中華民國 第 49 屆中小學科學展覽會

作品說明書

國中組　數學科

030410

平行周邊形

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關鍵詞：平行周邊形迴路

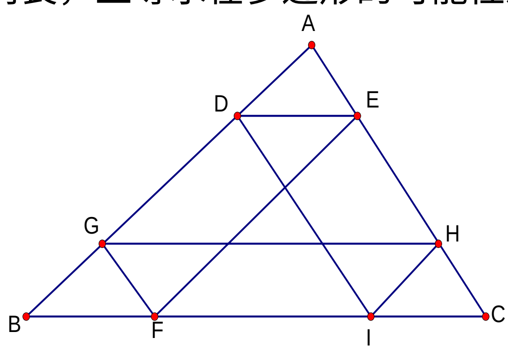
平　行　周　邊　形

摘 要

本研究是探討如何在座標平面上，利用直線方程式，尋求多邊形之內部封閉迴路，其迴路之線段都要平行周圍的邊，並探討其幾何性質。我們先從三角形邊上著手,利用相似形性質推論相關結果, 然後推廣至梯形,最後利用直線方程式推導任意四邊形(兩兩邊互不平行)之封閉迴路存在性與其斜率的關係。

壹、研究動機

在上數學課時, 老師講解一道數學試題:在ΔABC的三邊長分別為 5、6、7；在其邊上任一點D經過迴路$D \Rightarrow E \Rightarrow F \Rightarrow G \Rightarrow H \Rightarrow I \Rightarrow D$，試求其迴路的總路徑長為何?答案恰為$\Delta ABC$的周長 18；因此我們懷疑迴路的路徑是否是剛剛好等於ΔABC周長，並尋求在多邊形的可能性及幾何性質。



貳、研究目的

一、探討在ΔABC中,經過迴路$D \Rightarrow E \Rightarrow F \Rightarrow G \Rightarrow H \Rightarrow I \Rightarrow D$之路徑總長與三角形周長關係並尋求路徑所圍成的區域面積之最小值。

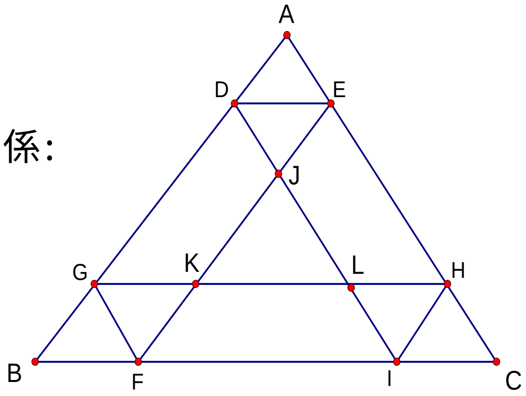
二、探討在梯形$ABCD$中，經過迴路$M \Rightarrow E \Rightarrow G \Rightarrow H \Rightarrow F \Rightarrow M$之路徑總長與梯形周長關係並尋求路徑所圍成的區域面積之最小值。

三、任意四邊形(兩兩互不平行)之封閉平行周邊形之迴路探討。

參、研究設備及器材：直尺、方格紙、Excel 軟體、GSP 繪圖軟體

肆、研究過程與方法

(一)、探討ΔABC邊上一點之平行周邊形迴路



圖(1)

1、 探討在ΔABC中,迴路徑總長與三角形周長關係:

如右圖(1),在$\overline{AB}$邊上任取一點 D;

因為 $\overline{DE}//\overline{BC}//\overline{GH};\overline{EF}//\overline{AB}//\overline{HI};\overline{FG}//\overline{AC}//\overline{DI};$

所以$\overline{DE}+\overline{GH}=\overline{BC}$($\because \overline{DE}=\overline{BF};\overline{GH}=\overline{FC}$)

同理,$\overline{GF}+\overline{DI}=\overline{AC}$($\because \overline{GF}=\overline{AE};\overline{DI}=\overline{EC}$);

$\overline{HI}+\overline{EF}=\overline{AB}$($\because \overline{HI}=\overline{AD};\overline{EF}=\overline{DB}$);

$\therefore (\overline{DE}+\overline{GH})+(\overline{GF}+\overline{DI})+(\overline{HI}+\overline{EF})=\overline{BC}+\overline{AC}+\overline{AB}=\Delta ABC$周長

我們簡記為$L_{D\to I}(\alpha)=L_{\Delta ABC}$ ($\alpha=\dfrac{\overline{AD}}{\overline{AB}}<1$)

$L_{D\to I}(\alpha)=\overline{DE}+\overline{EF}+\overline{FG}+\overline{GH}+\overline{HI}+\overline{ID}$;$L_{\Delta ABC}=\overline{AB}+\overline{BC}+\overline{CA}$

2、 探討在ΔABC中,迴路所圍成的區域面積與三角形面積關係:

如上圖(1),我們簡記為$A_{D\to I}(\alpha)=\Delta DEJ+\Delta GKF+\Delta LHI+\Delta JKL$

(1)在$\alpha\in(0,\dfrac{1}{3})$的條件下,討論$A_{D\to I}(\alpha)$ (令ΔABC面積$=1$平方單位)

$\because \overline{DE}//\overline{BC}\ \therefore \dfrac{\overline{AD}}{\overline{AB}}=\dfrac{\alpha}{1}\Rightarrow\dfrac{\Delta ADE}{\Delta ABC}=\dfrac{\Delta DEJ}{\Delta ABC}=\dfrac{\alpha^2}{1^2}$($\because \Delta ADE\cong\Delta DEJ\sim\Delta ABC$)

且$\Delta GKF\cong\Delta LHI\cong\Delta DEJ$($SSS$全等)$\therefore \dfrac{\Delta GKF}{\Delta ABC}=\dfrac{\Delta LHI}{\Delta ABC}=\dfrac{\Delta DEJ}{\Delta ABC}=\dfrac{\alpha^2}{1^2}=\alpha^2$

又$\Delta JKL\sim\Delta ABC$(AAA相似)所以$\dfrac{\Delta JKL}{\Delta ABC}=\dfrac{\overline{JK}^2}{\overline{AB}^2}=\dfrac{(1-3\alpha)^2}{1^2}$

所以迴路所圍成的區域面積$A_{D\to I}(\alpha)=\Delta DEJ+\Delta GKF+\Delta LHI+\Delta JKL$

與ΔABC的面積比值為

$$\frac{A_{D\to I}(\alpha)}{\Delta ABC}=\frac{A_{D\to I}(\alpha)}{1}=\frac{\Delta DEJ}{\Delta ABC}+\frac{\Delta GKF}{\Delta ABC}+\frac{\Delta LHI}{\Delta ABC}+\frac{\Delta JKL}{\Delta ABC}=\alpha^2+\alpha^2+\alpha^2+(1-3\alpha)^2$$

經過整理並且以配方法求得極值為

$$A_{D\to I}(\alpha)=12\alpha^2-6\alpha+1=12(\alpha-\frac{1}{4})^2+\frac{1}{4} \quad (\frac{\overline{JK}}{\overline{AB}}=1-3\alpha>0\text{；}\alpha<\frac{1}{3})$$

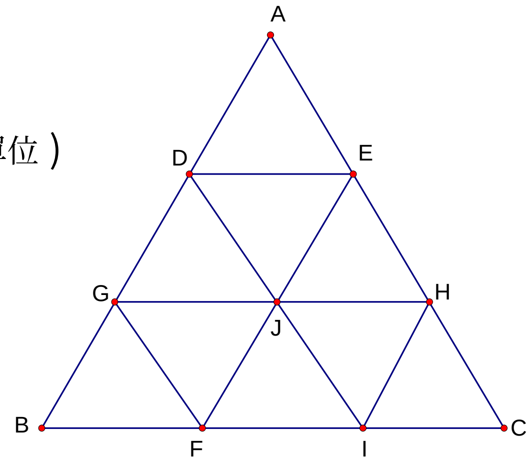
在$\alpha\in(0,\frac{1}{3})$的條件下；當$\alpha=\frac{1}{4}$時,有最小值$\frac{1}{4}$

(2)在$\alpha=\frac{1}{3}$的條件下,(令ΔABC面積$=1$平方單位)

討論$A_{D\to I}(\frac{1}{3})=\frac{1}{3}$如右圖(2)所示

$$A_{D\to I}(\frac{1}{3})=\Delta DEJ+\Delta GJF+\Delta JHI=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$$

$$L_{D\to I}(\frac{1}{3})=\overline{DE}+\overline{EF}+\overline{FG}+\overline{GH}+\overline{HI}+\overline{ID}=L_{\Delta ABC}$$



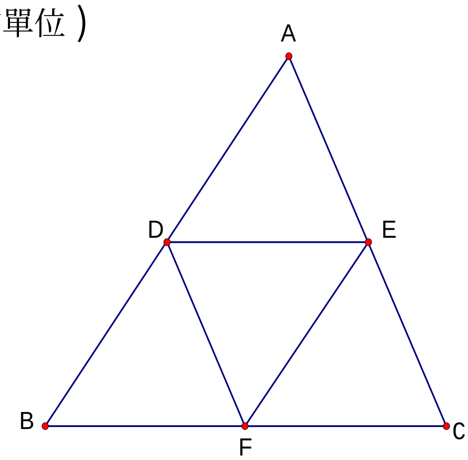
圖(2)

(3) 在$\alpha=\frac{1}{2}$的條件下,(令ΔABC面積$=1$平方單位)

討論$A_{D\to I}(\frac{1}{2})=\frac{1}{4}$如右圖(3)所示

$$A_{D\to I}(\frac{1}{2})=\Delta DEF=\frac{1}{4}\text{；}$$

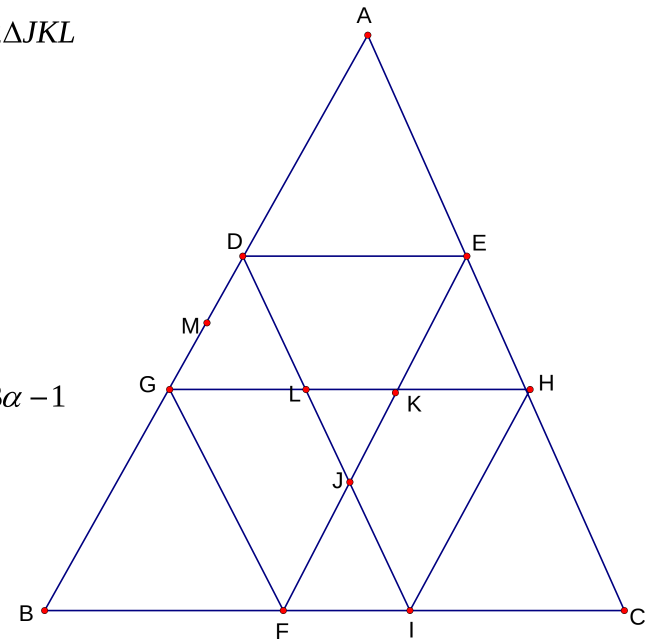
$$L_{D\to I}(\frac{1}{2})=\overline{DE}+\overline{EF}+\overline{FD}=\frac{1}{2}L_{\Delta ABC}$$



圖(3)

(4) 在 $\alpha \in (\frac{1}{3},\frac{1}{2})$ 的條件下，討論 $A_{D\to I}(\alpha)$ （令 ΔABC 面積 $=1$ 平方單位）

$A_{D\to I}(\alpha) = \Delta DEJ + \Delta GKF + \Delta LHI + \Delta JKL - 2\Delta JKL$



圖(4)

如右圖(4)所示，$\Delta DEJ \cong \Delta GKF \cong \Delta LHI$

$\therefore A_{D\to I}(\alpha) = 3\Delta DEJ - 2\Delta JKL$

$$\frac{\Delta DEJ}{\Delta ABC} = \frac{\Delta ADE}{\Delta ABC} = \frac{\overline{AD}^2}{\overline{AB}^2} = (\frac{\overline{AD}}{\overline{AB}})^2 = \alpha^2$$

又 $\overline{EK} = \overline{DG} = 1-2\alpha$ 且 $\overline{JK} = \alpha - (1-2\alpha) = 3\alpha - 1$

$$\therefore \frac{\Delta JKL}{\Delta DEJ} = \frac{(3\alpha-1)^2}{\alpha^2}; \frac{\Delta DEJ}{\Delta ABC} = \alpha^2$$

$$\therefore \frac{\Delta JKL}{\Delta ABC} = \frac{\Delta JKL}{\Delta DEJ} \times \frac{\Delta DEJ}{\Delta ABC} = (3\alpha-1)^2$$

$$\therefore A_{D\to I}(\alpha) = 3\alpha^2 - 2\times(3\alpha-1)^2 = -15\alpha^2 + 12\alpha - 2$$

配方法得 $\therefore A_{D\to I}(\alpha) = -15(\alpha - \frac{2}{5})^2 + \frac{2}{5}$

所以在 $\alpha \in (\frac{1}{3},\frac{1}{2})$ 的條件下；當 $\alpha = \frac{2}{5}$ 時 $A_{D\to I}(\alpha)$ 的最大值為 $\frac{2}{5}$

$L_{D\to I}(\alpha) = \overline{DE} + \overline{EF} + \overline{FG} + \overline{GH} + \overline{HI} + \overline{ID} = L_{\Delta ABC}$

3、結論：（ΔABC 面積 $=1$ 平方單位）

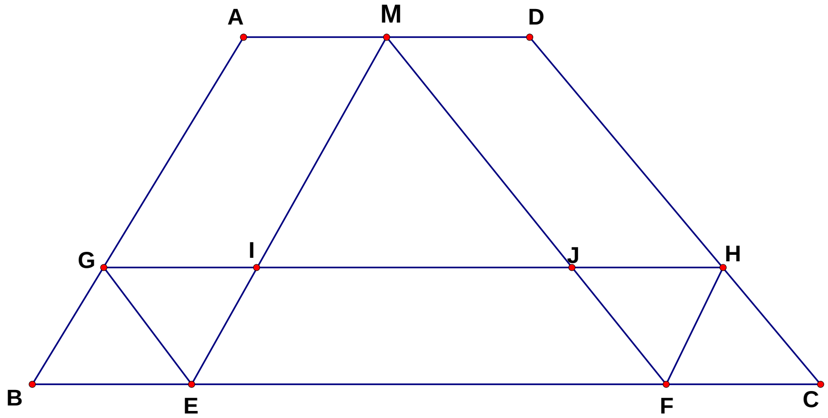
α 值及範圍	$\alpha \in (0,\frac{1}{3})$	$\alpha = \frac{1}{3}$	$\alpha \in (\frac{1}{3},\frac{1}{2})$	$\alpha = \frac{1}{2}$
$A_{D\to I}(\alpha)$	$A_{D\to I}(\alpha) = 12(\alpha-\frac{1}{4})^2 + \frac{1}{4}$	$A_{D\to I}(\frac{1}{3}) = \frac{1}{3}$	$A_{D\to I}(\alpha) = -15(\alpha-\frac{2}{5})^2 + \frac{2}{5}$	$A_{D\to F}(\frac{1}{2}) = \frac{1}{4}$
	$A_{D\to I}$ 極小值 $=\frac{1}{4}$		$A_{D\to I}$ 極大值 $=\frac{2}{5}$	
$L_{D\to I}(\alpha)$	$L_{D\to I}(\alpha) = L_{\Delta ABC}$	$L_{D\to I}(\frac{1}{3}) = L_{\Delta ABC}$	$L_{D\to I}(\alpha) = L_{\Delta ABC}$	$L_{D\to F}(\frac{1}{2}) = \frac{1}{2}L_{\Delta ABC}$

(二)、探討梯形 $ABCD$ 邊上一點之平行周邊形迴路存在性

1、探討:起點 M 是梯形 $\overline{AD}$ 之中點;是否可以構成一個封閉性的迴路?

已知: M 是梯形上底 $\overline{AD}$ 之中點; $\overline{ME}//\overline{AB}$, $\overline{MF}//\overline{DC}$, $\overline{EG}//\overline{CD}$, $\overline{FH}//\overline{BA}$

試證: $\overline{GH}//\overline{BC}$ ($\Leftrightarrow \overline{AG}:\overline{GB}=\overline{DH}:\overline{HC}$)如下圖(5)所示



說明(1)在平行四邊形 $ABEM$ 中

圖(5)

$\because \overline{AM}//\overline{BE},\overline{AB}//\overline{ME};\therefore \overline{AM}=\overline{BE},\overline{AG}=\overline{MI},\overline{GB}=\overline{IE}$

(2)在平行四邊形 $DCFM$ 中

$\because \overline{DM}//\overline{CF},\overline{DC}//\overline{MF};\therefore \overline{DM}=\overline{CF},\overline{DH}=\overline{MJ},\overline{HC}=\overline{JF}$

(3) $\because \overline{GE}//\overline{MF},\therefore \Delta IGE \sim \Delta IJM$ (AAA相似) $\therefore \overline{IG}:\overline{IJ}=\overline{IE}:\overline{IM}$且$\overline{IG}=\overline{AM}$

$\therefore \overline{AM}:\overline{IJ}=\overline{IE}:\overline{IM}$

(4) $\because \overline{HF}//\overline{ME},\therefore \Delta JHF \sim \Delta JIM$ (AAA相似) $\therefore \overline{JH}:\overline{JI}=\overline{JF}:\overline{JM}$且$\overline{JH}=\overline{MD}$

$\therefore \overline{MD}:\overline{JI}=\overline{JF}:\overline{JM}$

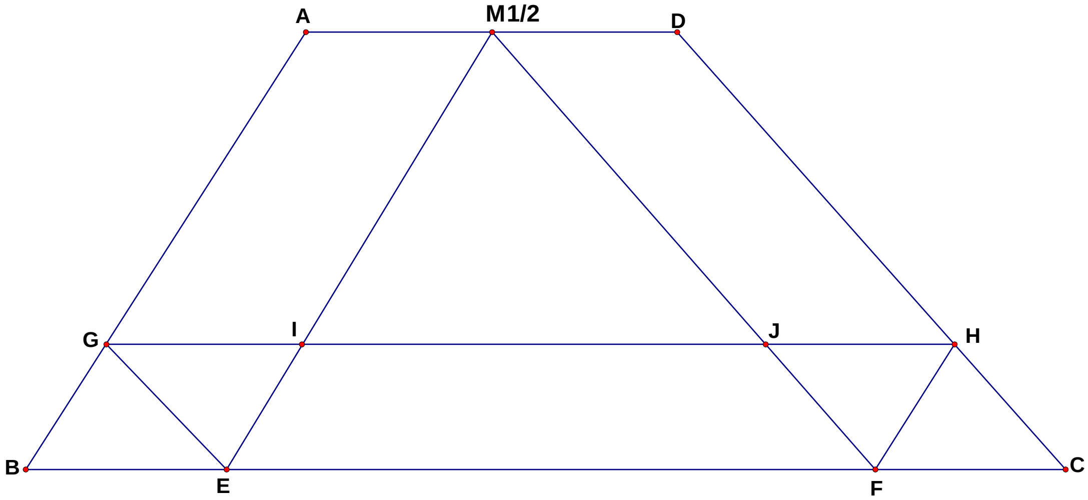
(5)又 M 是梯形上底 $\overline{AD}$ 之中點 $\therefore \overline{AM}=\overline{MD}=\frac{1}{2}\overline{AD}\ \therefore (3)=(4)$

$\therefore \overline{IE}:\overline{IM}=\overline{JF}:\overline{JM}$

(6)又 $\overline{IE}=\overline{GB};\overline{MI}=\overline{AG};\overline{JF}=\overline{HC};\overline{MJ}=\overline{DH}$

所以 $\therefore \overline{IE}:\overline{IM}=\overline{JF}:\overline{JM} \Leftrightarrow \overline{GB}:\overline{AG}=\overline{HC}:\overline{DH}\ \therefore \overline{GH}//\overline{BC}$

2、探討:梯形$ABCD$上底中點$M_{1/2}$之平行周邊形迴路之幾何性質



說明(1) $\Delta IGE \cong \Delta JHF(SSS全等)$　　圖(6)

$\because \overline{IG}=\overline{AM}=\frac{1}{2}\overline{AD}$; $\overline{JH}=\overline{MD}=\frac{1}{2}\overline{AD}$ $\therefore \overline{IG}=\overline{JH}$

在平行四邊形$GEFJ$中;$\overline{GE}=\overline{JF}$ (對邊相等)

在平行四邊形$IEFH$中;$\overline{IE}=\overline{HF}$ (對邊相等)

$\therefore \Delta IGE \cong \Delta JHF(SSS全等)$

(2) 梯形之平行周邊形迴路路徑長$L_{M1/2}$

令$\overline{AD}=a;\overline{BC}=b;\overline{AB}=c;\overline{DC}=d;\frac{\overline{AG}}{\overline{AB}}=\alpha<1$

$$\therefore L_{M1/2}=\overline{ME}+\overline{EG}+\overline{GH}+\overline{HF}+\overline{FM}=(\overline{ME}+\overline{MF}+\overline{GJ})+(\overline{EG}+\overline{JH}+\overline{HF})$$

$$=(\overline{ME}+\overline{MF}+\overline{EF})+(\overline{HC}+\overline{FC}+\overline{HF})=L_{\Delta MEF}+L_{\Delta HCF}$$

$$=c+d+(b-a)+(\frac{1}{2}a+\overline{HC}+\overline{HF})=c+d+(b-\frac{1}{2}a)+\overline{HC}+\overline{GB}$$

$\because \overline{GH}=\overline{BF}=b-\frac{1}{2}a$ 且 $\frac{\overline{AG}}{\overline{AB}}=\alpha<1$ 則 $\overline{AG}:\overline{GB}=\alpha:(1-\alpha)$

$$\therefore \overline{GH}=\frac{\alpha\times b+(1-\alpha)\times a}{\alpha+(1-\alpha)}=\frac{a+(b-a)\alpha}{1}=b-\frac{1}{2}a$$

$$\therefore (b-a)\alpha=b-\frac{3}{2}a \therefore \alpha=\frac{2b-3a}{2(b-a)}=1-\frac{1}{2}(\frac{a}{b-a}) \quad \therefore 1-\alpha=\frac{1}{2}(\frac{a}{b-a})$$

$\because \dfrac{\overline{AG}}{\overline{AB}} = \alpha$ <1 $\therefore \dfrac{\overline{GB}}{\overline{AB}} = 1-\alpha \therefore \overline{GB} = (1-\alpha)\overline{AB} = \dfrac{1}{2}(\dfrac{a}{b-a}) \times c$

同理 $\therefore \dfrac{\overline{HC}}{\overline{DC}} = 1-\alpha \therefore \overline{HC} = (1-\alpha)\overline{DC} = \dfrac{1}{2}(\dfrac{a}{b-a}) \times d$

$$\therefore L_{M1/2} = L_{\Delta MEF} + L_{HCF} = [c+d+(b-a)] + \left[\frac{1}{2}a + \frac{1}{2}(\frac{a}{b-a})c + \frac{1}{2}(\frac{a}{b-a})d\right]$$

$$= c+d+b-\frac{1}{2}a+\frac{1}{2}a(\frac{c+d}{b-a}) = a+b+c+d+\frac{a}{2}\left(\frac{c+d}{b-a}-3\right)$$

$$= (a+b+c+d)+\frac{a}{2}(\frac{c+d}{b-a}-3)$$

(3) 梯形之平行周邊形迴路所圍成的區域面積 $A_{M1/2}$

令梯形 $ABCD$面積 $=1$ 平方單位;梯形 ABCD 的高為 h

則 $\dfrac{(a+b)}{2}h = 1 \Rightarrow h = \dfrac{2}{(a+b)}$ 且 $\alpha = \dfrac{2b-3a}{2(b-a)} = 1-\dfrac{1}{2}(\dfrac{a}{b-a})$ $\quad \therefore 1-\alpha = \dfrac{1}{2}(\dfrac{a}{b-a})$

$$\Delta IGE = \Delta JHF = \frac{a}{2}\times(\frac{1-\alpha}{1})h\times\frac{1}{2} = \frac{a}{2}\times\left[\frac{1}{2}(\frac{a}{b-a})\right]\times\frac{2}{(a+b)}\times\frac{1}{2}$$

$$\therefore \Delta IGE = \Delta JHF = \frac{1}{4}(\frac{a^2}{b^2-a^2})$$

又 $\Delta IJM \sim \Delta IGE(AAA相似) \therefore \Delta IJM : \Delta IGE = \alpha^2 : (1-\alpha)^2$

$$\therefore \Delta IJM = \frac{\alpha^2}{(1-\alpha)^2}\times\Delta IGE = (\frac{\alpha}{1-\alpha})^2\times\frac{1}{4}(\frac{a^2}{b^2-a^2}) = \left[\frac{1-\frac{1}{2}(\frac{a}{b-a})}{\frac{1}{2}(\frac{a}{b-a})}\right]^2\times\frac{1}{4}(\frac{a^2}{b^2-a^2})$$

$$\therefore \Delta IJM = \frac{1}{4}\times\frac{(2b-3a)^2}{b^2-a^2}$$

區域面積 $A_{M1/2} = \Delta IGE + \Delta JHF + \Delta IJM$

$$= \frac{1}{4}(\frac{a^2}{b^2-a^2}) + \frac{1}{4}(\frac{a^2}{b^2-a^2}) + \frac{1}{4}\times\frac{(2b-3a)^2}{b^2-a^2}$$

$= \dfrac{1}{4}\times\dfrac{11a^2 12ab+4b^2}{b^2-a^2}$ 上下同除 b^2 (令 $\dfrac{a}{b} = t$ <1)

得 $A_{M1/2}(t) = \dfrac{1}{4}\times(\dfrac{11t^2-12t+4}{1-t^2})$

我們暫時令 $A_{M1/2}(t)=k$；$k\in$ 實數R

$\therefore \frac{1}{4}\times(\frac{11t^2-12t+4}{1-t^2})=k$ 交叉相乘得 $11t^2-12t+4=4k-4kt^2$

$\Rightarrow (11+4k)t^2-12t+(4-4k)=0$ 因為此一元二次方程式的根 $k\in R$

所以判別式 $D=(-12)^2-4(11+4k)\times(4-4k)\geq 0$

$\therefore 9-(1-k)\times(11+4k)\geq 0 \Rightarrow 4k^2+7k-2\geq 0$

$\therefore (4k-1)\times(k+2)\geq 0 \therefore k\geq\frac{1}{4}$ 或 $k\leq -2$（不合）

所以 $A_{M1/2}(t)=k$ 的最小值為 $\frac{1}{4}$

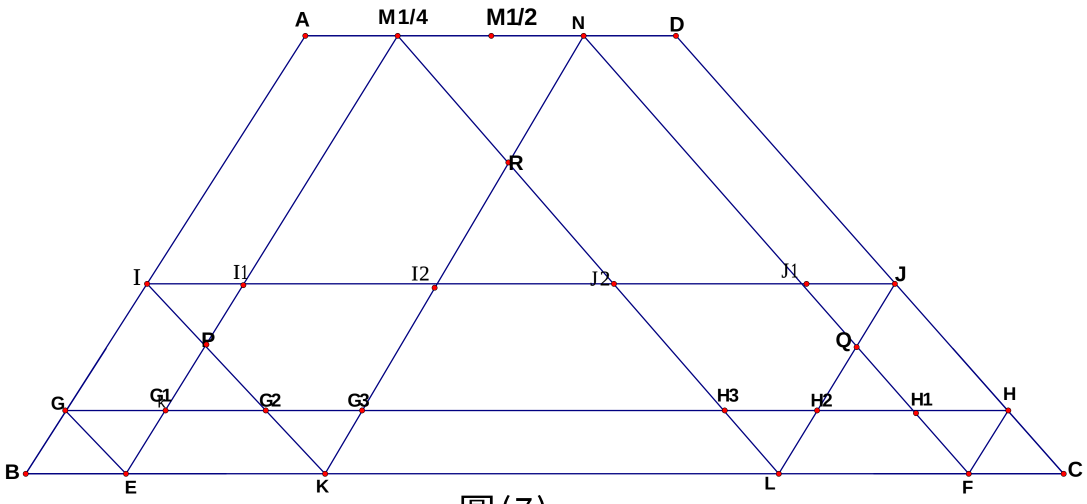
$\therefore \frac{1}{4}\times(\frac{11t^2-12t+4}{1-t^2})=\frac{1}{4}\Leftrightarrow 11t^2-12t+4=1-t^2$

$\therefore 12t^2-12t+3=0\Leftrightarrow 4t^2-4t+1=0$

$\therefore (2t-1)^2=0 \therefore t=\frac{1}{2}$

當 $t=\frac{a}{b}=\frac{1}{2}\Leftrightarrow b=2a$；$A_{M1/2}$ 的最小值為 $\frac{1}{4}$

3、探討：梯形 $ABCD$ 上底的點 $M_{1/4}$ 之平行周邊形迴路之幾何性質



圖(7)

(1) 梯形之平行周邊形迴路路徑長 $L_{M1/4}$

令 $\overline{AD}=a;\overline{BC}=b;\overline{AB}=c;\overline{DC}=d;\dfrac{\overline{AG}}{\overline{AB}}=\alpha<1$

$$L_{M1/4}=\overline{M_{1/4}E}+\overline{NK}+\overline{M_{1/4}L}+\overline{NF}+\overline{GE}+\overline{IK}+\overline{HF}+\overline{JL}+\overline{GH}+\overline{IJ}$$

$\overline{M_{1/4}E}=\overline{NK}=\overline{AB}=c$ 且 $\overline{M_{1/4}L}=\overline{NF}=\overline{CD}=d$

$\overline{GH}=\overline{BF}=b-\dfrac{1}{4}a$；$\overline{IJ}=\overline{BL}=b-\dfrac{3}{4}a$

$$\therefore L_{M1/4}=c+c+d+d+\overline{GE}+\overline{IK}+\overline{HF}+\overline{JL}+(b-\frac{1}{4}a)+(b-\frac{3}{4}a)$$

$$=2c+2d+2b-a+(\overline{GE}+\overline{IK})+(\overline{HF}+\overline{JL})$$

在 $\Delta BGE\sim\Delta BIK$ 中；$\overline{BE}:\overline{BK}=\dfrac{a}{4}:\dfrac{3a}{4}=1:3\Leftrightarrow\overline{IK}=3\overline{GE}\Rightarrow\overline{PK}=2\overline{GE}$

同理在 $\Delta CHF\sim\Delta CJL$ 中；$\overline{CF}:\overline{CL}=\dfrac{a}{4}:\dfrac{3a}{4}=1:3\Leftrightarrow\overline{JL}=3\overline{HF}\Rightarrow\overline{QL}=2\overline{HF}$

$\because\Delta EPK\sim\Delta EM_{1/4}L$；$\overline{EK}:\overline{EL}=\dfrac{a}{2}:(b-a)=\overline{PK}:d\Leftrightarrow\overline{PK}=\dfrac{ad}{2(b-a)}\Rightarrow\overline{GE}=\dfrac{ad}{4(b-a)}$

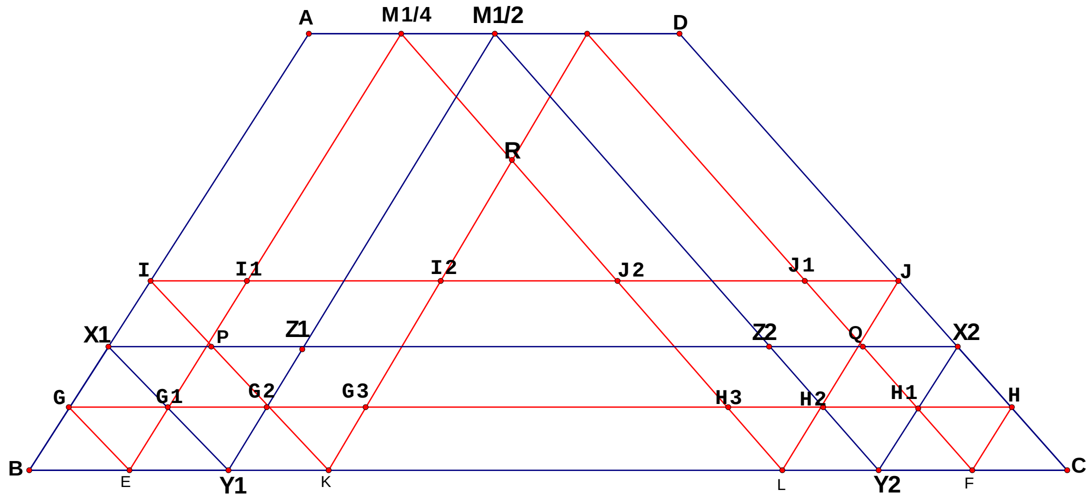
$\because\Delta FQL\sim\Delta FNK$；$\overline{FL}:\overline{FK}=\dfrac{a}{2}:(b-a)=\overline{QL}:c\Leftrightarrow\overline{QL}=\dfrac{ac}{2(b-a)}\Rightarrow\overline{HF}=\dfrac{ac}{4(b-a)}$

$$L_{M1/4}=2c+2d+2b-a+4\times\frac{ad}{4(b-a)}+4\times\frac{ac}{4(b-a)}=2(c+d)+(2b-a)+\frac{(c+d)a}{b-a}$$

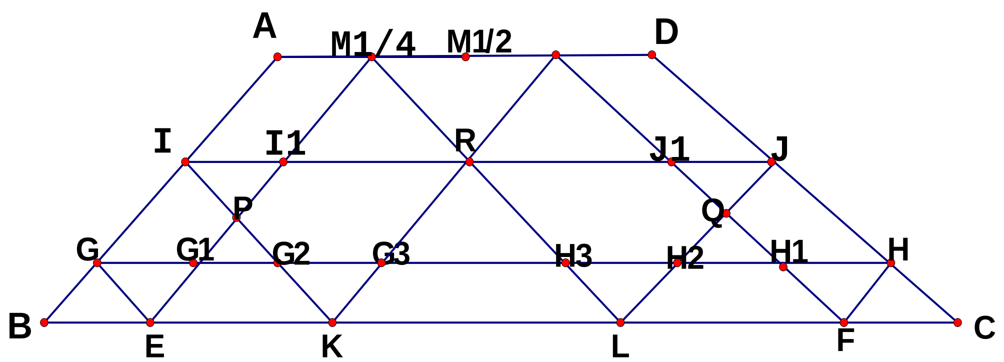
$$L_{M_{1/4}}=2c+2d+2b-a+\frac{ad}{b-a}+\frac{ac}{b-a}=2c+2d+2b+(\frac{a}{b-a})(a+c+d-b)$$

(2) 梯形之平行周邊形迴路所圍成的區域面積 $A_{M1/4}$

令梯形 $ABCD$面積 $=1$ 平方單位；　圖(8)



① 考慮 $\overline{I_2J_2}=0 \Leftrightarrow b-\frac{9}{4}a=0 \therefore b=\frac{9}{4}a$ 則上圖(8)退化成下圖(9)



圖(9)

則 ΔRI_2J_2退化至一點R

$$\therefore A_{M1/4}=(\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1)+(\Delta M_{1/2}RN+\Delta G_3RH_3+\Delta RJ_1N)$$
$$+(\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1)$$

且 $\Delta GEG_1 \cong \Delta G_1PG_2 \cong \Delta G_2KG_3 \cong \Delta IPI_1 \cong \Delta HFH_1 \cong \Delta H_1QH_2 \cong \Delta H_2LH_3 \cong \Delta JQJ_1$

在上圖(8),利用面積轉換可得 $\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\Delta X_1Y_1Z_1$

同理 $\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\Delta X_2Y_2Z_2$

我們利用在 $A_{M1/2}$ 結果，得知 $\Delta X_1Y_1Z_1=\dfrac{a^2}{4(b^2-a^2)}$ ；$\Delta X_2Y_2Z_2=\dfrac{a^2}{4(b^2-a^2)}$

所以 $\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\dfrac{a^2}{4(b^2-a^2)}$ ；

$\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\dfrac{a^2}{4(b^2-a^2)}$。

又 $\overline{I_1R}=\overline{J_1R}=\overline{G_3H_3}=\dfrac{a}{2}\ \therefore \Delta M_{1/2}RN\cong\Delta G_3RH_3\cong\Delta RJ_1N$ (ASA全等)

$\therefore \Delta M_{1/2}RN=\Delta G_3RH_3=\Delta RJ_1N=\dfrac{a^2}{4(b^2-a^2)}$

將 $b=\dfrac{9}{4}a$ 代入 $A_{M1/4}=\dfrac{5a^2}{4(b^2-a^2)}$ 得 $A_{M1/4}=\dfrac{5a^2}{4\left[(\frac{81}{16}a^2)-a^2\right]}=\dfrac{5a^2}{\frac{65a^2}{4}}=\dfrac{4}{13}$

② 考慮 $\overline{I_2J_2}>0\Leftrightarrow b-\dfrac{9}{4}a>0\Leftrightarrow b>\dfrac{9}{4}a\Leftrightarrow(\dfrac{a}{b})<\dfrac{4}{9}$ 令 $t=\dfrac{a}{b}<\dfrac{4}{9}$

如上圖(8)所示

$\therefore A_{M1/4}=(\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1)+(\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1)$
$+(\Delta I_1M_{1/4}J_2+\Delta I_2NJ_1+\Delta G_3RH_3)-2\Delta I_2RJ_2$

且 $\Delta GEG_1\cong\Delta G_1PG_2\cong\Delta G_2KG_3\cong\Delta IPI_1\cong\Delta HFH_1\cong\Delta H_1QH_2\cong\Delta H_2LH_3\cong\Delta JQJ_1$

利用面積轉換可得 $\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\Delta X_1Y_1Z_1$

同理 $\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\Delta X_2Y_2Z_2$

所以 $\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\dfrac{a^2}{4(b^2-a^2)}$ ；

$\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\dfrac{a^2}{4(b^2-a^2)}$

同理 $\Delta I_1M_{1/4}J_2\cong\Delta I_2NJ_1\cong\Delta G_3RH_3$ (ASA全等) 且 $\Delta X_1Y_1Z_1=\dfrac{a^2}{4(b^2-a^2)}$

因為 $\Delta G_3RH_3\sim\Delta I_2RJ_2\sim\Delta X_1Y_1Z_1$ 所以面積比為

$\Delta G_3RH_3:\Delta I_2RJ_2:\Delta X_1Y_1Z_1=\overline{G_3H_3}^2:\overline{I_2J_2}^2:\overline{X_1Z_1}^2$

又 $\overline{G_3H_3}=b-\dfrac{7}{4}a;\overline{I_2J_2}=b-\dfrac{9}{4}a;\overline{X_1Z_1}=\dfrac{a}{2}$

所以$\Delta G_3RH_3=\dfrac{(b-\frac{7}{4}a)^2}{b^2-a^2}$且$\Delta I_2RJ_2=\dfrac{(b-\frac{9}{4}a)^2}{b^2-a^2}$

$$\therefore A_{M1/4}=2\times\frac{\frac{a^2}{4}}{b^2-a^2}+3\times\frac{(b-\frac{7}{4}a)^2}{b^2-a^2}-2\times\frac{(b-\frac{9}{4}a)^2}{b^2-a^2}$$

經過化簡以後;得$A_{M1/4}=\dfrac{\frac{-7}{16}a^2-\frac{3}{2}ab+b^2}{b^2-a^2}$

再將分子分母上下同除b^2並令$\dfrac{a}{b}=t$且$t<\dfrac{4}{9}$

轉換成$A_{M1/4}(t)=\dfrac{\frac{-7}{16}t^2-\frac{3}{2}t+1}{1-t^2}$並令$A_{M1/4}(t)=k$

$\dfrac{\frac{-7}{16}t^2-\frac{3}{2}t+1}{1-t^2}=k$交叉相乘得$\dfrac{-7}{16}t^2-\dfrac{3}{2}t+1=k-kt^2$

$\Rightarrow -7t^2-24t+16=16k-16kt^2\Rightarrow(16k-7)t^2-24t+16(1-k)=0$

$\because t\in R$所以判別式$D=(-24)^2-4\times(16k-7)\times16(1-k)\geq0$

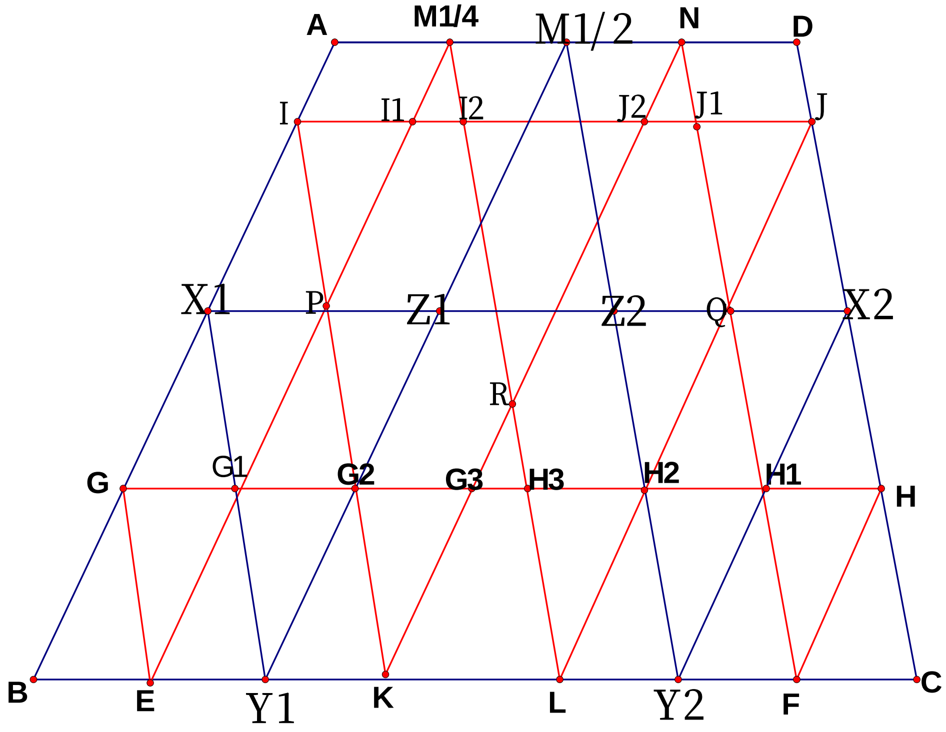
$\Rightarrow 9\geq(1-k)\times(16k-7)\Rightarrow16k^2-23k+16\geq0$

又因$(-23)^2-4\times16\times16=-495<0$;$k^2$的係數$16>0$

所以$16k^2-23k+16$恆正,k並無極值。

故在$t<\dfrac{4}{9}$的條件下$A_{M1/4}(t)=\dfrac{\frac{-7}{16}t^2-\frac{3}{2}t+1}{1-t^2}$並無極值。

③考慮$b-\frac{9}{4}a<0 \Leftrightarrow \frac{a}{b}>\frac{4}{9}$ 令$t=\frac{a}{b} \Leftrightarrow t>\frac{4}{9}$



圖(10)

如上圖(10)所示

$\therefore A_{M1/4}=(\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1)+(\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1)$
$+(\Delta I_1M_{1/4}J_2+\Delta I_2NJ_1+\Delta G_3RH_3)+\Delta I_2RJ_2$

且$\Delta GEG_1 \cong \Delta G_1PG_2 \cong \Delta G_2KG_3 \cong \Delta IPI_1 \cong \Delta HFH_1 \cong \Delta H_1QH_2 \cong \Delta H_2LH_3 \cong \Delta JQJ_1$

利用面積轉換可得$\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\Delta X_1Y_1Z_1$

同理$\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\Delta X_2Y_2Z_2$

所以$\Delta GEG_1+\Delta G_1PG_2+\Delta G_2KG_3+\Delta IPI_1=\frac{a^2}{4(b^2-a^2)}$；

$\Delta HFH_1+\Delta H_1QH_2+\Delta H_2LH_3+\Delta JQJ_1=\frac{a^2}{4(b^2-a^2)}$

同理$\Delta I_1M_{1/4}J_2 \cong \Delta I_2NJ_1 \cong \Delta G_3RH_3(ASA全等)$且$\Delta X_1Y_1Z_1=\frac{a^2}{4(b^2-a^2)}$

因為$\Delta G_3RH_3 \sim \Delta I_2RJ_2 \sim \Delta X_1Y_1Z_1$所以面積比為

$\Delta G_3RH_3:\Delta I_2RJ_2:\Delta X_1Y_1Z_1=\overline{G_3H_3}^2:\overline{I_2J_2}^2:\overline{X_1Z_1}^2$

又 $\overline{G_3H_3}=b-\frac{7}{4}a;\overline{I_2J_2}=\frac{9}{4}a-b;\overline{X_1Z_1}=\frac{a}{2}$

所以 $\Delta G_3RH_3=\dfrac{(b-\frac{7}{4}a)^2}{b^2-a^2}$ 且 $\Delta I_2RJ_2=\dfrac{(\frac{9}{4}a-b)^2}{b^2-a^2}$

$\therefore A_{M1/4}=2\times\dfrac{\frac{a^2}{4}}{b^2-a^2}+3\times\dfrac{(b-\frac{7}{4}a)^2}{b^2-a^2}+\dfrac{(\frac{9}{4}a-b)^2}{b^2-a^2}$

經過化簡以後;得 $A_{M1/4}=\dfrac{\frac{59}{4}a^2-15ab+4b^2}{b^2-a^2}$

再將分子分母上下同除 b^2 並令 $\frac{a}{b}=t$ 且 $t>\frac{4}{9}$

轉換成 $A_{M1/4}(t)=\dfrac{\frac{59}{4}t^2-15t+4}{1-t^2}$ 並令 $A_{M1/4}(t)=k$

交叉相乘得 $\frac{59}{4}t^2-15t+4=k-kt^2$

$\Rightarrow 59t^2-60t+16=4k-4kt^2\Rightarrow(59+4k)t^2-60t+4(4-k)=0$

$\because t\in R$ 所以判別式 $D=(-60)^2-4\times(59+4k)\times4(4-k)\geq0$

$\Rightarrow 225\geq(4-k)\times(59+4k)\Rightarrow 4k^2+43k-11\geq0$

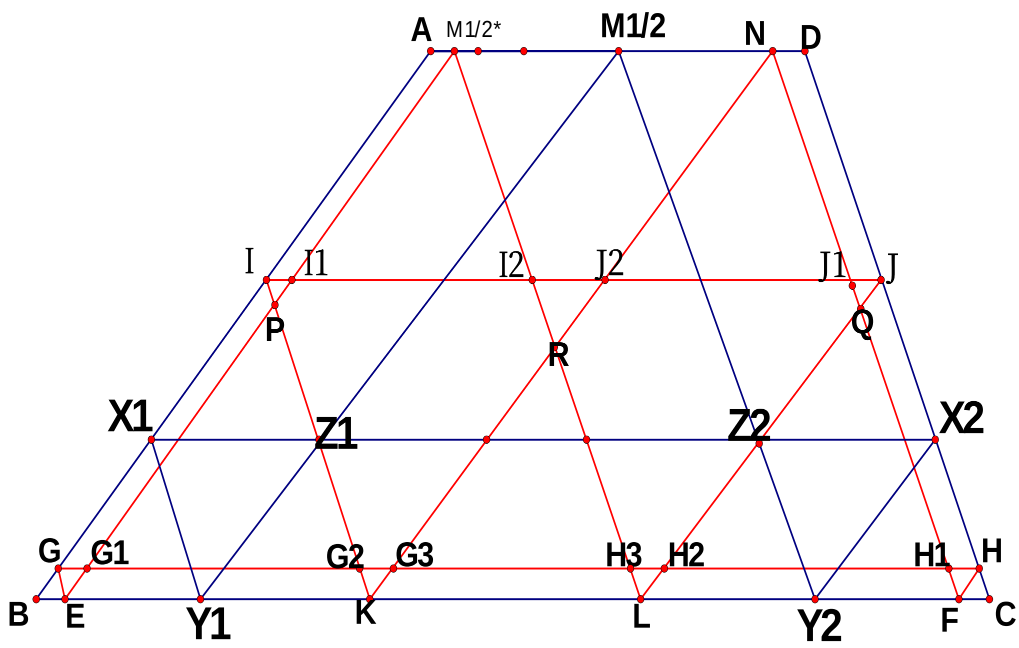
$(k+11)\times(4k-1)\geq0\ \therefore k\geq\frac{1}{4}$

若 $A_{M1/4}(t)=\dfrac{\frac{59}{4}t^2-15t+4}{1-t^2}=\frac{1}{4}\Leftrightarrow 59t^2-60t+16=1-t^2$

$\Rightarrow 60t^2-60t+15=0\Rightarrow 4t^2-4t+1=0\Rightarrow(2t-1)^2=0$

$\therefore t=\frac{1}{2}>\frac{4}{9}$ 時; $A_{M1/4}(t)$ 有最小值 $\frac{1}{4}$

4、探討:梯形 $ABCD$ 上底的點 $M_{1/2^n}$ 之平行周邊形迴路之幾何性質



圖(11)

(1) 梯形之平行周邊形迴路路徑長 $L_{M1/2^n}$

①如上圖(11) $\overline{AD}=a;\overline{BC}=b$;$\overline{AM_{1/2^n}}=\dfrac{1}{2^n}a$ 且 $\forall n\geq 2;n\in N$。

$$L_{M1/2^n}=\overline{M_{1/2^n}E}+\overline{NK}+\overline{M_{1/2^n}L}+\overline{NF}+\overline{GE}+\overline{IK}+\overline{HF}+\overline{JL}+\overline{GH}+\overline{IJ}$$

$$\overline{M_{1/2^n}E}=\overline{NK}=\overline{AB}=c \text{ 且 } \overline{M_{1/2^n}L}=\overline{NF}=\overline{CD}=d$$

$$\overline{GH}=\overline{BF}=b-\frac{1}{2^n}a\text{ ; }\overline{IJ}=\overline{BL}=b-(a-\frac{1}{2^n}a)=b-a+\frac{1}{2^n}a$$

$$\therefore L_{M1/4}=c+c+d+d+\overline{GE}+\overline{IK}+\overline{HF}+\overline{JL}+(b-\frac{1}{4}a)+(b-\frac{3}{4}a)$$

$$=2c+2d+2b-a+(\overline{GE}+\overline{IK})+(\overline{HF}+\overline{JL})$$

②$\because \overline{HF} // \overline{JL} \Leftrightarrow \overline{HF} : \overline{JL} = \overline{CF} : \overline{CL} = \frac{1}{2^n} a : (a - \frac{1}{2^n} a) = 1 : (2^n - 1)$

我們暫令$\overline{HF} = x$，則$\overline{JL} = (2^n - 1)x \Rightarrow \overline{QL} = (2^n - 2)x$

又$\overline{QL} // \overline{NK} \Leftrightarrow \overline{QL} : \overline{NK} = \overline{FL} : \overline{FK} \Leftrightarrow (2^n - 2)x : c = (a - \frac{2}{2^n} a) : (b - a)$

$\Rightarrow x = \frac{1}{2^n} \times \frac{ac}{b-a} \Rightarrow \overline{HF} = \frac{1}{2^n} \times \frac{ac}{b-a}$；$\overline{JL} = (2^n - 1) \times \frac{1}{2^n} \times \frac{ac}{b-a}$

$\therefore \overline{HF} + \overline{JL} = 2^n x = \frac{ac}{b-a}$

③$\because \overline{GE} // \overline{IK} \Leftrightarrow \overline{GE} : \overline{IK} = \overline{BE} : \overline{BK} = \frac{1}{2^n} a : (a - \frac{1}{2^n} a) = 1 : (2^n - 1)$

我們暫令$\overline{GE} = y$，則$\overline{IK} = (2^n - 1)y \Rightarrow \overline{PK} = (2^n - 2)y$

又$\overline{PK} // \overline{M_{1/2^n} L} \Leftrightarrow \overline{PK} : \overline{M_{1/2^n} L} = \overline{EK} : \overline{EL} \Leftrightarrow (2^n - 2)y : d = (a - \frac{2}{2^n} a) : (b - a)$

$\Rightarrow y = \frac{1}{2^n} \times \frac{ad}{b-a} \Rightarrow \overline{GE} = \frac{1}{2^n} \times \frac{ad}{b-a}$；$\overline{IK} = (2^n - 1) \times \frac{1}{2^n} \times \frac{ad}{b-a}$

$\therefore \overline{GE} + \overline{IK} = 2^n y = \frac{ad}{b-a}$

④$L_{M_{1/2^n}} = 2c + 2d + 2b - a + (\overline{GE} + \overline{IK}) + (\overline{HF} + \overline{JL})$

$$= 2c + 2d + 2b - a + \frac{ad}{b-a} + \frac{ac}{b-a} = 2c + 2d + 2b + (\frac{a}{b-a})(a + c + d - b)$$

所以$L_{M_{1/2^n}}$ 為一定值($n \ge 2$之自然數)

(2) 梯形之平行周邊形迴路之區域面積 $A_{M1/_{2}^{n}}$

如上圖(11)所示；$\overline{GG_1}=\overline{G_2G_3}=\overline{II_1}=\overline{HH_1}=\overline{H_2H_3}=\overline{JJ_1}=\frac{a}{2^n}$；

$\overline{G_1G_2}=\overline{H_1H_2}=a-\frac{3}{2n}a$；$\overline{I_1I_2}=\overline{J_1J_2}=\overline{G_3H_3}=\left[(b-2a)+\frac{a}{2^n}\right]$；$\overline{I_2J_2}=3(a-\frac{a}{2^n})-b$

$$\therefore A_{M1/2^N}=(\Delta GEG_1+\Delta G_2KG_3+\Delta IPI_1+\Delta HFH_1+\Delta H_2LH_3+\Delta JQJ_1)+(\Delta PG_1G_2+\Delta QH_1H_2)$$

$$+(\Delta I_1M_{1/2^N}I_2+\Delta J_2NJ_1+\Delta G_3RH_3)+\Delta I_2RJ_2$$

且 $\Delta GEG_1\cong\Delta G_2KG_3\cong\Delta IPI_1\cong\Delta HFH_1\cong\Delta H_2LH_3\cong\Delta JQJ_1$ (ASA 全等)

同理 $\Delta PG_1G_2\cong\Delta QH_1H_2$ (ASA 全等) $\Delta I_1M_{1/2^N}I_2\cong\Delta J_2NJ_1\cong\Delta G_3RH_3$ (ASA 全等)

我們利用 $\Delta GEG_1\sim\Delta PG_1G_2\sim\Delta RG_3H_3\sim\Delta RI_2J_2\sim\Delta X_1Y_1Z_1$

又 $\Delta X_1Y_1Z_1=\dfrac{(\frac{a}{2})^2}{b^2-a^2}$；所以 $\Delta GEG_1=\dfrac{(\frac{a}{2^n})^2}{b^2-a^2}$；$\Delta PG_1G_2=\dfrac{\left[(a-\frac{3a}{2^n})\right]^2}{b^2-a^2}$

$\Delta RG_3H_3=\dfrac{\left[(b-2a)+\frac{a}{2^n}\right]^2}{b^2-a^2}$ 且 $\Delta RI_2J_2=\dfrac{\left[3(a-\frac{a}{2^n})-b\right]^2}{b^2-a^2}$

故 $A_{M1/_{2}^{n}}=\dfrac{1}{b^2-a^2}\left\{6\times(\frac{a}{2^n})^2+2(a-\frac{3a}{2^n})^2+3\left[(b-2a)+\frac{a}{2^n}\right]^2+\left[3(a-\frac{a}{2^n})-b\right]^2\right\}$

我們將 $A_{M_{1/2^N}}$ 分子分母上下同除 b^2 並令 $\frac{a}{b}=t$ $x_n=\frac{1}{2^n}$ 轉換成

$$A_{M_{1/2^N}}(t)=\frac{1}{1-t^2}\left[36t^2x_n{}^2-(42t^2-12t)x_n+(15t^2-10t+2)\right]$$

配方得 $A_{M_{1/2^N}}(t)=\dfrac{1}{1-t^2}\left[36t^2(x_n-\frac{7t-2}{12t})^2+\frac{11t^2-12t+4}{4}\right]$

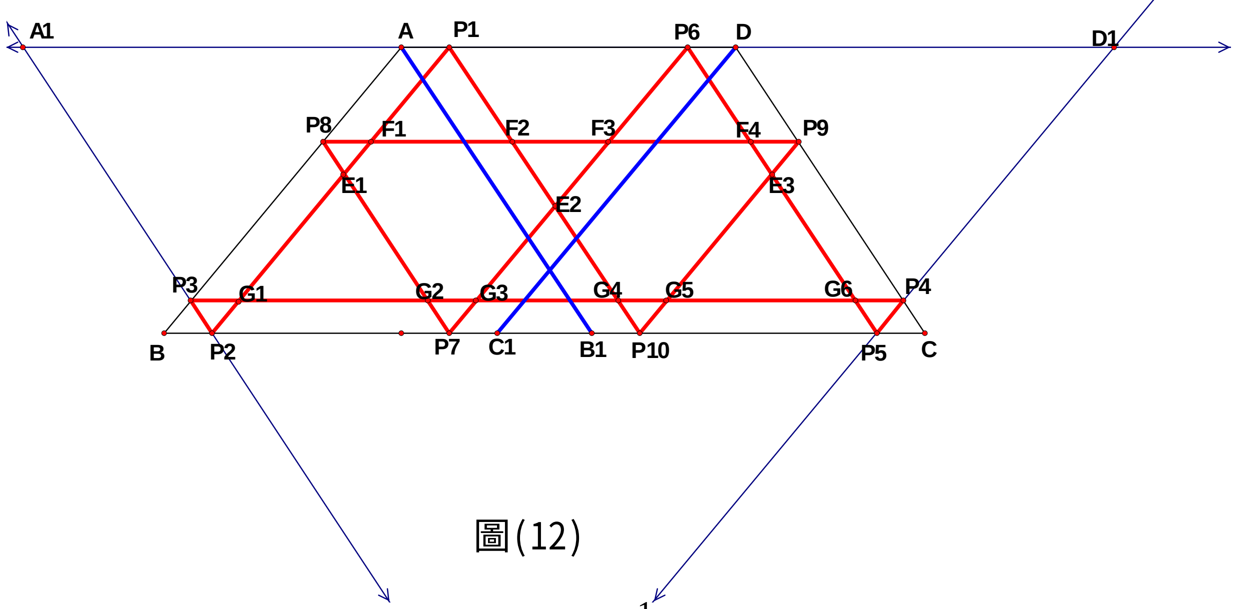
所以當 $x_n=\dfrac{7t-2}{12t}$ $A_{M_{1/2^N}}$ 有最小值 $k=\dfrac{11t^2-12t+4}{4-4t^2}$ 交叉相乘得

$(11+4k)t^2-12t+4(1-k)=0$，利用判別式 $D\geq0\Rightarrow4k^2+7k-2\geq0$

$\Rightarrow(k+2)(4k-1)\geq0\Rightarrow k\geq\frac{1}{4}$ 最小值 $k=\dfrac{11t^2-12t+4}{4-4t^2}=\frac{1}{4}\Rightarrow t=\frac{a}{b}=\frac{1}{2}\Rightarrow b=2a$

$\therefore x_n=\dfrac{3.5-2}{6}=\frac{1}{4}\Rightarrow\frac{1}{2^n}=\frac{1}{4}\Rightarrow n=2$ 所以當 $b=2a;n=4$ 時 $A_{\overline{AP_1}}(\frac{1}{n}a)$ 會有最小值 $\frac{1}{4}$。

5、探討:梯形 $ABCD$ 上底的點 P_1 ($\overline{AP_1}=\frac{1}{n}a$)之平行周邊形迴路之幾何性質



圖(12)

(1) 梯形之平行周邊形迴路路徑長 $L_{\overline{AP_1}}(\frac{1}{n}a)$；$n=3,5,7,.....$

①如圖(12)所示 $\overline{AD}=a;\overline{BC}=b$；$\overline{AB}=c;\overline{DC}=d$ $\overline{AP_1}=\frac{1}{n}a$ 令 $\frac{\overline{AP_3}}{\overline{AB}}=\frac{\overline{DP_4}}{\overline{DC}}=\alpha<1$

$\Delta G_1P_3P_2\cong\Delta G_3G_2P_7\cong\Delta G_5G_4P_{10}\cong\Delta P_4G_6P_5\cong\Delta F_1P_8E_1\cong\Delta P_9F_4E_3(SAS全等)$

$\Delta E_1G_1G_2\cong\Delta E_3G_5G_6(SAS全等)$ 且 $\Delta P_1F_1F_2\cong\Delta P_6F_3F_4\cong\Delta E_2G_3G_4(SAS全等)$

假設 $\overline{P_4P_5}=x;\overline{P_9P_{10}}=y$ $\overline{P_2P_3}=z;\overline{P_7P_8}=w$ 且 $\overline{DC_1}=\overline{P_6P_7}=\overline{P_1P_2}=\overline{AB}=c$

$$\overline{P_1P_6}=(1-\frac{2}{n})a=\overline{P_{10}P_5}\Rightarrow\overline{P_7P_{10}}=b-2a(1-\frac{1}{n})\Rightarrow\overline{DD_1}=b-a-\frac{1}{n}a$$

$$\because\Delta P_4CP_5\sim\Delta P_4D_1D\Rightarrow\frac{\overline{CP_4}}{\overline{P_4D}}=\frac{\overline{CP_5}}{\overline{DD_1}}\Rightarrow\frac{1-\alpha}{\alpha}=\frac{(1/n)a}{b-a-(1/n)a}\Rightarrow\frac{1-\alpha}{1}=\frac{(1/n)a}{b-a}$$

$$\because\overline{P_4P_5}//\overline{DC_1}\Rightarrow\frac{x}{c}=\frac{1-\alpha}{1}=\frac{(1/n)a}{b-a}\Rightarrow x=\frac{ac}{n(b-a)}\text{ 同理, }y=\frac{(n-1)ac}{n(b-a)}\ \therefore x+y=\frac{ac}{(b-a)}$$

$$\because\overline{P_2P_3}//\overline{AB_1}\Rightarrow\frac{z}{d}=\frac{1-\alpha}{1}=\frac{(1/n)a}{b-a}\Rightarrow z=\frac{ad}{n(b-a)}\text{ 同理, }w=\frac{(n-1)ad}{n(b-a)}\ \therefore z+w=\frac{ad}{(b-a)}$$

②$\because\overline{P_3P_4}//\overline{P_8P_9}//\overline{AD}//\overline{BC}\ \therefore\overline{P_3P_4}+\overline{P_8P_9}=\frac{(1-\alpha)a+\alpha b}{1}+\frac{(n-1)(1-\alpha)a+[1-(n-1)(1-\alpha)]b}{1}$

$$\therefore\overline{P_3P_4}+\overline{P_8P_9}=n(1-\alpha)a+[2-n(1-\alpha)]b=\frac{a^2}{b-a}+(2-\frac{a}{b-a})b=\frac{a^2-3ab+2b^2}{b-a}=2b-a$$

③$\therefore L_{\overline{AP_1}}(\frac{1}{n}a)=2c+2d+\frac{ac}{b-a}+\frac{ad}{b-a}+2b-a=2c+2d+2b+(\frac{a}{b-a})(a+c+d-b)$

(2) 梯形之平行周邊形迴路區域面積 $A_{\overline{AP_1}}(\frac{1}{n}a)$

如上圖(12)所示； $\overline{AP_1}=\overline{DP_6}=\overline{G_1P_3}=\overline{G_2G_3}=\overline{G_4G_5}=\overline{P_4G_6}=\overline{F_1P_8}=\overline{F_4P_9}=\frac{a}{n}$

令梯形 $ABCD$ 面積為 1 平方單位,梯形高為 h ;則 $\frac{(a+b)}{2}\times h=1\Rightarrow h=\frac{2}{a+b}$

① $\Delta G_1P_3P_2=(\frac{a}{n})\times(1-\alpha)h\times\frac{1}{2}=(\frac{a}{n})\times\frac{a}{n(b-a)}\times\frac{2}{(a+b)}\times\frac{1}{2}=\frac{1}{n^2}(\frac{a^2}{b^2-a^2})$

② $\because\ \Delta G_1G_2E_1\sim\Delta G_1P_2P_3\ \therefore\overline{G_1P_3}:\overline{G_1G_2}=(\frac{a}{n}):(a-\frac{3a}{n})=1:(n-3)\ \therefore\Delta G_1P_2P_3=\frac{(n-3)^2}{n^2}(\frac{a^2}{b^2-a^2})$

③ $\because\ \Delta G_3E_2G_4\sim\Delta G_1P_2P_3\ \therefore\overline{G_1P_3}:\overline{G_3G_4}=(\frac{a}{n}):(b-2a+\frac{a}{n})\Rightarrow 1+n(\frac{b}{a}-2)$

$$\therefore\Delta G_3E_2G_4=\frac{\left[1+n(\frac{b}{a}-2)\right]^2}{n^2}(\frac{a^2}{b^2-a^2})$$

④ $\because\ \Delta F_3E_2F_2\sim\Delta G_1P_2P_3\ \therefore\overline{G_1P_3}:\overline{F_3F_2}=(\frac{a}{n}):(3a-\frac{3a}{n}-b)\Rightarrow -3+n(3-\frac{b}{a})$

$$\therefore\Delta F_3F_2E_2=\frac{\left[-3+n(3-\frac{b}{a})\right]^2}{n^2}(\frac{a^2}{b^2-a^2})$$

⑤ $\therefore\ A_{\overline{AP_1}}(\frac{1}{n}a)=\frac{a^2}{b^2-a^2}\left\{\frac{6}{n^2}+2(1-\frac{3}{n})^2+3\left[(\frac{b}{a}-2)+\frac{1}{n}\right]^2+\left[(3-\frac{b}{a})-\frac{3}{n}\right]^2\right\}$

令 $t=\frac{b}{a}$ 定值代換

得 $A_{\overline{AP_1}}(\frac{1}{n}a)=\frac{1}{t^2-1}\left\{\frac{6}{n^2}+2(1-\frac{3}{n})^2+3\left[(t-2)+\frac{1}{n}\right]^2+\left[(3-t)-\frac{3}{n}\right]^2\right\}$

化簡成 $A_{\overline{AP_1}}(\frac{1}{n}a)=\frac{1}{t^2-1}\left[\frac{36}{n^2}+\frac{12t-42}{n}+(4t^2-18t+23)\right]$

再令 $x_n=\frac{1}{n}$ 代換

得 $A_{\overline{AP_1}}(x_na)=\frac{1}{t^2-1}\left[36{x_n}^2+(12t-42)x_n+(4t^2-18t+23)\right]$

配方得 $A_{\overline{AP_1}}(x_na)=\frac{36}{t^2-1}(x_n-\frac{7-2t}{12})^2+\frac{12t^2-44t+43}{4t^2-4}$

當 $x_n=\frac{7-2t}{12}$ 時 $A_{\overline{AP_1}}(\frac{1}{n}a)$ 有最小值 $\frac{12t^2-44t+43}{4t^2-4}$;

並令$\dfrac{12t^2-44t+43}{4t^2-4}=k\in R$實數

$\Rightarrow (12-4k)t^2-44t+(43+4k)=0 \because t\in R$判別式$D=44^2-4(12-4k)(43+4k)\geq 0$

$4k^2+31k-8\geq 0 \Leftrightarrow (k+8)(4k-1)\geq 0 \Rightarrow k\leq -8$(不合)或$k\geq \dfrac{1}{4}$

若k的最小值為$\dfrac{1}{4}$時,則$\dfrac{12t^2-44t+43}{4t^2-4}=\dfrac{1}{4}\Rightarrow t=2$

$t=\dfrac{b}{a}=2\Rightarrow b=2a$及$x_n=\dfrac{1}{n}=\dfrac{7-2t}{12}\Rightarrow \dfrac{1}{n}=\dfrac{7-2\times 2}{12}=\dfrac{1}{4}\Rightarrow n=4$

所以當$b=2a;n=4$時$A_{\overline{AP_1}}(\dfrac{1}{n}a)$會有最小值$\dfrac{1}{4}$。

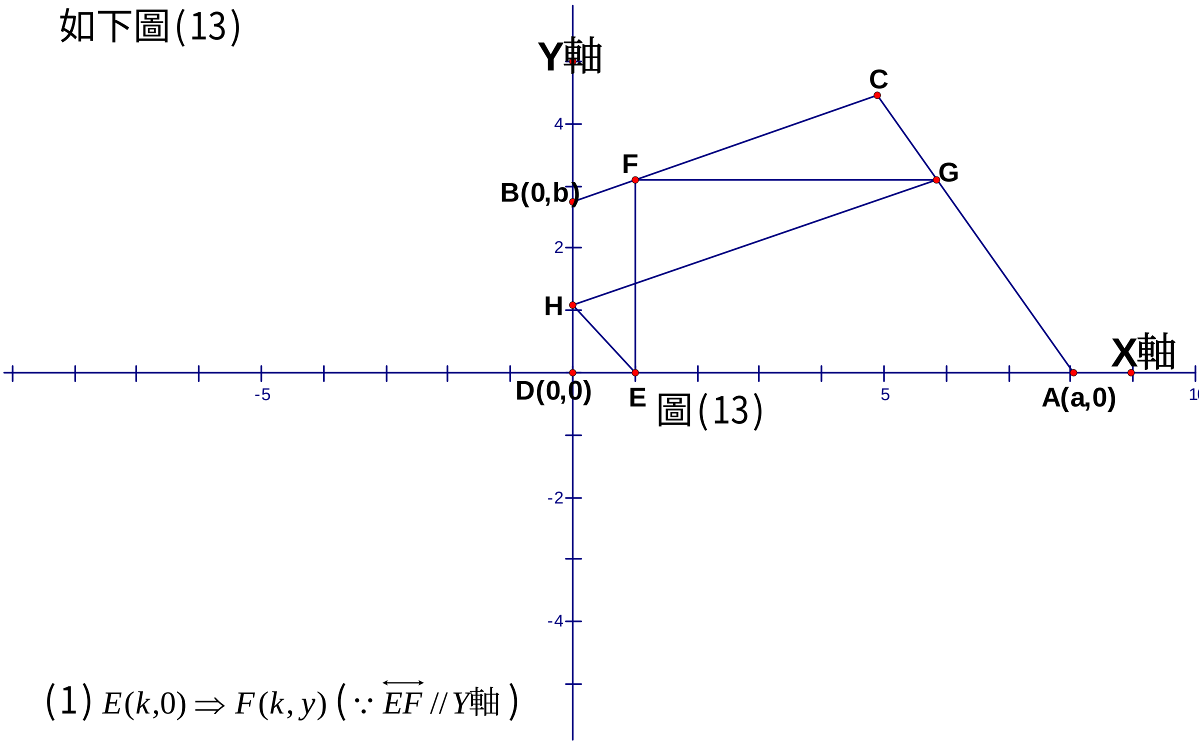
(三)、任意四邊形(兩兩互不平行)之封閉平行周邊形之迴路探討。

1、探討:四邊形 ABCD 之迴路存在性(兩兩互不平行且有一內角為直角)

我們利用平面直角座標,將四邊形 ABCD 量化處理

令 $L_{\overline{BC}}: y-b=m_1x \Leftrightarrow y=m_1x+b$;$L_{\overline{CA}}: y=m_2(x-a) \Leftrightarrow y=m_2x-am_2$

如下圖(13)



圖(13)

(1) $E(k,0) \Rightarrow F(k,y)$ ($\because \overleftrightarrow{EF} // Y$軸)

設 $F(k,y)$ 代入 $L_{\overline{BC}}: y=m_1x+b$ 得 $y=m_1k+b \Rightarrow F(k,m_1k+b)$

(2) $F(k,m_1k+b) \Rightarrow G(x,m_1k+b)$ ($\because \overleftrightarrow{FG} // X$軸)

設 $G(x,m_1k+b)$ 代入 $L_{\overline{CA}}: y=m_2(x-a)$ 得 $m_1k+b=m_2(x-a)$

$$\Rightarrow x=\frac{m_1k+m_2a+b}{m_2} \Rightarrow \therefore G(\frac{m_1k+m_2a+b}{m_2}, m_1k+b)$$

(3) $G(\frac{m_1k+m_2a+b}{m_2}, m_1k+b) \Rightarrow H(0,y)$ ($\because \overline{GH} // \overline{BC}$)

令 $x=0$ 代入 $L_{\overline{GH}}: y-(m_1k+b)=m_1(x-\frac{m_1k+m_2a+b}{m_2})$

得 $y=m_1(-\frac{m_1k+m_2a+b}{m_2})+(m_1k+b)=\frac{m_1(m_2-m_1)}{m_2}k+\frac{(m_2-m_1)}{m_2}b-m_1a$

(4) $H(0,\dfrac{m_1(m_2-m_1)}{m_2}k+\dfrac{(m_2-m_1)}{m_2}b-m_1a)\Rightarrow E(k,0)$ ($\because \overline{HE}\,//\,\overline{AC}$)

令 $y=0$ 代入 $L_{\overline{EH}}:y-(\dfrac{m_1(m_2-m_1)}{m_2}k+\dfrac{(m_2-m_1)}{m_2}b-m_1a)=m_2x$

得 $m_1a+\dfrac{(m_1-m_2)}{m_2}b+\dfrac{m_1(m_1-m_2)}{m_2}k=m_2x$

又 $k=x=\dfrac{m_1m_2a+(m_1-m_2)b+(m_1-m_2)m_1k}{{m_2}^2}$

$\therefore {m_2}^2k=m_1m_2a+(m_1-m_2)b+(m_1-m_2)m_1k$

$\therefore m_1m_2a+(m_1-m_2)b=k({m_2}^2+m_1m_2-{m_1}^2)\ \therefore k=\dfrac{m_1m_2a+(m_1-m_2)b}{{m_2}^2+m_1m_2-{m_1}^2}$

(5)存在性的探討:若 k 存在,則 $0<k=\dfrac{m_1m_2a+(m_1-m_2)b}{{m_2}^2+m_1m_2-{m_1}^2}<a$

令分母 $T_m={m_2}^2+m_1m_2-{m_1}^2\neq 0$ 令 $t=\dfrac{m_2}{m_1}$

則對不等式 ${m_2}^2+m_1m_2-{m_1}^2\neq 0$ 兩邊同除 ${m_1}^2$

得 $(\dfrac{m_2}{m_1})^2+(\dfrac{m_2}{m_1})-1\neq 0\Leftrightarrow t^2+t-1\neq 0$,解出 $t\neq\dfrac{-1\pm\sqrt{5}}{2}\to(\dfrac{m_2}{m_1})\neq\dfrac{-1\pm\sqrt{5}}{2}$

①若 $T_m={m_2}^2+m_1m_2-{m_1}^2>0$, $t=\dfrac{m_2}{m_1}\in(-\infty,\dfrac{-1-\sqrt{5}}{2})$

$k=\dfrac{m_1m_2a+(m_1-m_2)b}{{m_2}^2+m_1m_2-{m_1}^2}<a\Rightarrow m_1m_2a+(m_1-m_2)b<a{m_2}^2+m_1m_2a-a{m_1}^2$

$\Rightarrow -(m_2-m_1)b<a({m_2}^2-{m_1}^2)\Rightarrow -(m_2-m_1)b<a(m_2-m_1)(m_2+m_1)$

其中圖(13)所示: $m_1\neq m_2$ 且 $m_1>0$; $m_2<0\Rightarrow m_2-m_1<0$ 且 $m_1m_2<0$

$\therefore -b>a(m_2+m_1)\Rightarrow -(\dfrac{b}{a})>m_2+m_1\Rightarrow m_{\overline{AB}}>m_{\overline{AC}}+m_{\overline{CB}}$

又 $0<k=\dfrac{m_1m_2a+(m_1-m_2)b}{{m_2}^2+m_1m_2-{m_1}^2}\Rightarrow m_1m_2a+(m_1-m_2)b>0\Rightarrow m_1m_2a>(m_2-m_1)b$

將不等式兩邊同除 m_1m_2 得 $a<(\dfrac{1}{m_1}-\dfrac{1}{m_2})b\Leftrightarrow\dfrac{1}{b/a}<\dfrac{1}{m_1}-\dfrac{1}{m_2}$

$\Leftrightarrow\dfrac{1}{m_2}<\dfrac{1}{m_1}+(\dfrac{1}{-b/a})\Leftrightarrow\dfrac{1}{m_{\overline{AC}}}<\dfrac{1}{m_{\overline{CB}}}+\dfrac{1}{m_{\overline{AB}}}$

②若 $T_m = m_2^{\ 2} + m_1 m_2 - m_1^{\ 2} < 0$，$t = \dfrac{m_2}{m_1} \in (\dfrac{-1-\sqrt{5}}{2}, 0)$

$$k = \frac{m_1 m_2 a + (m_1 - m_2) b}{m_2^{\ 2} + m_1 m_2 - m_1^{\ 2}} < a \quad \Rightarrow -(\frac{b}{a}) < m_2 + m_1 \Rightarrow m_{\overline{AB}} < m_{\overline{AC}} + m_{\overline{CB}}$$

又 $0 < k = \dfrac{m_1 m_2 a + (m_1 - m_2) b}{m_2^{\ 2} + m_1 m_2 - m_1^{\ 2}} \quad \Rightarrow \dfrac{1}{m_2} > \dfrac{1}{m_1} + (\dfrac{1}{-b/a}) \Leftrightarrow \dfrac{1}{m_{\overline{AC}}} > \dfrac{1}{m_{\overline{CB}}} + \dfrac{1}{m_{\overline{AB}}}$

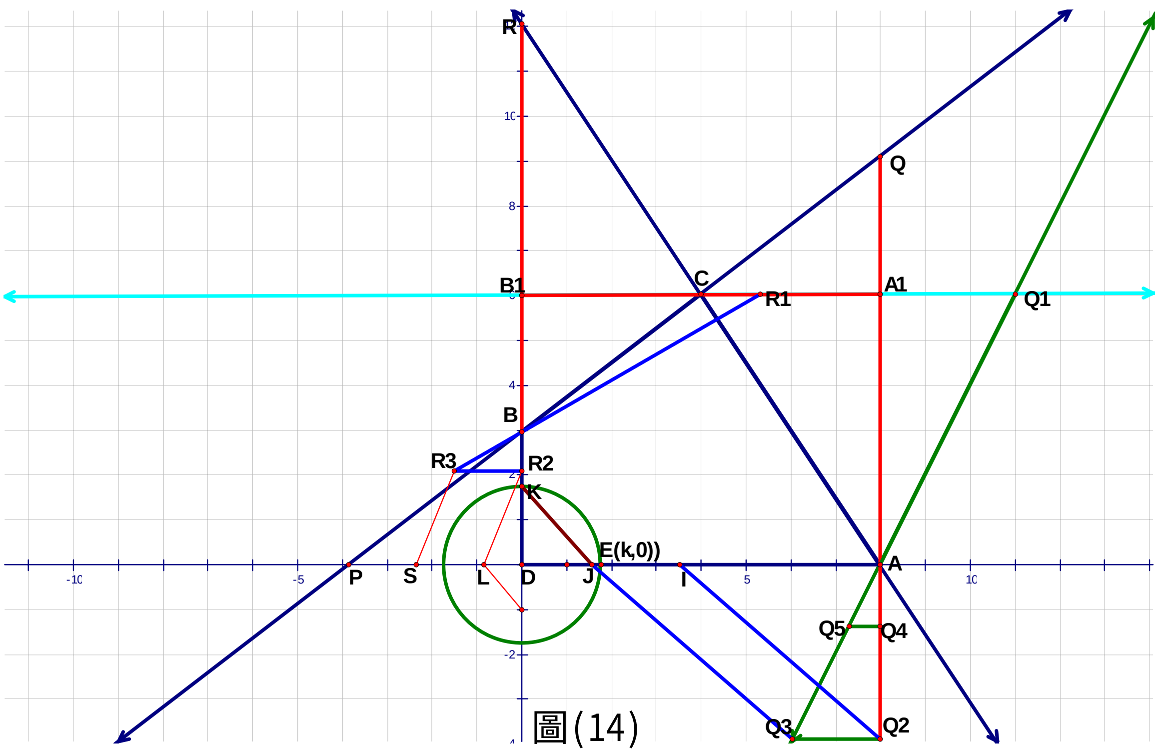
(6)探討四邊形 ABCD 之平行迴路起點 $E(k,0)$ 作圖

①先將 $k = \dfrac{m_1 m_2 a + (m_1 - m_2) b}{m_2^{\ 2} + m_1 m_2 - m_1^{\ 2}}$ 轉換成

$\dfrac{(\dfrac{m_2}{m_1})a + (1 - \dfrac{m_2}{m_1})\dfrac{b}{m_1}}{(\dfrac{m_2}{m_1})^2 + (\dfrac{m_2}{m_1}) - 1} = \dfrac{ta + (1-t)b_x}{t^2 + t - 1} = \dfrac{a - b_x + (1/t)b_x}{t - (1/t) + 1}$ 其中 $t = \dfrac{m_2}{m_1} < 0$，$b_x = \dfrac{b}{m_1}$，假設

$\overline{BB_1} = y_1$，$\overline{B_1C} = x_1$，$\overline{AA_1} = y_2$，$\overline{A_1C} = x_2$ 且 $m_1 = \dfrac{y_1}{x_1} > 0$，$m_2 = \dfrac{-y_2}{x_2} < 0$

$a = x_1 + x_2$，$b = y_2 - y_1$



圖(14)

②平行迴路起點$E(k,0)$作圖

(a)延伸$\overleftrightarrow{AC},\overleftrightarrow{BC}$分別交 Y 軸與鉛直線$x=a$於$R,Q$兩點，

(b)過 C 點作平行$\overline{AD}$之平行線分別交 Y 軸與鉛直線$x=a$於B_1,A_1兩點，

則$\overline{A_1Q}=\dfrac{x_2y_1}{x_1};\overline{B_1R}=\dfrac{x_1y_2}{x_2}$

(c)在$\overline{A_1B_1}$線上取兩點Q_1,R_1使得$\overline{A_1Q}=\overline{A_1Q_1}=\dfrac{x_2y_1}{x_1};\overline{B_1R}=\overline{B_1R_1}=\dfrac{x_1y_2}{x_2}$

(d)在鉛直線$x=a$上取兩點Q_2,Q_4使得$\overline{AQ_4}=1;\overline{AQ_2}=b_x$；連接$\overline{AQ_1}$

並延伸$\overline{AQ_1}$交Q_2,Q_4之水平線於Q_3,Q_5；則$\overline{Q_4Q_5}=\dfrac{x_2y_1}{x_1y_2};\overline{Q_2Q_3}=\dfrac{x_2y_1}{x_1y_2}b_x$

(e)在$\overline{AD}$上取一點 I 使得$\overline{AI}=\overline{AQ_2}=b_x$；並連接$\overline{Q_2I}$，過$Q_3$作平行$\overline{Q_2I}$

之平行線交$\overline{AD}$於J；則$\overline{IJ}=\overline{Q_2Q_3}=(-1/t)b_x$；所以$\overline{DJ}=a-b_x+(1/t)b_x$

(f)同理在 Y 軸上取$\overline{BR_2}=1$並連接$\overline{BR_1}$，過R_2作平行線交$\overline{BR_1}$延長線於R_3；

則$\overline{R_2R_3}=\dfrac{x_1y_2}{x_2y_1}=-t$，並在$\overline{PD}$上取$\overline{SL}=\overline{R_2R_3}$；所以$\overline{DL}=1-(-t)+(-1/t)$

(g)連接$\overline{LD_1}$，並過J作平行$\overline{LD_1}$之平行線交 Y 軸於K

$\because \Delta LDD_1 \sim \Delta JDK \ \therefore \overline{LD}:\overline{DD_1}=\overline{DJ}:\overline{DK}$

$\Rightarrow 1+t-(1/t):1=a-b_x+(1/t)b_x:\overline{DK}$

$\Rightarrow \overline{DK}=\dfrac{a-b_x+(1/t)b_x}{1+t-1/t}$

(h)最後以D為圓心，$\overline{DK}$為半徑畫圓，交$\overline{AD}$於E點，則$E(k,0)$

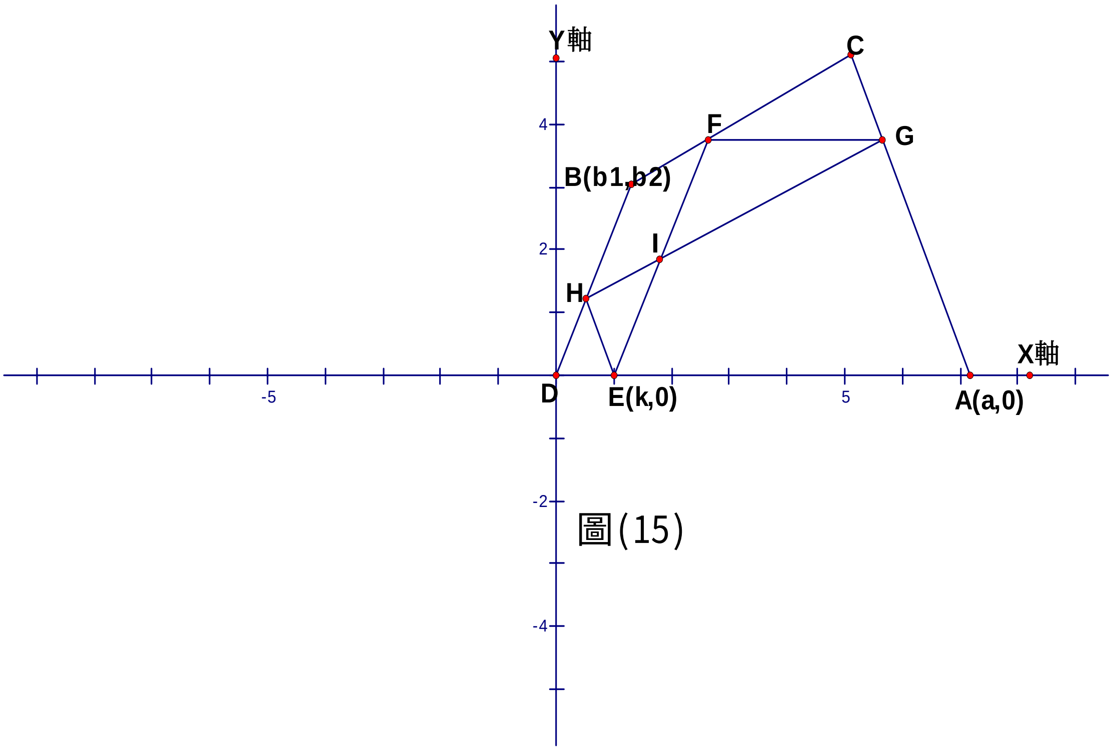
即為所求。

2、探討:四邊形 ABCD 之迴路(兩兩互不平行且內角均不為直角)

我們利用平面直角座標,將四邊形 ABCD 量化處理，如下圖(15)：

假設 $A(a,0)$、$B(b_1,b_2)$，$L_{\overline{BD}}: y=m_1x$；$L_{\overline{BC}}: y=m_2(x-b_1)+b_2$；$L_{\overline{CA}}: y=m_3(x-a)$

令起點 $E(k,0)$ 在 x 軸上其中 $k\in(0,a)$



圖(15)

(1) $E(k,0)\Rightarrow F(x,y)$ ($\because \overline{EF}\,//\,\overline{BD}$) $\therefore m_{\overline{EF}}=m_1$

$$\begin{cases} y=m_1(x-k) \\ y=m_2(x-b_1)+b_2 \end{cases} \Leftrightarrow L_{\overline{EF}}\text{相交}L_{\overline{BC}}\text{於 } F\left(\frac{m_1k-m_2b_1+b_2}{m_1-m_2},\frac{m_1m_2(k-b_1)+m_1b_2}{m_1-m_2}\right)$$

(2) $F\left(\frac{m_1k-m_2b_1+b_2}{m_1-m_2},\frac{m_1m_2(k-b_1)+m_1b_2}{m_1-m_2}\right)\Rightarrow G\left(x,\frac{m_1m_2(k-b_1)+m_1b_2}{m_1-m_2}\right)$代入

$L_{\overline{CA}}: y=m_3(x-a)$ $\therefore G\left(\frac{m_1m_2(k-b_1)+m_1b_2}{m_3(m_1-m_2)}+a,\frac{m_1m_2(k-b_1)+m_1b_2}{m_1-m_2}\right)$

(3) $G\left(\frac{m_1m_2(k-b_1)+m_1b_2}{m_3(m_1-m_2)}+a,\frac{m_1m_2(k-b_1)+m_1b_2}{m_1-m_2}\right)\Rightarrow H(x,y)$

$\because \overline{GH}\,//\,\overline{BC}$ $\therefore m_{\overline{GH}}=m_2$

$$\begin{cases} y = m_2\left[x - (\dfrac{m_1m_2(k-b_1)+m_1b_2}{m_3(m_1-m_2)} + a)\right] + \dfrac{m_1m_2(k-b_1)+m_1b_2}{(m_1-m_2)} \\ y = m_1x \end{cases} \Leftrightarrow L_{\overline{GH}}\text{相交}L_{\overline{BD}}\text{於}H$$

$H(x_0, m_1x_0)$ 其中 $x_0 = \dfrac{m_1(m_3-m_2)[m_2(k-b_1)+b_2] - am_2m_3(m_1-m_2)}{m_3(m_1-m_2)^2}$

(4) $H(x_0, m_1x_0) \Rightarrow E(k,0)$ ($\because \overline{EH} // \overline{AC}$) $\therefore m_{\overline{EH}} = m_3$

$$\begin{cases} y = m_3[x - x_0] + m_1x_0 \\ y = 0 \end{cases} \Leftrightarrow L_{\overline{EH}}\text{相交 } x\text{軸於 } E(k,0)\text{點}$$

則 $k = \dfrac{m_1(m_3-m_2)(m_1-m_3)(b_2-m_2b_1) + am_2m_3(m_1-m_2)(m_3-m_1)}{m_1m_2(m_3-m_2)(m_3-m_1) - m_3^{\ 2}(m_1-m_2)^2}$

(5)存在性的探討:若 k 存在,則令分母為 T_m

則 $T_m = m_1m_2(m_3-m_2)(m_3-m_1) - m_3^{\ 2}(m_1-m_2)^2 \neq 0$

將 T_m 乘開為: $T_m = 3m_1m_2m_3^{\ 2} - m_1^{\ 2}m_2m_3 - m_1m_2^{\ 2}m_3 + (m_1^{\ 2}m_2^{\ 2} - m_2^{\ 2}m_3^{\ 2} - m_3^{\ 2}m_1^{\ 2})$

$$= (m_1m_2m_3)^2\left[(\frac{3}{m_1m_2} - \frac{1}{m_2m_3} - \frac{1}{m_1m_3}) + (\frac{1}{m_3^{\ 2}} - \frac{1}{m_1^{\ 2}} - \frac{1}{m_2^{\ 2}})\right]$$

$$= (m_1m_2m_3)^2\left[(\frac{1}{m_1m_2} - \frac{1}{m_2m_3} - \frac{1}{m_1m_3}) + (\frac{1}{m_3^{\ 2}}) - (\frac{1}{m_1^{\ 2}} - \frac{2}{m_1m_2} + \frac{1}{m_2^{\ 2}})\right]$$

$$= (m_1m_2m_3)^2\left[(\frac{1}{m_1m_2} - \frac{1}{m_2m_3} - \frac{1}{m_1m_3}) + (\frac{1}{m_3^{\ 2}}) - (\frac{1}{m_1} - \frac{1}{m_2})^2\right]$$

$$= (m_1m_2m_3)^2\left[\frac{1}{m_2}(\frac{1}{m_1} - \frac{1}{m_3}) + \frac{1}{m_3}(\frac{1}{m_3} - \frac{1}{m_1}) - (\frac{1}{m_1} - \frac{1}{m_2})^2\right]$$

$$= (m_1m_2m_3)^2\left[(\frac{1}{m_1} - \frac{1}{m_3}) \times (\frac{1}{m_2} - \frac{1}{m_3}) - (\frac{1}{m_1} - \frac{1}{m_2})^2\right]$$

$T_m \neq 0$;則 $(\dfrac{1}{m_1} - \dfrac{1}{m_3}) \times (\dfrac{1}{m_2} - \dfrac{1}{m_3}) \neq (\dfrac{1}{m_1} - \dfrac{1}{m_2})^2$

$\therefore (\dfrac{1}{m_{\overline{BD}}} - \dfrac{1}{m_{\overline{AC}}}) \times (\dfrac{1}{m_{\overline{BC}}} - \dfrac{1}{m_{\overline{AC}}}) \neq (\dfrac{1}{m_{\overline{BD}}} - \dfrac{1}{m_{\overline{BC}}})^2$

①若 $T_m = (m_1m_2m_3)^2\left[(\frac{1}{m_1}-\frac{1}{m_3})\times(\frac{1}{m_2}-\frac{1}{m_3})-(\frac{1}{m_1}-\frac{1}{m_2})^2\right]>0$

$T_m>0 \Rightarrow (\frac{1}{m_1}-\frac{1}{m_3})\times(\frac{1}{m_2}-\frac{1}{m_3})>(\frac{1}{m_1}-\frac{1}{m_2})^2$

(a)先處理 $k<a$ 的情形(令 $b_y=b_2-m_2b_1$)

$$m_1(m_3-m_2)(m_1-m_3)(b_2-m_2b_1)+am_2m_3(m_1-m_2)(m_3-m_1)<a\times T_m$$

$$\Rightarrow m_1(m_3-m_2)(m_1-m_3)\times b_y < a\times m_1(m_3-m_2)(2m_2m_3-m_1m_2-m_1m_3)$$

$$\Rightarrow m_1(m_2-m_3)(m_1-m_3)\times(-b_y) < a\times m_1(m_2-m_3)(m_1m_2+m_1m_3-2m_2m_3)$$

,由圖(15)可知 $m_1>0$;$m_2>0$;$m_3<0$,並延伸 $\overleftrightarrow{BC}$ 交 Y 軸於

$P(0,b_2-m_2b_1)=P(0,b_y)$ 且 $m_{\overline{AP}}=\frac{-b_y}{a}$

將不等式兩邊同除 $m_1(m_2-m_3)$ 可得 $(m_1-m_3)\times(-b_y)<a\times m_1(m_2+m_3)$

$$\Rightarrow (\frac{-b_y}{a})<\frac{m_1m_2+m_1m_3-2m_2m_3}{m_1-m_3} \quad \Rightarrow m_{\overline{AP}}<\frac{m_{\overline{BD}}m_{\overline{BC}}+m_{\overline{BD}}m_{\overline{AC}}-2m_{\overline{BC}}m_{\overline{AC}}}{m_{\overline{BD}}-m_{\overline{AC}}}$$

(b)再處理 $0<k$ 的情形($T_m>0$)

$$\Rightarrow 0<m_1(m_3-m_2)(m_1-m_3)b_y+am_2m_3(m_1-m_2)(m_3-m_1)$$

$\Rightarrow 0<(m_1-m_3)\lfloor m_1(m_3-m_2)b_y+am_2m_3(m_2-m_1)\rfloor$ 同除 $(m_1-m_3)>0$

$\Rightarrow m_1(m_3-m_2)(-b_y)<am_2m_3(m_2-m_1)$;再除 $(m_3-m_2)<0$

$$\Rightarrow \frac{-b_y}{a}>\frac{m_2m_3(m_2-m_1)}{m_1(m_3-m_2)} \Rightarrow m_{\overline{AP}}>\frac{m_{\overline{BC}}m_{\overline{AC}}(m_{\overline{BC}}-m_{\overline{BD}})}{m_{\overline{BD}}(m_{\overline{AC}}-m_{\overline{BC}})}$$

$\therefore T_m>0$ 時, $\frac{m_{\overline{BC}}m_{\overline{AC}}(m_{\overline{BC}}-m_{\overline{BD}})}{m_{\overline{BD}}(m_{\overline{AC}}-m_{\overline{BC}})}<m_{\overline{AP}}<\frac{m_{\overline{BD}}m_{\overline{BC}}+m_{\overline{BD}}m_{\overline{AC}}-2m_{\overline{BC}}m_{\overline{AC}}}{m_{\overline{BD}}-m_{\overline{AC}}}$

②若 $T_m<0$ 可得 $\frac{m_{\overline{BD}}m_{\overline{BC}}+m_{\overline{BD}}m_{\overline{AC}}-2m_{\overline{BC}}m_{\overline{AC}}}{m_{\overline{BD}}-m_{\overline{AC}}}<m_{\overline{AP}}<\frac{m_{\overline{BC}}m_{\overline{AC}}(m_{\overline{BC}}-m_{\overline{BD}})}{m_{\overline{BD}}(m_{\overline{AC}}-m_{\overline{BC}})}$

伍、研究結果

一、在 ΔABC 中，經過迴路 $D \Rightarrow E \Rightarrow F \Rightarrow G \Rightarrow H \Rightarrow I \Rightarrow D$ 之路徑總長等於外圍 ΔABC 周長 $L_{D\to I}(\alpha)=L_{\Delta ABC}$（$\alpha=\dfrac{\overline{AD}}{\overline{AB}}$），其路徑所圍成的區域面積如下表格(5)所示：(令 ΔABC面積 $=1$平方單位)

表格(5) α 範圍	$\alpha \in (0,\frac{1}{3})$	$\alpha=\frac{1}{3}$	$\alpha \in (\frac{1}{3},\frac{1}{2})$	$\alpha=\frac{1}{2}$
$A_{D\to I}(\alpha)$	$A_{D\to I}(\alpha)=12(\alpha-\frac{1}{4})^2+\frac{1}{4}$	$A_{D\to I}(\frac{1}{3})=\frac{1}{3}$	$A_{D\to I}(\alpha)=-15(\alpha-\frac{2}{5})^2+\frac{2}{5}$	$A_{D\to F}(\frac{1}{2})=\frac{1}{4}$
	$A_{D\to I}$ 極小值$=\frac{1}{4}$		$A_{D\to I}$ 極大值$=\frac{2}{5}$	
$L_{D\to I}(\alpha)$	$L_{D\to I}(\alpha)=L_{\Delta ABC}$	$L_{D\to I}(\frac{1}{3})=L_{\Delta ABC}$	$L_{D\to I}(\alpha)=L_{\Delta ABC}$	$L_{D\to F}(\frac{1}{2})=\frac{1}{2}L_{\Delta ABC}$

二、在梯形 $ABCD$ 中，經迴路 $M\Rightarrow E\Rightarrow G\Rightarrow H\Rightarrow F\Rightarrow M$之路徑長與梯形周長關係和區域面積之極值；如下表格(6)所示：(令 梯形$ABCD$面積 $=1$平方單位)

$\overline{AM}_{1/2^n}$	$\frac{1}{2}a$	$\frac{1}{4}a$			$\frac{1}{2^n}a$	$\frac{1}{n}a$
L (路徑)	$a+b+c+d+\frac{a}{2}(\frac{c+d}{b-a}-3)$	$2b+2c+2d+\frac{a}{b-a}(a+c+d-b)$				
$t=\frac{a}{b}$	$t=\frac{1}{2}$	$t<\frac{4}{9}$	$t=\frac{4}{9}$	$t>\frac{4}{9}$	$t=\frac{1}{2}$及 $n=2$	$t=\frac{1}{2}$及 $n=4$
$A(t)$ 面積極值	$A(t)$ 最小值$\frac{1}{4}$	$A(t)$ 無極值	$A(t)$ 最小值 $\frac{4}{13}$	$A(t)$ 最小值 $\frac{1}{4}$	$A(t)$ 最小值 $\frac{1}{4}$	$A(t)$ 最小值 $\frac{1}{4}$

三、任意四邊形(兩兩互不平行)之封閉平行周邊形之迴路探討。

1、四邊形 ABCD 之迴路存在性(兩兩互不平行且<u>有一內角為直角</u>)

其直線斜率關係為:

(1)當 $t=\dfrac{m_{\overline{AC}}}{m_{\overline{CB}}}\in(-\infty,\dfrac{-1-\sqrt{5}}{2})$ 時,滿足 $m_{\overline{AB}}>m_{\overline{AC}}+m_{\overline{CB}}$ 且 $\dfrac{1}{m_{\overline{AC}}}<\dfrac{1}{m_{\overline{CB}}}+\dfrac{1}{m_{\overline{AB}}}$

(2)當 $t=\dfrac{m_{\overline{AC}}}{m_{\overline{CB}}}\in(\dfrac{-1-\sqrt{5}}{2},0)$ 時,滿足 $m_{\overline{AB}}<m_{\overline{AC}}+m_{\overline{CB}}$ 且 $\dfrac{1}{m_{\overline{AC}}}>\dfrac{1}{m_{\overline{CB}}}+\dfrac{1}{m_{\overline{AB}}}$

2、四邊形 ABCD 之迴路存在性(兩兩互不平行且<u>內角均不為直角</u>)

其直線斜率關係為: $(\dfrac{1}{m_{\overline{BD}}}-\dfrac{1}{m_{\overline{AC}}})\times(\dfrac{1}{m_{\overline{BC}}}-\dfrac{1}{m_{\overline{AC}}})\neq(\dfrac{1}{m_{\overline{BD}}}-\dfrac{1}{m_{\overline{BC}}})^2$

(1)當 $(\dfrac{1}{m_1}-\dfrac{1}{m_3})\times(\dfrac{1}{m_2}-\dfrac{1}{m_3})>(\dfrac{1}{m_1}-\dfrac{1}{m_2})^2$

滿足: $\dfrac{m_{\overline{BC}}m_{\overline{AC}}(m_{\overline{BC}}-m_{\overline{BD}})}{m_{\overline{BD}}(m_{\overline{AC}}-m_{\overline{BC}})}<m_{\overline{AP}}<\dfrac{m_{\overline{BD}}m_{\overline{BC}}+m_{\overline{BD}}m_{\overline{AC}}-2m_{\overline{BC}}m_{\overline{AC}}}{m_{\overline{BD}}-m_{\overline{AC}}}$

(2)當 $(\dfrac{1}{m_1}-\dfrac{1}{m_3})\times(\dfrac{1}{m_2}-\dfrac{1}{m_3})<(\dfrac{1}{m_1}-\dfrac{1}{m_2})^2$

滿足: $\dfrac{m_{\overline{BD}}m_{\overline{BC}}+m_{\overline{BD}}m_{\overline{AC}}-2m_{\overline{BC}}m_{\overline{AC}}}{m_{\overline{BD}}-m_{\overline{AC}}}<m_{\overline{AP}}<\dfrac{m_{\overline{BC}}m_{\overline{AC}}(m_{\overline{BC}}-m_{\overline{BD}})}{m_{\overline{BD}}(m_{\overline{AC}}-m_{\overline{BC}})}$

3. 四邊形 ABCD 之平行迴路以任意一點 $K(k_1,0)$ 為起點,利用 Excel 軟體計算出其平行迴路會收斂至最短路徑 $E\Rightarrow F\Rightarrow G\Rightarrow H\Rightarrow E(k,0)$

(1)<u>有一內角為直角:</u> $k=\dfrac{m_1m_2a+(m_1-m_2)b}{m_2{}^2+m_1m_2-m_1{}^2}$

(2)<u>內角均不為直角:</u> $k=\dfrac{m_1(m_3-m_2)(m_1-m_3)(b_2-m_2b_1)+am_2m_3(m_1-m_2)(m_3-m_1)}{m_1m_2(m_3-m_2)(m_3-m_1)-m_3{}^2(m_1-m_2)^2}$

陸、參考資料及其他

朱建正（民 96.2）。國三下數學教科書（2 版）。臺南市：翰林

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【評語】030410

1. 本作品頗具創意，內容豐富，證明嚴謹。

2. 惟實用性稍不足。