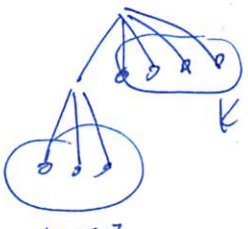
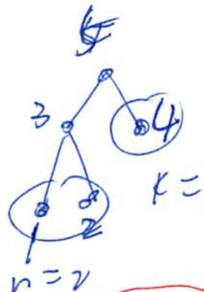
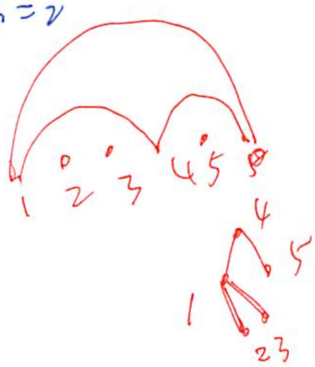
2014/05/06.

①

目標：求  問此類樹的 # linear extensions.

例 



linear extensions 有

12345
12435
21345
21435
14235
24135
41235
42135

共有 8 個 $= a_{21}$

$$a_{n,k}=\frac{(n+k+1)!}{n+1}$$

n \ k	0	1	2	3	4	5	6
0	1	2	6	24	120	720	5040
1	1	3	12	60	360	2520	
2	2	8	40	240	1680	13440	
3	6	30	180	1260	10080		
4	24	144	1008	8064			
5	120	840	6720				
6	720	5760					
7	5040						

A77102

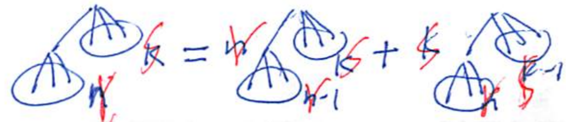
~~$\frac{n!}{0!}$ $\frac{n!}{1!}$~~ $\frac{n!}{2!}$ $\frac{n!}{3!}$ $\frac{n!}{4!}$

猜

$$a_{n,k}=\sum_{m}\binom{k}{m}(n+k-m)!\,m!$$

驗 check

1. $a_{n,0}=n!$ →  n點 有 $n!$ 法 ✓
2. $a_{0,k}=(k+1)!$ →  k點 有 $(k+1)!$ 法 ✓
3. $a_{n,k}=n\,a_{n-1,k}+k\,a_{n,k-1}$ ✓



②

- $a_{n,k}$ 滿足 (1):

$$a_{n,0} = \sum_m \binom{0}{m} (n - m)!\, m!$$
$$= n! \quad ✓$$

∑ 只有 $m=0$ 一項

- $a_{n,k}$ 滿足 (2):

$$a_{0,k} = \sum_m \binom{k}{m} (k-m)!\, m! \qquad m = 0 \ldots k$$
$$= \sum_m \frac{k!}{\cancel{m!}\,\cancel{(k-m)!}} \cancel{(k-m)!}\,\cancel{m!}$$
$$= (k+1)\,k! = (k+1)! \quad ✓$$

- $a_{n,k}$ 滿足 (3).

$$\underbrace{\sum_{m\geq 0} \binom{k}{m} (n+k-m)!}_{①} \overset{?}{=} \underbrace{n \sum_{m\geq 0} \binom{k}{m} (n-1+k-m)!\, m!}_{②} + \underbrace{k \sum_{m\geq 0} \binom{k-1}{m} (n+k-1-m)!\, m!}_{③}$$

① $= k!\left(n! + \frac{(n+1)!}{1!} + \frac{(n+2)!}{2!} + \cdots + \frac{(n+k)!}{k!} \right)$

② $= nk!\left((n-1)! + \frac{n!}{1!} + \frac{(n+1)!}{2!} + \cdots + \frac{(n+k-1)!}{k!} \right)$

③ $= k!\left[n! + \frac{(n+1)!}{1!} + \cdots + \frac{(n+k-1)!}{(k-1)!} \right]$

故只要证

$$① = k!\Big[\underbrace{n! + \frac{(n+1)!}{1!} + \cdots + \frac{(n+k-1)!}{(k-1)!}}_{②} + \underbrace{\frac{n+k!}{(k)!}}_{③}\Big]$$

只要证明

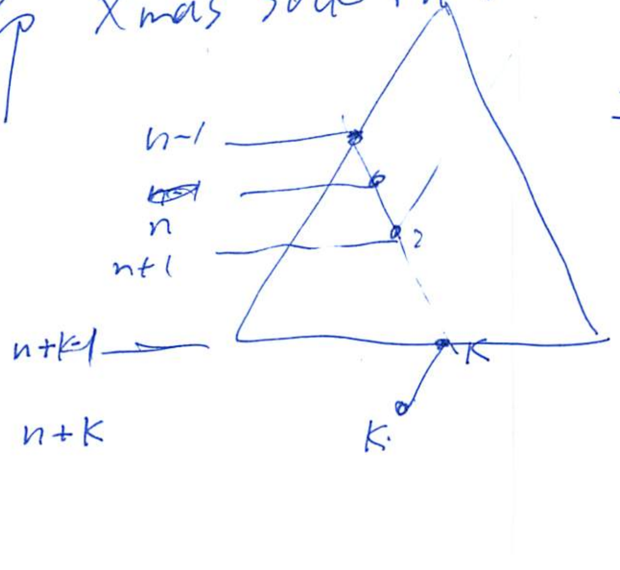
$$\cancel{k!}\,\frac{(n+k)!}{k!} \overset{?}{=} n\cdot \cancel{k!}\Big[(n-1)! + \frac{n!}{1!} + \frac{(n+1)!}{2!} + \cdots + \frac{(n+k-1)!}{k!}\Big]$$

$$\Longrightarrow \frac{(n+k)!}{k!\,n!} \overset{?}{=} \frac{n}{n}\Big[\frac{(n-1)!}{(n-1)!} + \frac{n!}{(n-1)!\,1!} + \frac{(n+1)!}{(n-1)!\,2!} + \cdots + \frac{(n+k-1)!}{(n-1)!\,k!}\Big]$$

$$\Longrightarrow \binom{n+k}{k} \overset{?}{=} \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k}$$

但此即 Xmas sock 恒!!

Yes!



④

母式 学生の作法

証
$$\frac{(n+k)!}{k!} \overset{?}{=} n!\left((n-1)! + \frac{n!}{1!} + \frac{(n+1)!}{2!} + \cdots + \frac{(n+k-1)!}{k!}\right).$$

方法：逐项相加。~~觀~~ 觀察变化。

claim:
$$n!\left((n-1)! + \cdots + \frac{(n+r-1)!}{r!}\right) = \frac{(n+r)!}{r!}$$

用歸納法

$r=0$ 時成立

設 r 時成立.

則 $r+1$ 時

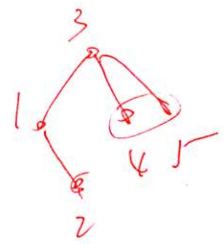
為 $\underbrace{n!\left((n-1)! + \cdots + \frac{(n+r-1)!}{r!}\right.}\left. + \frac{(n+r)!}{(r+1)!}\right)$

$$= \frac{(n+r)!}{r!} + n!\frac{(n+r)!}{(r+1)!}$$

$$= \frac{(n+r+1)!}{(r+1)!}\left(\underbrace{\frac{r+1}{n+r+1} + n\frac{1}{(n+r+1)}}_{=1}\right)$$

$$= \frac{(n+r+1)!}{(r+1)!} \checkmark.$$

(4)



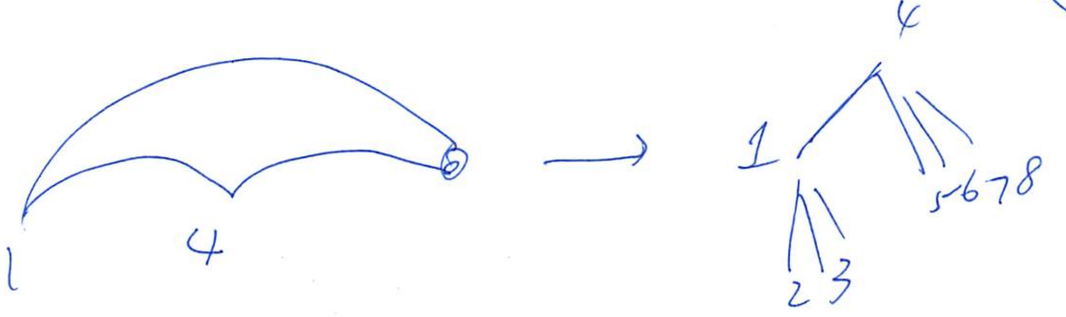
12個

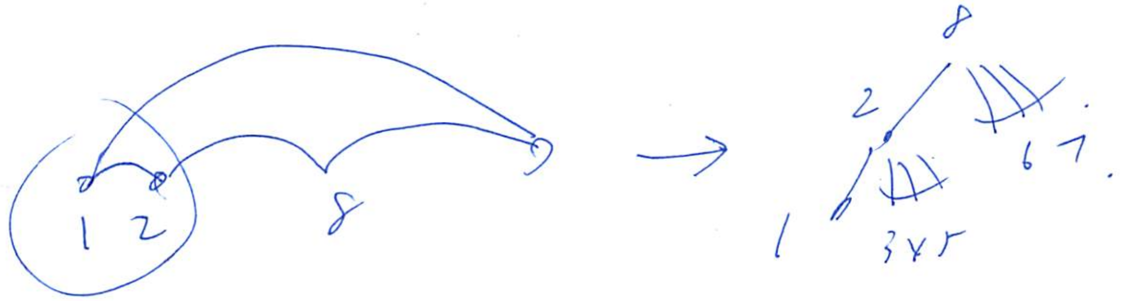
24513
25413
42513
45213
52413
54213
42153
52143

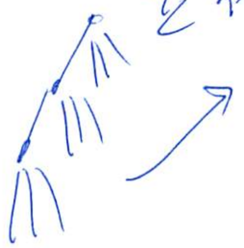
21453
21543
24453
~~24513~~
~~25413~~
25143

困难有

5.







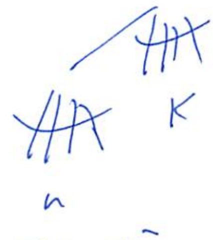
立体!!

就难以控制了.

目前能做的只是

转化成 tree 后

为



的形式

general approach needed

"elementary"

need NEW idea!!

否则又做到这里了

Book 3 of Taco page 70 5.1.4 exercise 20

Knuth 有一版的证明

但是无法用在这里