子的編號數所成的集合為 $\{X_1,X_2,...,X_k\}$ ，令 $Y_i=X_i$-i，i=1,2,⋯k。我們去掉 $(Y_1,Y_2,Y_3,...,Y_k)$ 中的偶數，最後我們把剩下的數相同的放在一堆，假設共有 r 堆，每堆的個數為 $Z_1,Z_2,Z_3,...,Z_r$ ，則我們就說：我們得到棋子數的數列 $(Z_1,Z_2,Z_3,...,Z_r)$ 。將這個棋子數的數列 $(Z_1,Z_2,Z_3,...,Z_r)$ 中的各整數 Z 寫成二進制形式(不同位數的按最高位數塡 0 補齊)，按位對齊後逐位計算該位中 1 的個數—偶數個時記為“偶”，奇數個時記為“奇”，稱所得到的串 “偶”“奇”字列為該棋子數的數列的“逐位和”。

**例一**. 假設 $k$ 個棋子的編號數所成的集合為(4,5,6,8,9,11,12,13,14,21,22,23,25,27)，則 $(Y_1,Y_2,Y_3...,Y_k)=$ (3,3,3,4,4,5,5,5,5,11,11,11,12,13)，去掉 $(Y_1,Y_2,Y_3,...,Y_k)$ 中的偶數，我們得到(3,3,3,5,5,5,5,11,11,11,13)，最後我們有棋子數的數列(3,4,3,1)。將這個棋子數的數列(3,4,3,1)中的各整數寫成二進制形式(011,100,011,001)，按 $(Y_1,Y_2,Y_3,...,Y_k)$ 位對齊微逐位計算該位中 1 的個數 最後我們得到這個數列的“逐位和”為(奇，偶，奇)。

**例二.** 假設 $k$ 個棋子的編號數所成的集合爲(5,6,8,9,12,13,14,21,23,24)，則 $(Y_1, Y_2, Y_3, ..., Y_k)$ ={4,4,5,5,7,7,7,13,14,14}，去掉中的偶數，我們得到(5,5,7,7,7,13)，最後我們有棋子數的數列(2,3,1)。將這個棋子數的數列(2,3,1)中的各整數寫成二進制形式(10,11,01)，按位對齊微逐位計算該位中 1 的個數 最後我們得到這個數列的“逐位和” 爲(偶，偶)。

當某人把放在編號爲 $X_i$ 方格的棋子跳過前面的 p 個棋子移到它左邊最近的空格，$1 \le p \le k$，則此人把 $(Y_1, Y_2, Y_3, ..., Y_k)$ 中 $Y_j$ 變成 $Y_j - 1$， $j = i-p, i-p+1, ..., i,$ 這樣造成棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 中只把一個 $Z_j$ 的值變成 $Z_j$ +p+1 或 $Z_j$ -p-1。

例如在例一中，$k$ 個棋子的編號數所成的集合爲(4,5,6,8,9,11,12,13,14,21,22,23,25,27), 當某人把放在編號爲 13 ( $X_8$ =13) 方格的棋子跳過前面的 2 個棋子移到它左邊最近的空格 10，則 $Y_6 = Y_7 = Y_8 = 5$ 都會變成 4，因此 $(Y_1, Y_2, Y_3 ..., Y_k)$ = (3,3,3,4,4,5,5,5,5,11,11, 11,12,13) 變成 (3,3,3,4,4,4,4,4,5,11,11,11,12,13)，即我們把棋子數的數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 由 (3,4,3,1)(3,1,3,1)。

例如在例二中，$k$ 個棋子的編號數所成的集合爲(5,6,8,9,12,13,14,21,23,24)，當某人把放在編號爲 24 ( $X_{10}$ =24) 方格的棋子跳過前面的 1 個棋子移到它左邊最近的空格 22，則 $Y_9 = Y_{10} = 14$ 都會變成 13，因此 $(Y_1, Y_2, Y_3 ..., Y_k)$ ={4,4,5,5,7,7,7,13,14,14}變成 ={4,4,5,5,7,7,7,13,13,13}，即我們把棋子數的數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 由 (2,3,1) 變成(2,3,3)。

反過來說，如果我們想把棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 其中一個 $Z_j$ 的值減去某個數 q ( $Z_j \ge$ q)，我們可以找到相對的移動棋子的方法，使得新的棋子數數列恰好是我們所要的。首先我們要找出對應到 $Z_j$ 的那些 $Y_i$ 的值，在從這些 $Y_i$ 的值中找出前 q 個值來，再找出它們對應 q 個 $X_i$ 的值，然後我們把編號數爲 q 個 $X_i$ 的值中最後一個 $X_i$ 方格的棋子跳過前面的 q-1 個棋子移到它左邊最近的空格。例如在例一中，$k$ 個棋子的編號數所成的集合爲(4,5,6,8,9,11,12,13,14,21,22, 23,25,27), 如果我們想把棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 由 (3,4,3,1) 變成(3,4,1,1)，我們知道對應到 $Z_j$ =3 的是 $Y_{10} = Y_{11} = Y_{12}$ =11，在從這三個 $Y_i$ 的值中找出前二個 $Y_{10} = Y_{11}$ =11 來，它們對應的 $X_i$ 值是 $X_{10}$ 和 $X_{11}$ ( $X_{11}$ = 22)，然後我們把編號數爲 22 方格的棋子跳過前面的 1 個棋子移到它左邊最近的空格編號數爲 20，這樣新的棋子數數列就變成(3,4,1,1)。

如果所有的 $Y_j$ 全是偶數，因此所有 $Z_j$ 的值全爲 0，我們想把棋子數數列變成(1)，我們只要把前面的方格是空格的棋子移到這個空格即可。例如 $k$ 個棋子的編號數所成的集合爲

(5,6,9,12,13)，$(Y_1, Y_2, Y_3 ..., Y_k)$ ={4,4,6,8,8}全是偶數，我們只要編號數為 5 方格的棋子移到空格 4 即可 (移動編號數為 9 或 12 方格的棋子也可以)。

**定理一：假設 $k$ 個棋子的編號數所成的集合為 $\{X_1, X_2, ..., X_k\}$，棋子數數列的 "逐位和" 定義如上。 如果棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 的 "逐位和" 全是偶數(包含所有 $Z_j$ 的值全為 0)，則函數 $f_A(X_1, X_2, ..., X_k)$ 的值為 1，否則函數 $f_A(X_1, X_2, ..., X_k)$ 的值為 0。**

**證明：**我們對 $X_1 + X_2 + X_3 + \cdots + X_k$ 的值作數學歸納法來證明:

"函數 $f_A(X_{1,} X_2, ..., X_k)$ 的值為 1 若且為若

棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 的 "逐位和" 全是偶數(包含所有 $Z_j$ 的值全為 0)"。

由於篇幅關係，只寫出大略的證明。

一、當 $X_1 + X_2 + X_3 + \cdots + X_k$ =3+4+5+⋯+(k+2)時，所有的 $Y_j$ = 2 全是偶數，因此所有 $Z_j$ 的值全為 0，它的 "逐位和" 全是偶數，顯然地甲是輸家，乙是贏家。所以函數 $f_A(X_1, X_2, ..., X_k)$ 的值為 1，因此定理成立。

二、假設 $X_1 + X_2 + X_3 + \cdots + X_k$ = m 時，定理成立。

三、當 $X_1 + X_2 + X_3 + \cdots + X_k$ = m +1 時，我們考慮兩種情形：

1. 棋子數數列的 "逐位和" 全是偶數。如果甲把放在編號為 $X_i$ 方格的棋子跳過前面的 p 個棋子移到它左邊最近的空格，$1 \le p \le k$，則甲把 $(Y_1, Y_2, Y_3, ..., Y_k)$ 中 $Y_j$ 變成 $Y_j - 1$，$j = i - p, i - p + 1, ..., i,$ 這樣造成棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 中只改變一個 $Z_j$ 的值( $Z_j$ 變成 $Z_j$ +p+1 或 $Z_j$ -p-1)。所以新的棋子數數列 $(Z'_1, Z'_2, Z'_3, ..., Z'_r)$ 的 "逐位和" 不會全是偶數，因此函數 $f_A(X_1, X_2, ..., X_k)$ 的值為 1。

2. 當棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 的 "逐位和" 不全是偶數。根據 Nim 遊戲的玩法，我們可以只把其中一個 $Z_j$ 的值減去某個數，使得新的棋子數數列 $(Z'_1, Z'_2, Z'_3, ..., Z'_r)$ 的 "逐位和" 全是偶數，我們可以找到相對的移動棋子的方法，使得新的棋子數數列恰好是我們所要的，所以函數 $f_A(X_1, X_2, ..., X_k)$ 的值為 0。

由數學歸納法得知定理成立。

在例一中 $k$ 個棋子的編號數所成的集合爲(4,5,6,8,9,11,12,13,14,21,22,23,25,27)，它的棋子數的數列之“逐位和”爲(奇，偶，奇)，所以函數 $f_A(4,5,6,8,9,11,12,13,14,21,22,23,25,27) = 0$。

在例二中 $k$ 個棋子的編號數)所成的集合爲(5,6,8,9,12,13,14,21,23,24)，它的棋子數的數列之“逐位和”爲(偶，偶)。所以函數 $f_A(5,6,8,9,12,13,14,21,23,24) = 1$。

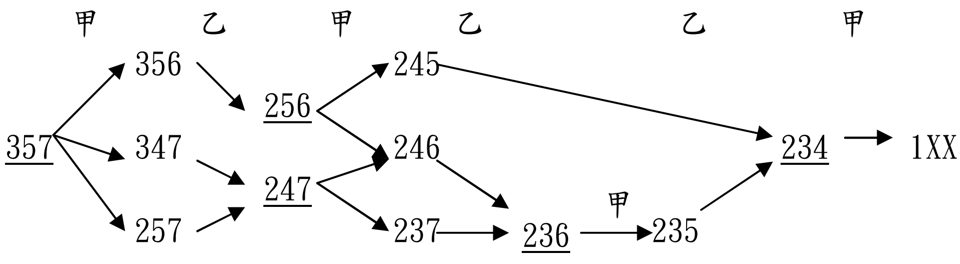
## 5、遊戲 A,B,C 之間的關係

前面我們討論的“輸贏結果”是：甲、乙兩個人之中，誰先將 $k$ 個棋子中任意一個棋子移到第一個方格，誰就是贏家。本節我們將對“結果規定”加以變化，採用 B,C,D 的“輸贏結果”規定，同時研究遊戲 A,B,C,D 之間的關係。

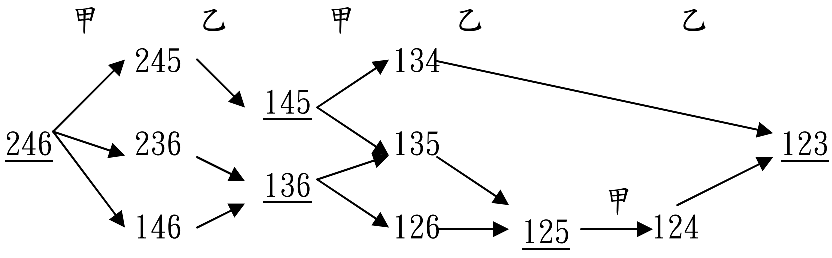
當採用遊戲 B,C,D 的“輸贏結果”時，令 $k$ 個棋子的編號數所成的集合爲 $S = \{X_1, X_2, ..., X_k\}$，輪到甲(先手)移動棋子時，如果他把放在編號爲 $X_i$ 方格的棋子移到它左邊最近的空格，$1 \le i \le k$，此時我們把移動後的 $k$ 個棋子的編號數所成的集合記爲 $S_i$。我們也有

$$f_t(S) = 1 - \max\{f_t(S_i) \mid 1 \le i \le k\}.$$

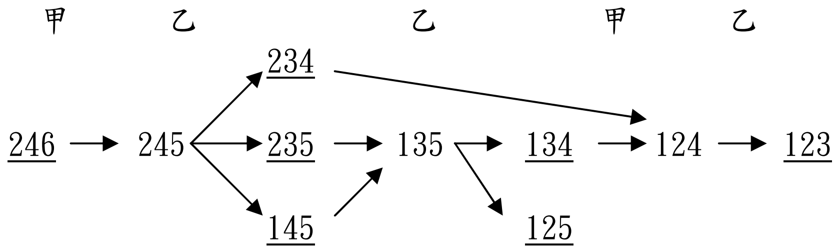
觀察了幾個例子的樹狀圖形(如圖二，三，四)，我們看到 $f_B$，$f_C$ 與 $f_A$ 的樹狀圖形(如圖一，二，三)幾乎一樣。從 $f_A$ 的樹狀圖形中每個數字都減去 1，2 就得到 $f_B$，$f_C$ 的樹狀圖形。但是 $f_D$ 與 $f_A$ 的樹狀圖形(如圖一，四)卻不一樣。




**圖二.** $f_B(3,5,7)$ **的樹狀圖形**




**圖三.** $f_C\,(2,4,6)$ **的樹狀圖形**




**圖四.** $f_D\,(2,4,6)$ **的樹狀圖形**

下面的定理是我們得到遊戲 A,B,C 之間的關係。

**定理二：有一個**$1\times n$**的方格表中放置 k 個棋子，這 k 個棋子的編號數所成的集合爲** $\{X_1,X_2,\ldots,X_k\}$ **，其中** $1<X_1<X_2<\cdots<X_k\le n$ **。則**

$f_c(X_1,X_2,\ldots,X_k)=f_B(X_1+1,X_2+1,\ldots,X_k+1)=f_A(X_1+2,X_2+2,\ldots,X_k+2)$ 。

**證明：**研究遊戲 B 的函數 $f_B$ 的值時，由於輪到某一個人移動棋子，如果他把棋子移到編號爲 1 的方格，則他就是輸家。所以甲、乙兩個人都避免把棋子移到編號爲 1 的方格，也就是說大家都只會把棋子移到編號≥ 2 的方格，直到所有棋子放在編號爲 $2,3,4,\cdots,(k+1)$ 爲止; 也就是說大家都希望把所有棋子放在編號爲 $2,3,4,\cdots,(k+1)$ 。研究遊戲 A 的函數 $f_A$ 的值時，輪到移動棋子的人只好被迫把某一個棋子移到編號爲 1 的方格。由於輪到某一個人移動棋子，如果他把棋子移到編號爲 2 的方格，則他就是輸家。所以甲、乙兩個人都避免把棋子移到編號爲 2 的方格，除非他被迫把某一個棋子移到編號爲 2 的方格。也就是說大家都希望把所有棋子放在編號爲 $3,4,5,\cdots,(k+2)$ 。所以我們有

$$f_c(X_1,X_2,\ldots,X_k)=f_B(X_1+1,X_2+1,\ldots,X_k+1)=f_A(X_1+2,X_2+2,\ldots,X_k+2)$$ 。

由於遊戲 D 遊戲 A,B,C 之間的關係不明顯。找函數 $f_D(X_1, X_2,..., X_k)$ 的值要花費一番功夫，必須重新從樹狀圖著偶手，我們知道

| |
|---|
| $$f_D(a,b,c)=\begin{cases}\chi(a+b+c) & if 1\le a<b-1<c-2;\\ \chi(a(1-c)) & if 1\le a=b-1<c-2,\\ \chi((1-a)c) & if 1\le a<b-1=c-2;\\ 0 & if 1\le a=b-1=c-2.\end{cases}$$ |
| $$f_D(a,b,c,d)=\begin{cases}\chi(a+b+c+d) & if 1\le a<b-1<c-2<d-3\\ \chi(a(c+d-1)) & if 1\le a=b-1<c-2<d-3\\ \chi(c(a+d-1)) & if 1\le a<b-1=c-2<d-3\\ \chi(c(a+b-1)) & if 1\le a<b-1<c-2=d-3\\ \chi(ad) & if 1\le a=b-1=c-2<d-3\\ \chi(c(a-1)) & if 1\le a<b-1=c-2=d-3\\ \chi(bd) & if 1\le a=b-1<c-2=d-3\\ 0 & if 1\le a=b-1=c-2=d-3\end{cases}$$ |

觀察函數 $f_D(X_1, X_2, X_3)$ 和 $f_D(X_1, X_2, X_3, X_4)$ 的值之後，我們也注意到相鄰的兩個棋子的棋子編號是否差 1 扮演著重要的角色，仿照找函數 $f_A(X_1, X_2,..., X_k)$ 的過程，把棋子編號 $X_i$ 減去 i， 也就是說令 $Y_i = X_i$ -i，i=1,2,⋯k。 當 $k = 3,4$ 時，我們重新觀察和分析 $(Y_1,Y_2,Y_3...,Y_k)$ ，藉著 $(Y_1,Y_2,Y_3...,Y_k)$ 的值，我們重新整理函數 $f_D(X_1, X_2,..., X_k)$ 的值，我們得到 $f_D(X_1, X_2,..., X_k) = 1$ 的充要條件:

1.當 $k = 3$ 時， $(Y_1,Y_2,Y_3)$ 中必須沒有相同的奇數，且恰好有奇數個奇數;

2.當 $k = 4$ 時，$(Y_1,Y_2,Y_3,Y_4)$ 中必須沒有大於二個相同的奇數，且

a.如果有相同的奇數， 則恰好必須有兩對(不可以是四個相同的奇數)，

b.如果沒有相同的奇數，則恰好有奇數個奇數;

由於我們發現函數 $f_A(X_1, X_2,..., X_k)$ 的值與“輸贏結果的規定是誰拿到最後的一個棋子，誰就是贏家”的 Nim 遊戲有密切的關係，很快地，我們也猜測函數 $f_D(X_1, X_2,..., X_k)$ 的正確答案與“假設共有 r 堆棋子，每堆棋子的個數爲 $Z_1,Z_2,Z_3...,Z_r$ ，甲、乙二個人輪流拿走棋子，每次每個人只能在同一堆棋子中拿走一些棋子(至少一個，也可以拿走整堆)，輸贏結果的規定是誰拿到最後的一個棋子，誰就是輸家”的 Nim 遊戲有密切的關係，仿照 Nim 遊戲的證明，我們得到 $f_D(X_1, X_2,..., X_k)$ 的正確答案，同時得到遊戲 A 與 D 之間的關係，終於解開了一年多以來在我心中的困惑。

**定理三：** 假設 $k$ 個棋子的編號數所成的集合為 $\{X_1, X_2, ..., X_k\}$，棋子數數列 $(Z_1, Z_2, Z_3 ..., Z_r)$ 的定義如定理一。我們把 $(Z_1, Z_2, Z_3 ..., Z_r)$ 的情形分下列二種來討論函數 $f_D(X_1, X_2, ..., X_k)$ 的值。

一、當 $(Z_1, Z_2, Z_3, ..., Z_r) = (1,1,1,...,1)$。若 r 是奇數，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 1;若 r 是偶數(包含所有 $Z_j$ 的值全為 0)，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 0。

二、當 $(Z_1, Z_2, Z_3 ..., Z_r)$ 中至少有一個數大於 1，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值等於函數 $f_A(X_1, X_2, ..., X_k)$ 的值。也就是說: 如果棋子數數列的 "逐位和" 全是偶數，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 1，否則函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 0。

**證明：** 我們對 $X_1 + X_2 + X_3 + \cdots + X_k$ 的值作數學歸納法來證明:

" 函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 1 若且為若棋子數數列 $(Z_1, Z_2, Z_3, ..., Z_r)$ 是下列兩種情形：

1. (1,1,1,...,1) 且 r 是奇數；

2. 至少有一個 $Z_j$ 大於 1 且它的 "逐位和" 全是偶數(包含所有 $Z_j$ 的值全為 0) "

由於篇幅關係，只寫出大略的證明。

一、當 $X_1 + X_2 + X_3 + \cdots + X_k$=3+4+5+⋯+(k+2)時，所有的 $Y_j$=2 全是偶數，因此所有 $Z_j$ 的值全為 0，它的 "逐位和" 全是偶數，顯然地甲是輸家，乙是贏家。所以函數 $f_D(X_1, X_2, ..., X_k)$ 的值為 1，因此定理成立。

二、假設 $X_1 + X_2 + X_3 + \cdots + X_k = \mathrm{m}$ 時，定理成立。

三、當 $X_1 + X_2 + X_3 + \cdots + X_k = \mathrm{m} + 1$ 時，我們考慮四種情形：

1.當 $(Z_1, Z_2, Z_3, ..., Z_r) = (1,1,1,...,1)$ 且 r 是奇數。如果甲把放在編號為 $X_i$ 方格的棋子移到它左邊最近的空格，則甲在 $(Z_1, Z_2, Z_3 ..., Z_r)$ 中只改變一個 $Z_j$ 的值( $Z_j$ 變成 $Z_j$+p 或 $Z_j$-1)。這樣造成 $f_D(S'_i)$ 的值恆為 0，所以原來 $f_D(X_1, X_2, ..., X_k)$ 的值為 1。

2.當 $(Z_1, Z_2, Z_3 ..., Z_r)$ 中至少有一個數大於 1(不可能恰好只有一個)，而且棋子數數列的 "逐位和" 全是偶數，如果甲把放在編號為 $X_i$ 方格的棋子移到它左邊最近的空格後，在 $(Z_1, Z_2, Z_3, ..., Z_r)$ 中一個 $Z_j$ 的值變成 $Z_j$+p 或 $Z_j$-p。這樣造成新棋子數數列 $(Z'_1, Z'_2, Z'_3, ...)$ 的 "逐位和" 不可能全是偶數。

由數學歸納法知道，上述情形 $f_D(S'_i)$ 的值恆為 0，所以 $f_D(X_1, X_2,..., X_k)$ 的值為 1。

3.當 $(Z_1,Z_2,Z_3,...,Z_r)=(1,1,1,...,1)$ 且 r 是偶數(包含所有 $Z_j$ 的值全為 0)，。甲可以把某個編號為 $X_i$ 方格的棋子移到它左邊最近的空格，使得 $(Z_1,Z_2,Z_3...,Z_r)$ 中只改變一個 $Z_j$ 的值成 $Z_j$-1 或 $Z_j$+1。也就是說新的棋子數數列 $(Z'_1,Z'_2,Z'_3,...)$ 變成 $(1,1,1,...,1)$，其中 1 的個數是奇數，

4.當 $(Z_1,Z_2,Z_3...,Z_r)$ 中至少有一個數大於 1，而且棋子數數列的 "逐位和" 不全是偶數，如果甲把放在編號為 $X_i$ 方格的棋子移到它左邊最近的空格後，則甲在 $(Z_1,Z_2,Z_3...,Z_r)$ 中只改變一個 $Z_j$ 的值( $Z_j$ 變成 $Z_j$ + p 或 $Z_j$ - p)。如果

(1) $(Z_1,Z_2,Z_3...,Z_r)$ 中只有一個數大於 1，則甲可以找到相對的移動棋子的方法，使得新的棋子數數列 $(Z'_1,Z'_2,Z'_3,...)$ 變成 $(1,1,1,...,1)$ ，其中 1 的個數是奇數

(2) $(Z_1,Z_2,Z_3...,Z_r)$ 中至少有二個數大於 1，根據 Nim 遊戲的玩法，我們可以只把其中一個 $Z_j$ 的值減去某個數，使得新的棋子數數列 $(Z'_1,Z'_2,Z'_3,...)$ 的 "逐位和" 全是偶數，這樣造成 $f_D(S'_i)$ 的值為 1，所以原來 $f_D(X_1, X_2,..., X_k)$ 的值為 0。

由數學歸納法得知定理成立

# 伍、研究結果與討論

一、假設 $k$ 個棋子的編號數所成的集合為 $\{X_1, X_2,..., X_k\}$ ，令 $Y_i = X_i$ -i，i=1,2,⋯k. 我們去掉 $(Y_1,Y_2,Y_3...,Y_k)$ 中的偶數，最後我們把剩下的數相同的放在一堆，假設共有 r 堆，每堆的個數為 $Z_1,Z_2,Z_3...,Z_r$ ，則我們就說：我們得到棋子數的數列 $(Z_1,Z_2,Z_3...,Z_r)$ 。將這個棋子數的數列 $(Z_1,Z_2,Z_3...,Z_r)$ 中的各整數 $Z_i$ 寫成二進制形式(不同位數的按最高位數填 0 補齊)，按位對齊微逐位計算該位中 1 的個數—偶數個時記為 "偶" ，奇數個時記為 "奇" ，稱所得到的串 "偶" "奇" 字列為該棋子數的數列的 "逐位和" 。 我們得到

定理一: 假設 $k$ 個棋子的編號數所成的集合為 $\{X_1, X_2, \ldots, X_k\}$，棋子數數列的 "逐位和" 定義如上。 如果棋子數數列 $(Z_1,Z_2,Z_3...,Z_r)$ 的 "逐位和" 全是偶數(包含所有 $Z_j$ 的值全為 0)，則函數 $f_A(X_1, X_2,..., X_k)$ 的值為 1，否則函數 $f_A(X_1, X_2,..., X_k)$ 的值為 0。

二、遊戲A,B,C,D之間的關係：

定理二： $f_c(X_1, X_2, \ldots, X_k) = f_B(X_1+1, X_2+1, \ldots, X_k+1) = f_A(X_1+2, X_2+2, \ldots, X_k+2)$ 。

遊戲C與D之間的關係"輸贏結果的規定是誰最後的一個棋子，誰就是贏家"的Nim遊戲與"輸贏結果的規定是誰最後的一個棋子，誰就是輸家"的Nim遊戲的關係。

定理三： 假設 $k$ 個棋子的編號數所成的集合爲 $\{X_1, X_2, ..., X_k\}$，棋子數數列 $(Z_1, Z_2, Z_3..., Z_r)$ 的定義如定理一。我們把 $(Z_1, Z_2, Z_3..., Z_r)$ 的情形分下列二種來討論函數 $f_D(X_1, X_2, ..., X_k)$ 的值。

一、當 $(Z_1, Z_2, Z_3, ..., Z_r) = (1,1,1,...,1)$。若 r 是奇數 ，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值爲1;若 r 是偶數(包含所有 $Z_j$ 的值全爲0)，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值爲0。

二、當 $(Z_1, Z_2, Z_3..., Z_r)$ 中至少有一個數大於1，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值等於函數 $f_A(X_1, X_2, ..., X_k)$ 的值。也就是說: 如果棋子數數列的"逐位和"全是偶數，則函數 $f_D(X_1, X_2, ..., X_k)$ 的值爲1，否則函數 $f_D(X_1, X_2, ..., X_k)$ 的值爲0。

# 陸、結論與應用

一、在這個研究的過程中，使我覺得作研究需要認真觀察，精於選例子，努力想像，善於聯想，大膽猜測，靈活轉換，嚴格推理，更重要的是要有不屈不撓的毅力及持久的耐心。

二、作完這個研究，使我更熟悉樹狀圖的運用和利用樹狀圖來找出大部份遊戲的必勝策略。

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# Winning Strategies for Games Played with Chips

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**Abstract**

We consider a game played with chips on a strip of squares. The squares are labeled, left to right, with $1, 2, 3, \ldots$, and there are $k$ chips initially placed on distinct squares. Two players take turns to move one of these chips to the next empty square to its left. In this project, we study four different games according to the following rules:

Game $A$: the player who places a chip on square 1 wins;

Game $B$: the player who places a chip on square 1 loses;

Game $C$: the player who finishes up with chips on $12\ldots k$ wins;

Game $D$: the player who finishes up with chips on $12\ldots k$ loses.

After studying the cases $k = 3, 4, 5$ and 6 for Game $A$ and the relation among these four games, we are led to discover the winning strategy of each game for any positive integer $k$. The strategies of Games $A$, $B$ and $C$ are closely related through a forward or backward shifting in position. We also found that such strategies are similar to the type of Nim game that awards the player taking the last chip. Game $D$ is totally different from the rest. To solve this game, we investigate the Nim game that declares the player taking the last chips loser. Amazingly, the strategies of two Nim games can be concisely linked by two equations. Through these two Nim games, we not only find the winning strategy of Game $D$ but also the precise relation between Game $D$ and all others.

**Keywords:** tree diagrams; directed graphs; winning strategy; Nim game

# 1 Motivation

I got an interesting math puzzle from my father as follows:

> A strip of squares labeled, left to right, by $1, 2, \ldots, 8$ with three chips settled on squares $4, 6$ and $8$, initially. Two player take turns to move one chip each time to the next empty square on its left. The game ends with the winner who places a chip on square 1. Who can win this game, the first player or the second?

It spent not much time for me to solve this problem by using a tree diagram, but some native questions came up to me:

> Is there a general winning strategy for any position of three chips? How about we play with more chips on the strip? For $k$ chips and any given position of these chips, is there a general winning strategy? We might claim the player putting a chip on square 1 loser. So we could play different types of such game by changing the rule.

After consulting my math teacher and my father, I started a long-term mathematical quest for solving several general cases of this problem.

# 2 Purpose

Let us consider a general game played with $k$ chips on a strip of squares which are labeled with $1, 2, 3, \ldots$. The chips are located on distinct squares labeled with $P_1, P_2, \ldots, P_k$. Two players, Alice and Bob, take turns to move one of these chips to the next empty square on its left. The rule allows at most one chip per square and allows one chip to jump over the ones laid adjacently on its left. In this project, we would like to study the general cases such that $k$ can be any positive integer and consider four different games according to the following rules:

$A$. the player who places a chip on square 1 wins;

$B$. the player who places a chip on square 1 loses;

$C$. the player who finishes up with chips on $12\ldots k$ wins;

$D$. the player who finishes up with chips on $12\ldots k$ loses.

We should call them Games $A, B, C$, and $D$, respectively. It is same to say that Games $C$ and $D$ are end up with no chips can be moved again.

# 3 Procedures

## (1) Notations

Let $\mathbb{N}$ and $\mathbb{P}$ denote the sets of nonnegative and positive integers, respectively. Also let $\mathbb{B}$ denote the *binary set* $\{0,1\}$ with *Boolean sum* $+_b$ such that $0 +_b 0 = 1 +_b 1 = 0$ and $0 +_b 1 = 1 +_b 0 = 1$.

Initially, $k$ chips are located on the *position array* $P = P_1P_2 \ldots P_k$ where the labels $P_1, P_2, \ldots, P_k \in \mathbb{P}$ are ordered increasingly. The chip on square $m$ is called an *m-chip*. To avoid confusion, sometimes we will also write $P = \{P_1, P_2, \ldots, P_k\}$ as a set. Let $P[i]$ denote the new position array after moving $P_i$-chip to the next empty square on its left. Note that we should always list the labels of $P[i]$ with increasing order. For example, let $P = 24568$ then $P[3] = 23468$ and $P[5] = 24567$. We also define $P[i,j]$ to be the new position array after moving $P_i$-chip and then $P[i]_j$ by the two players. Generally, $P[i_1, i_2, \ldots, i_k]$ means the new position array after moving $P_{i_1}$-chip, $P[i_1]_{i_2}$-chip,..., and $P[i_1, i_2, \ldots, i_{k-1}]_{i_k}$-chip sequentially. Adopting the last example, we have $P[3,4,2] = 12458$.

We should define a function, called *game function*, for each game to tell whether a position array is 'win-able' or not. Let $G \in \{A, B, C, D\}$, we set the game function $f_G : \mathcal{P} \to \mathbb{B}$ where $\mathcal{P} \subset \mathbb{P}^k$ denotes the family of all position arrays. Given any position array $P$, if the next player has a strategy to win Game $G$, then we define $f_G(P) = 0$. On the other hand, whatever chip the next player would move in advance the second player has a winning strategy, then we define $f_G(P) = 1$. For example, $f_A(2P_2P_3 \ldots P_k) = 0$ and $f_A(345 \ldots (k+2)) = 1$.

## (2) Tree Diagrams and Directed Graphs of Game $A$

In this project, I first focused on Game $A$. For abbreviation, let us use A and B to represent Alice and Bob, respectively. Let us use $P \xrightarrow{\text{A}} P'$ (resp. $P \xrightarrow{\text{B}} P'$) to denote the action changing a position array $P$ to a new array $P'$ by Alice (resp. by Bob) where $\longrightarrow$ is a *directed edge*.

I used a tree diagram to solve the puzzle posted at the very beginning, i.e., to find the value $f_A(468)$. Obviously, the player facing a position array $3P_1P_2 \ldots P_k$ would disincline to move 3-chip, so some moves can be ignored in the following diagram.
From this diagram, we not only find $f_A(468) = 1$ but also know that

$$\begin{aligned} f_A(367) &= f_A(358) = f_A(347) = f_A(345) = 1, \text{ and} \\ f_A(467) &= f_A(458) = f_A(368) = f_A(357) \\ &= f_A(356) = f_A(348) = f_A(346) = 0. \end{aligned}$$

Suppose $G \in \{A, B, C, D\}$. Let $\mathcal{W}_G$ denote the set of all position arrays such that

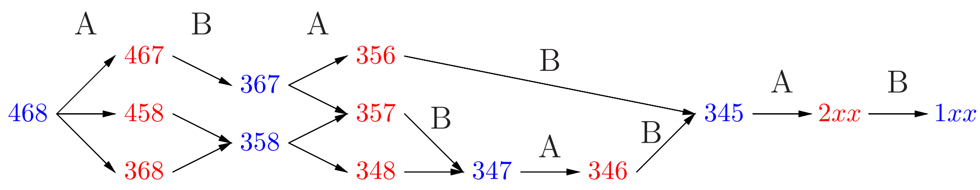


Figure 1: Tree diagram for $f_A(468)$

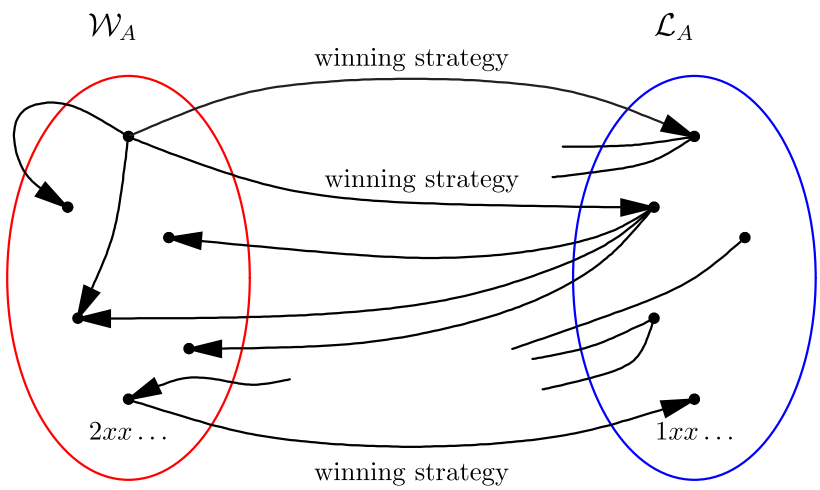


Figure 2: Directed graph formed by $\mathcal{W}_A$ and $\mathcal{L}_A$

the next player has a winning strategy for Game $G$, i.e., $\mathcal{W}_G = {f_G}^{-1}(0)$. Similarly, let $\mathcal{L}_G = {f_G}^{-1}(1)$. We can use a directed graph shown in Figure 2 to demonstrate the game. Except the position array in the initial condition (2) who has no out-edge, every position array of $\mathcal{W}_G$ has at least an out-edge pointing into $\mathcal{L}_G$. On the other hand, every position array of $\mathcal{L}_G$ has its out-edges all pointing into $\mathcal{W}_G$. With this idea, we can re-define each game function as following recursive formula:

$$f_G(P) = 1 - \max\{f_G(P[i]) \mid 1 \leq i \leq k\}, \tag{1}$$

with initial condition of each game

$$\begin{aligned} f_A(1P_2P_3 \dots P_k) &= 1; \\ f_B(1P_2P_3 \dots P_k) &= 0; \\ f_C(12 \dots k) &= 1; \\ f_D(12 \dots k) &= 0. \end{aligned} \tag{2}$$

The reason is easy. If some $i$ makes $P[i] = 1$, the next player would definitely move $i$-chip to win. On the other hand, when every $P[i]$ equals 0, whatever chip the next player move, the following player would still have winning strategy.

From above initial condition, it is easy to find $f_G$ for some trivial position arrays:

$$\begin{aligned} f_A(2P_1P_2\dots P_k) &= 0; & \\ f_B(23\dots(k+1)) &= 1; & \\ f_C(\{1,2,\dots,k,(k+1)\}-\{i\}) &= 0 & \text{for } i=1,2,\dots,k; \\ f_C(12\dots(k-1)(k+2n+b)) &= b+_b 1 & \text{for any } n\in\mathbb{N} \text{ and } b\in\mathbb{B}; \\ f_D(12\dots(k-1)(k+2n+b)) &= b & \text{for any } n\in\mathbb{N} \text{ and } b\in\mathbb{B}. \end{aligned} \tag{3}$$

## (3) The Game Function $f_A(P_1P_2P_3)$

By using Eq. (1) and tree diagrams to investigate many examples, I found the first conjecture which was then proved by induction:

**Proposition 3.1** *Game A has* $\mathcal{W}_A = \{2P_2P_3\}\cup\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3\cup\mathcal{W}_3$ *and* $\mathcal{L}_A=\{1P_2P_3\}\cup\mathcal{L}_1\cup\mathcal{L}_2\cup\mathcal{L}_3\cup\mathcal{L}_3$ *where*

$$\begin{aligned} \mathcal{W}_1 &= \{P_1P_2P_3 \mid 3\le P_1<P_2-1<P_3-2 \text{ and } P_1+P_2+P_3 \text{ is odd}\}; \\ \mathcal{W}_2 &= \{P_1P_2P_3 \mid 3\le P_1=P_2-1<P_3-2 \text{ and } P_1\times P_3 \text{ is even}\}; \\ \mathcal{W}_3 &= \{P_1P_2P_3 \mid 3\le P_1<P_2-1=P_3-2 \text{ and } P_1\times P_3 \text{ is even}\}; \\ \mathcal{W}_4 &= \{P_1P_2P_3 \mid 3\le P_1=P_2-1=P_3-2 \text{ and } P_1 \text{ is even}\}; \\ \mathcal{L}_1 &= \{P_1P_2P_3 \mid 3\le P_1<P_2-1<P_3-2 \text{ and } P_1+P_2+P_3 \text{ is even}\}; \\ \mathcal{L}_2 &= \{P_1P_2P_3 \mid 3\le P_1=P_2-1<P_3-2 \text{ and } P_1\times P_3 \text{ is odd}\}; \\ \mathcal{L}_3 &= \{P_1P_2P_3 \mid 3\le P_1<P_2-1=P_3-2 \text{ and } P_1\times P_3 \text{ is odd}\}; \\ \mathcal{L}_4 &= \{P_1P_2P_3 \mid 3\le P_1=P_2-1=P_3-2 \text{ and } P_1 \text{ is odd}\}. \quad \blacksquare \end{aligned}$$

The belonging of position arrays in $\{2P_2P_3\}$ and $\{1P_2P_3\}$ is directly from (2) and (3). As for the other sets $\mathcal{W}_i$ and $\mathcal{L}_i$, originally, I proved their classification by using induction on $P_1+P_2+P_3$.

After several attempts on simplifying the above proposition, it did come out with an easy formula to treat the case when $P_1\ge 3$.

$$f_A(P_1P_2P_3) = \begin{cases} 1-\chi(P_1\times P_2\times P_3) & \text{if } P_1<P_2-1<P_3-2; \\ \chi(P_1\times P_3) & \text{if } P_1=P_2-1<P_3-2 \text{ or } \\ & \quad\; P_1<P_2-1=P_3-2; \\ \chi(P_1) & \text{if } P_1=P_2-1=P_3-2, \end{cases}$$

where $\chi:\mathbb{N}\to\mathbb{B}$ with $\chi(n)=1$ if $n$ is odd and $\chi(n)=0$ if $n$ is even. It is much easier to prove this formula by the directed graph of Game $A$.

## (4) The Game Function $f_A(P_1P_2\dots P_k)$ with $k = 4, 5, 6$

Obviously, both players are unwilling to move a chip onto square 2 otherwise the following player would definitely win by moving this chip again. In particular, when $P_1 = 3$ no one want to move this 3-chip; so in this case we can ignore 3-chip as well as square 3. With the following formula we represent this idea:

$$f_A(3P_2P_3\dots P_k) = f_A((P_2-1)(P_3-1)\dots(P_k-1)). \tag{4}$$

This formula can help us understanding $f_A(P_1P_2\dots P_k)$ from $f_A(P_1P_2\dots P_{k-1})$. Besides, through the observation of $f_A(P_1P_2P_3)$ in Proposition 3.1, I found a key point—the difference $P_{i+1}-P_i$ being 1 or not plays an important role for the game function $f_A$. According to these two concepts I looked into a large amount of examples and, after a whole-month exploration, I did found a formula of $f_A(P_1P_2P_3P_4)$ which contains 16 different cases. Suppose $P_1 \geq 3$, a general formula of $f_A(P_1P_2P_3P_4)$ was obtained as follows and the proof could by done again by induction or a directed graph.

$$f_A(P_1P_2P_3P_4) = \begin{cases} \chi(P_1+P_2+P_3+P_4-1) & \text{if} \quad \alpha<\beta<\gamma<\delta; \\ \chi(P_1(P_3+P_4)) & \text{if} \quad \alpha=\beta<\gamma<\delta; \\ \chi(P_3(P_1+P_4)) & \text{if} \quad \alpha<\beta=\gamma<\delta; \\ \chi(P_3(P_1+P_2)) & \text{if} \quad \alpha<\beta<\gamma=\delta; \\ \chi(P_1(P_4-1)) & \text{if} \quad \alpha=\beta=\gamma<\delta; \\ \chi(P_1P_3) & \text{if} \quad \alpha<\beta=\gamma=\delta; \\ \chi(P_1+P_3-1) & \text{if} \quad \alpha=\beta<\gamma=\delta; \\ \chi(P_1) & \text{if} \quad \alpha=\beta=\gamma=\delta, \end{cases}$$

where $\alpha = P_1$, $\beta = P_2 - 1$, $\gamma = P_3 - 2$, and $\delta = P_4 - 3$.

Then it took me long time to study game functions $f_A(P_1P_2P_3P_4P_5)$ and $f_A(P_1P_2P_3P_4P_5P_6)$ and it turned out that the former consists 32 cases and the latter 64 shown as follows. For convenience, set $\alpha = P_1$, $\beta = P_2 - 1$, $\gamma = P_3 - 2$, $\delta = P_4 - 3$, $\epsilon = P_5 - 4$, and $\iota = P_6 - 5$. Also we assume $P_1 \geq 3$ and only list the subsets $\mathcal{W}_i$ of the win-able position arrays but skip $\mathcal{L}_i$.

### When will $f_A(P_1P_2P_3P_4P_5)$ equals 0:

$$\begin{aligned} \mathcal{W}_1 &= \{P \mid \alpha<\beta<\gamma<\delta<\epsilon \textit{ and } P_1+P_2+P_3+P_4+P_5 \textit{ is even}\}; \\ \mathcal{W}_2 &= \{P \mid \alpha=\beta<\gamma<\delta<\epsilon \textit{ and } P_1\times(P_3+P_4+P_5-1) \textit{ is even}\}; \\ \mathcal{W}_3 &= \{P \mid \alpha<\beta=\gamma<\delta<\epsilon \textit{ and } P_3\times(P_1+P_4+P_5-1) \textit{ is even}\}; \\ \mathcal{W}_4 &= \{P \mid \alpha<\beta<\gamma=\delta<\epsilon \textit{ and } P_3\times(P_1+P_2+P_5-1) \textit{ is even}\}; \\ \mathcal{W}_5 &= \{P \mid \alpha<\beta<\gamma<\delta=\epsilon \textit{ and } P_5\times(P_1+P_2+P_3-1) \textit{ is even}\}; \end{aligned}$$

$$\mathcal{W}_6 = \{P \mid \alpha = \beta = \gamma < \delta < \epsilon \textit{ and } P_1 \times (P_4 + P_5) \textit{ is even}\};$$
$$\mathcal{W}_7 = \{P \mid \alpha < \beta = \gamma = \delta < \epsilon \textit{ and } P_3 \times (P_1 + P_5 - 1) \textit{ is even}\};$$
$$\mathcal{W}_8 = \{P \mid \alpha < \beta < \gamma = \delta = \epsilon \textit{ and } P_3 \times (P_1 + P_2) \textit{ is even}\};$$
$$\mathcal{W}_9 = \{P \mid \alpha = \beta < \gamma = \delta < \epsilon \textit{ and } P_5 \times (P_1 + P_4) \textit{ is even}\};$$
$$\mathcal{W}_{10} = \{P \mid \alpha = \beta < \gamma < \delta = \epsilon \textit{ and } P_3 \times (P_1 + P_4) \textit{ is even}\};$$
$$\mathcal{W}_{11} = \{P \mid \alpha < \beta = \gamma < \delta = \epsilon \textit{ and } P_1 \times (P_3 + P_4) \textit{ is even}\};$$
$$\mathcal{W}_{12} = \{P \mid \alpha = \beta = \gamma = \delta < \epsilon \textit{ and } P_1 \times P_5 \textit{ is even}\};$$
$$\mathcal{W}_{13} = \{P \mid \alpha = \beta = \gamma < \delta = \epsilon \textit{ and } P_1 \times P_5 \textit{ is even}\};$$
$$\mathcal{W}_{14} = \{P \mid \alpha = \beta < \gamma = \delta = \epsilon \textit{ and } P_1 \times P_5 \textit{ is even}\};$$
$$\mathcal{W}_{15} = \{P \mid \alpha < \beta = \gamma = \delta = \epsilon \textit{ and } P_1 \times P_5 \textit{ is even}\};$$
$$\mathcal{W}_{16} = \{P \mid \alpha = \beta = \gamma = \delta = \epsilon \textit{ and } P_1 \textit{ is even}\}.$$

**When will $f_A(P_1P_2P_3P_4P_5P_6)$ equals 0:**

$$\mathcal{W}_1 = \{P \mid \alpha < \beta < \gamma < \delta < \epsilon < \iota \textit{ and } P_1 + P_2 + P_3 + P_4 + P_5 + P_6 \textit{ is even}\};$$
$$\mathcal{W}_2 = \{P \mid \alpha = \beta < \gamma < \delta < \epsilon < \iota \textit{ and } P_1 \times (P_3 + P_4 + P_5 + P_6 - 1) \textit{ is even}\};$$
$$\mathcal{W}_3 = \{P \mid \alpha < \beta = \gamma < \delta < \epsilon < \iota \textit{ and } P_3 \times (P_1 + P_4 + P_5 + P_6 - 1) \textit{ is even}\};$$
$$\mathcal{W}_4 = \{P \mid \alpha < \beta < \gamma = \delta < \epsilon < \iota \textit{ and } P_3 \times (P_1 + P_2 + P_5 + P_6 - 1) \textit{ is even}\};$$
$$\mathcal{W}_5 = \{P \mid \alpha < \beta < \gamma < \delta = \epsilon < \iota \textit{ and } P_5 \times (P_1 + P_2 + P_3 + P_6 - 1) \textit{ is even}\};$$
$$\mathcal{W}_6 = \{P \mid \alpha < \beta < \gamma < \delta < \epsilon = \iota \textit{ and } P_5 \times (P_1 + P_2 + P_3 + P_4 - 1) \textit{ is even}\};$$
$$\mathcal{W}_7 = \{P \mid \alpha = \beta = \gamma < \delta < \epsilon < \iota \textit{ and } P_1 \times (P_4 + P_5 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_8 = \{P \mid \alpha < \beta = \gamma = \delta < \epsilon < \iota \textit{ and } P_3 \times (P_1 + P_5 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_9 = \{P \mid \alpha < \beta < \gamma = \delta = \epsilon < \iota \textit{ and } P_3 \times (P_1 + P_2 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_{10} = \{P \mid \alpha < \beta < \gamma < \delta = \epsilon = \iota \textit{ and } P_5 \times (P_1 + P_2 + P_3 - 1) \textit{ is even}\};$$
$$\mathcal{W}_{11} = \{P \mid \alpha = \beta < \gamma = \delta < \epsilon < \iota \textit{ and } (P_1 + P_4) \times (P_5 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_{12} = \{P \mid \alpha = \beta < \gamma < \delta = \epsilon < \iota \textit{ and } (P_1 + P_4) \times (P_3 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_{13} = \{P \mid \alpha = \beta < \gamma < \delta < \epsilon = \iota \textit{ and } (P_1 + P_6) \times (P_3 + P_4) \textit{ is even}\};$$
$$\mathcal{W}_{14} = \{P \mid \alpha < \beta = \gamma < \delta = \epsilon < \iota \textit{ and } (P_1 + P_6) \times (P_3 + P_4) \textit{ is even}\};$$
$$\mathcal{W}_{15} = \{P \mid \alpha < \beta = \gamma < \delta < \epsilon = \iota \textit{ and } (P_1 + P_4) \times (P_3 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_{16} = \{P \mid \alpha < \beta < \gamma = \delta < \epsilon = \iota \textit{ and } (P_1 + P_2) \times (P_3 + P_6) \textit{ is even}\};$$
$$\mathcal{W}_{17} = \{P \mid \alpha = \beta = \gamma = \delta < \epsilon < \iota \textit{ and } P_1 \times (P_5 + P_6 - 1) \textit{ is even}\};$$
$$\mathcal{W}_{18} = \{P \mid \alpha < \beta = \gamma = \delta = \epsilon < \iota \textit{ and } P_3 \times (P_1 + P_6 - 1) \textit{is even}\};$$
$$\mathcal{W}_{19} = \{P \mid \alpha < \beta < \gamma = \delta = \epsilon = \iota \textit{ and } P_5 \times (P_1 + \beta) \textit{ is even}\};$$
$$\mathcal{W}_{20} = \{P \mid \alpha = \beta = \gamma < \delta = \epsilon < \iota \textit{ and all } P_2, P_4, P_6 \textit{ are either even or odd}\};$$

$$\begin{aligned}
\mathcal{W}_{21} &= \{P \mid \alpha = \beta = \gamma < \delta < \epsilon = \iota \textit{ and all } P_2, P_4, P_6 \textit{ are either even or odd}\};\\
\mathcal{W}_{22} &= \{P \mid \alpha = \beta < \gamma = \delta = \epsilon < \iota \textit{ and all } P_2, P_4, P_6 \textit{ are either even or odd}\};\\
\mathcal{W}_{23} &= \{P \mid \alpha = \beta < \gamma < \delta = \epsilon = \iota \textit{ and all } P_1, P_3, P_5 \textit{ are either even or odd}\};\\
\mathcal{W}_{24} &= \{P \mid \alpha < \beta = \gamma = \delta < \epsilon = \iota \textit{ and all } P_1, P_3, P_5 \textit{ are either even or odd}\};\\
\mathcal{W}_{25} &= \{P \mid \alpha < \beta = \gamma < \delta = \epsilon = \iota \textit{ and all } P_1, P_3, P_5 \textit{ are either even or odd}\};\\
\mathcal{W}_{26} &= \{P \mid \alpha = \beta < \gamma = \delta < \epsilon = \iota \textit{ and } P_1 + P_3 + P_5 \textit{ is even}\};\\
\mathcal{W}_{27} &= \{P \mid \alpha = \beta = \gamma = \delta = \epsilon < \iota \textit{ and } P_1 \times P_6 \textit{ is even}\};\\
\mathcal{W}_{28} &= \{P \mid \alpha = \beta = \gamma = \delta < \epsilon = \iota \textit{ and } P_1 \times P_6 \textit{ is even}\};\\
\mathcal{W}_{29} &= \{P \mid \alpha = \beta = \gamma < \delta = \epsilon = \iota \textit{ and } P_1 + P_4 \textit{ is even}\};\\
\mathcal{W}_{30} &= \{P \mid \alpha = \beta < \gamma = \delta = \epsilon = \iota \textit{ and } P_1 \times P_6 \textit{ is even}\};\\
\mathcal{W}_{31} &= \{P \mid \alpha < \beta = \gamma = \delta = \epsilon = \iota \textit{ and } P_1 \times P_6 \textit{ is even}\};\\
\mathcal{W}_{32} &= \{P \mid \alpha = \beta = \gamma = \delta = \epsilon = \iota \textit{ and } P_1 \textit{ is even}\}.
\end{aligned}$$

## (5) The Game Function $f_A(P_1P_2\dots P_k)$ for any $k$

The results of the previous two subsections suggest that $f_A(P_1P_2\dots P_k)$ would have a formula with $2^k$ cases. Of course, such complicate formula is not easy to accomplish and we would like to find a more tidy result. Back to the key point mentioned in the last subsection, I found a surprising change on all previous results while investigating a new array $Q = Q_1Q_2\dots Q_k$ where $Q_i = P_i - i$ for $i = 1, 2, \dots, k$. Notice that $0 \le Q_1 \le Q_2 \le \dots \le Q_k$ and we will use $\mathcal{Q}$ to denote the family of all such arrays. Let us define a new game function $F_G : \mathcal{Q} \to \mathbb{B}$ by $F_G(Q) = f_G(P)$. Now the previous results about $f_A$ can be briefly represented as the next proposition.

**Proposition 3.2** *Besides the trivial cases* $F_A(0Q_2Q_3\dots Q_k) = 1$ *and* $F_A(1Q_2Q_3\dots Q_k) = 0$*, the function* $F_A(Q) = 1$ *if and only if* $Q_1 \ge 3$ *and*

1. *when* $k = 3$*, in* $\{Q_1, Q_2, Q_3\}$ *there are no identical odd numbers and the cardinality of odds must be even;*
2. *when* $k = 4$ *or* $5$*, in* $Q$ *the cardinality of any identical odd numbers is not more than two and*
   - (a) *if an identical odd pair exists, there must be two different pairs, or*
   - (b) *if no identical odd pair exists, the cardinality of odds must be even;*
3. *when* $k = 6$*, in* $Q$ *the cardinality of any identical odd numbers is not more than three and*

(a) *if there are three identical odds, then*

i. *there are exactly two groups with three identical odds, or*

ii. *there are one such group, another identical odd pair and a single odd;*

(b) *if no group of three exists but an odd pair do exist, then there must be two different pairs and the cardinality of odd numbers must be even;*

(c) *if there are no identical odds, then the cardinality of odd numbers must be even.*

■

The argument in the above proposition implies that we should only consider odd numbers in $Q$ but ignore all evens. This new idea unveils the close relationship between Game $A$ and the well-known Nim game which awards the player taking the last chip. To discuss our final neat result of Game $A$, we need the binary function $b(n)$ that represents the binary bit-string of a integer $n$. For example, $b(13) = 1101$. We also need a special operation $\oplus$ over bit-strings such that $b_1b_2\dots b_k \oplus d_1d_2\dots d_k = (b_1 +_b d_1)(b_2 +_b d_2)\dots(b_k +_b d_k)$ where $+_b$ is the Boolean sum. For example, $11010 \oplus 01110 = 10100$. Let $O = \{O_1, O_2, \dots, O_r\}$ be the set of the cardinalities of all identical odd numbers in $\{Q_1, Q_2, \dots, Q_k\}$, and let $\Phi(O) = b(O_1) \oplus b(O_2) \oplus \dots \oplus b(O_r)$.

**Theorem 3.3** *The game function $F_A$ satisfies $F_A(0Q_2Q_3\dots Q_k) = 1$, $F_A(1Q_2Q_3\dots Q_k) = 0$ and when $Q_1 \geq 2$*

$$F_A(Q_1Q_2\dots Q_k) = \begin{cases} 1 & \text{if every bit of string } \Phi(O) \text{ is } 0; \\ 0 & \text{else.} \end{cases} \quad \blacksquare \tag{5}$$

We did get a great improvement by using the new type of game function, so we should focus on $F_G$ from now on.

**Example 3.4** *Let $P = \{4, 5, 6, 8, 9, 11, 12, 13, 14, 21, 22, 23, 25, 27\}$. So $Q = \{3, 3, 3, 4, 4, 5, 5, 5, 5, 11, 11, 11, 12, 13\}$ and we get $\{3, 3, 3, 5, 5, 5, 5, 11, 11, 11, 13\}$ after ruling out even numbers. Clearly, $O = \{3, 4, 3, 1\} = \{011, 100, 011, 001\}_b$ and $\Phi(O) = 101$; thus $f_A(P) = F_A(Q) = 0$.*

**Example 3.5** *Suppose $P = \{5, 6, 8, 9, 12, 13, 14, 21, 23, 24\}$. We have $Q = \{4, 4, 5, 5, 7, 7, 7, 13, 14, 14\}$ and $O = \{2, 3, 1\} = \{10, 11, 01\}_b$. Since $\Phi(O) = 000$, the game function $f_A(P) = F_A(Q) = 1$.*

In Eq. (5), the operation $\oplus$ over the set $O = \{O_1, O_2, \dots, O_r\}$ is quite familiar in the famous Nim game which is stated as follows.

There are $r$ heaps of chips each of which has $O_i$ chips ($1 \leq i \leq r$). Two players take turns to move away a part of or even all of chips in any single heap until the last chip(s) is moved away by the winner.

It is well known that a player has a winning strategy if and only if the string $\Phi(O)$ consists at least a 1, i.e., the game function of this Nim game is defined by

$$N_w(O) = \begin{cases} 1 & \textit{if every bit of string } \Phi(O) \textit{ is } 0; \\ 0 & \textit{else.} \end{cases}$$

Therefore, we conclude that $F_A(Q) = N_w(O)$ for $Q_1 \geq 2$.

## (6) The Relation among Games $A$, $B$ and $C$

While studied Game $A$, I also stared looking into the other games. Here are three examples of tree diagrams for Games $B$, $C$ and $D$ in Figures 3–5.

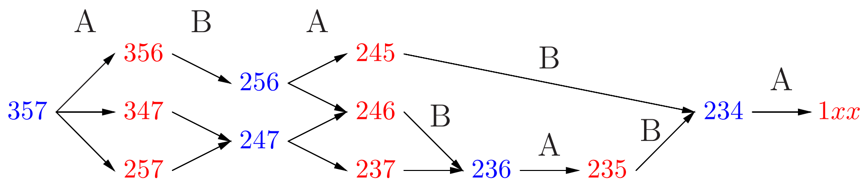


Figure 3: Tree diagram for $f_B(357)$

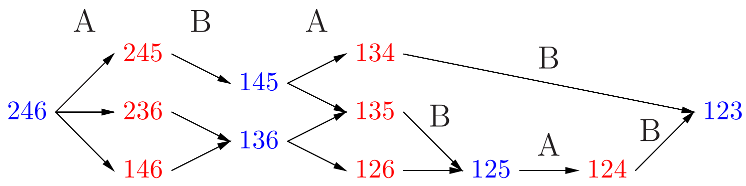


Figure 4: Tree diagram for $f_C(246)$

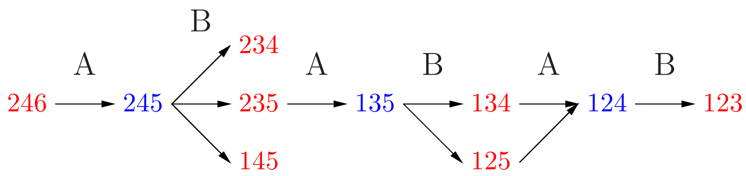


Figure 5: Tree diagram for $f_D(246)$

The next lemma illustrates the interesting relation among Games $A$, $B$ and $C$ through the game functions:

**Lemma 3.6** *Suppose $P = P_1P_2 \dots P_k$ is a position array. The game functions $f_A$, $f_B$ and $f_C$ satisfy the following equation:*

$$\begin{aligned} f_C(P_1P_2 \dots P_k) &= f_B((P_1+1)(P_2+1)\dots(P_k+1)) \\ &= f_A((P_1+2)(P_2+2)\dots(P_k+2)) \end{aligned}$$

**Proof.** For Game $B$, each players is unwilling to move a chip onto square 1 unless he/she faces the position array $23\dots(k+1)$. That means both players wish to accomplish with $P = 23\dots(k+1)$. Notice that, for Game $C$, both players wish to accomplish with $P = 12\dots k$; therefore $f_B((P_1+1)(P_2+1)\dots(P_k+1)) = f_C(P_1P_2\dots P_k)$.

For Game $A$, no player would like to move a chip onto square 2 otherwise the following player will definitely win by moving this chip again. That means both players wish to accomplish with $P = 34\dots(k+2)$. Thus $f_A((P_1+2)(P_2+2)\dots(P_k+2)) = f_C(P_1P_2\dots P_k)$. ■

From above theorem, a direct consequence about $F_G$ is:

**Theorem 3.7** *Suppose $Q = Q_1Q_2\dots Q_k$ is any position array. The game functions $F_A$, $F_B$ and $F_C$ satisfy the following equation:*

$$\begin{aligned} F_C(Q_1Q_2 \dots Q_k) &= F_B((Q_1+1)(Q_2+1)\dots(Q_k+1)) \\ &= F_A((Q_1+2)(Q_2+2)\dots(Q_k+2)). \end{aligned}$$

According to the above equation, we derived the winning strategies of Games $B$ and $C$ as follows:

**Corollary 3.8** *Let $Q = Q_1Q_2\dots Q_k$ be any position array. The game function $F_C$ satisfies*

$$F_C(Q_1Q_2\dots Q_k) = \begin{cases} 1 & \text{if every bit of string } \Phi(O) \text{ is } 0; \\ 0 & \text{else.} \end{cases} \quad ■ \qquad (6)$$

Let $E = \{E_1, E_2, \dots, E_r\}$ be the set of the cardinalities of all identical even numbers in $\{Q_1, Q_2, \dots, Q_k\}$.

**Corollary 3.9** *Let $Q = Q_1Q_2\dots Q_k$ be any position array. The game function $F_B$ satisfies $F_B(0Q_2Q_3\dots Q_k) = 0$ and when $Q_1 \geq 1$*

$$F_B(Q_1Q_2\dots Q_k) = \begin{cases} 1 & \text{if every bit of string } \Phi(E) \text{ is } 0; \\ 0 & \text{else.} \end{cases} \quad ■ \qquad (7)$$

## (7) Game $D$ and the Nim Game

It is amazing that Game $D$ is a totally different game—well, at least to me and my teacher. It took me several months to study this game by the similar method that I treated on Game $A$. Unfortunately, it was nearly in vain until the unveiling of the relation between Game $A$ and Nim game. As a kind of detour, I started to look into a different type of Nim game:

> There are $r$ heaps of chips each of which has $O_i$ chips ($1 \leq i \leq r$). Two players take turns to move away a part of or even all of chips in any single heap until the last chip is moved away by the loser.

Let us use $N_l$ to denote the game function of this Nim game. After searching many references, I did not find anything about $N_l$, even thought most books did mention that this Nim game is totally different from the ordinary Nim game. For this reason, it was worth my while for the quest of $N_l$.

From many experiences of practicing games, I just spent a little of time to investigate $N_l$. The result about $N_l$ was again amazing to me since it was so closed to $N_w$.

**Theorem 3.10** *The winning strategies of the two types of Nims satisfy the following formula:*

$$N_l(O) = \begin{cases} N_w(O) & \text{if } O_i \geq 2 \text{ for some } i; \\ 1 - N_w(O) & \text{if } O_i \leq 1 \text{ for all } i. \end{cases} \tag{8}$$

**Proof.** In case that $O_i = 1$ for all $i$, obviously, $N_w(O) = 0$ if and only if $r$ is odd and $N_l(O) = 0$ if and only if $r$ is even; therefore the second equation holds.

Suppose $O_i \geq 2$ for exactly one fixed $i$. Note that $N_w(O)$ must be zero, because the bit string $\chi(O)$ cannot be all zeros. It depends on how many 1 in $O$. If $O$ has odd number of 1, the next player should move all chips in the $i$th heap; if $O$ has even number of 1, the next player should leave the $i$th heap with one chip. In the both cases, the following player will face a new array containing exactly odd number of 1 and he/she will definitely lose. Therefore we derive $N_l(O) = N_w(O) = 0$.

Now we assume that there are at least two $i$'s such that $O_i \geq 2$. If $N_w(O) = 0$ then the next player would leave a new array $O'$ by moving some chips so that $N_w(O) = 1$ and still $O'_j \geq 2$ for some $j$. Clearly such move could be used for the other type of Nim. On the other hand, if $N_w(O) = 1$ then every kind of move by the next player will always get a new $O'$ with $N_w(O') = 0$ and still $O'_j \geq 2$ for some $j$. For both case, we just demonstrate $N_l(O) = N_w(O)$ by induction. ∎

With the help of the game function $N_l$, I finally found out the winning strategy of Game $D$ as follows:

**Theorem 3.11** *Let $Q = Q_1Q_2 \dots Q_k$ be any position array and let $O = \{O_1, O_2, \dots, O_r\}$ be the set of the cardinalities of all identical odd numbers in $Q$. The game function $F_D$ satisfies*

$$F_D(Q_1Q_2 \dots Q_k) = N_l(O). \qquad \blacksquare \tag{9}$$

# 4 Summary

Here we would like to outline the results of the whole project. For simplicity, we rather focus on the game functions $F_A, F_B, F_C$ and $F_D$ than the original ones $f_G$. One of the fundamental result is the relation among Games $A$, $B$ and $C$:

**Theorem 1** *Suppose $Q = Q_1Q_2 \dots Q_k$ is any position array. The game functions $F_A$, $F_B$ and $F_C$ satisfy the following equation:*

$$\begin{aligned} F_C(Q_1Q_2 \dots Q_k) &= F_B((Q_1+1)(Q_2+1)\dots(Q_k+1)) \\ &= F_A((Q_1+2)(Q_2+2)\dots(Q_k+2)). \quad \blacksquare \end{aligned}$$

The winning strategy of Game $A$ is well studied at the beginning of this project. Then we used the above relation to obtain the winning strategies of Games $B$ and $C$. Here we summarize Theorem 3.3, Corollaries 3.8 and 3.9 as follows by using the game function of ordinary Nim (see page 10.)

**Theorem 2** *Let $Q = Q_1Q_2 \dots Q_k$ be any position array and suppose $Q_1 \geq 2$ for $F_A$ and $Q_1 \geq 1$ for $F_B$. The game functions $F_A, F_B$ and $F_C$ satisfy*

$$\begin{aligned} F_A(Q) &= F_C(Q) = N_w(O) \qquad \text{and} \\ F_B(Q) &= N_w(E), \end{aligned}$$

*where $O = \{O_1, O_2, \dots, O_r\}$ and $E = \{E_1, E_2, \dots, E_r\}$ are the sets of the cardinalities of all identical odd and even numbers, respectively, in $\{Q_1, Q_2, \dots, Q_k\}$.*

Even though Game $D$ is different from the other games, we still found a connection through two different types of Nim games (see the second type of Nim on page 12.) In the following, the first theorem is the relation of two types of Nims and the second theorem is the winning strategy of Game $D$.

**Theorem 3** *The winning strategies of the two types of Nims satisfy the following formula:*

$$N_l(O) = \begin{cases} N_w(O) & \text{if } O_i \geq 2 \text{ for some } i; \\ 1 - N_w(O) & \text{if } O_i \leq 1 \text{ for all } i. \end{cases} \qquad \blacksquare$$

**Theorem 4** *Let $Q = Q_1Q_2 \dots Q_k$ be any position array. The game function $F_D$ satisfies*

$$F_D(Q) = N_l(O),$$

*where $O = \{O_1, O_2, \dots, O_r\}$ is the set of the cardinalities of all identical odd numbers in $\{Q_1, Q_2, \dots, Q_k\}$.* ■

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# Software

QBasic.

# 評　　　語

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